Least-squares Fourier phase estimation from the modulo 2π bispectrum phase

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The recovery of Fourier phases from measurements of the bispectrum occupies a vital role in many astronomical speckle imaging schemes. In a recent paper [J. Opt. Soc. Am. A **7**, 14 (1990)] it was suggested that a least-squares solution to this problem must fail if the bispectrum phase is known only modulo 2π. Here an alternative nonlinear least-squares algorithm is presented that differs from the linear method discussed in the aforementioned paper and that permits the fitting of Fourier phases directly to modulo 2π measurements of the bispectrum phase, thus eliminating any need for phase unwrapping. Numerical simulations of this method confirm that it is reliable and robust in the presence of noise and verify its enhanced performance when compared with a linear least-squares method that includes the unwrapping of the bispectral phase before Fourier phase retrieval.

1. INTRODUCTION

Despite the widespread use of bispectral-based algorithms in current astronomical speckle imaging experiments, there still appears to be no consensus as to the optimum procedure to be adopted for the recovery of images from measurements of the bispectrum. One aspect of this problem that has received considerable attention is the question of Fourier phase retrieval from the measured bispectrum phases. Broadly speaking, two basic strategies have been proposed: those that use recursive schemes, which make explicit use of the linear combinations of Fourier phases contained in each bispectrum phase[1,2] and derive the source Fourier phases sequentially, and those that attempt to fit the complete set of source Fourier phases simultaneously to the bispectrum phases in a least-squares sense.[3]

By virtue of its being the argument of a product of three complex visibilities, the bispectrum phase is, by definition, a modulo 2π quantity, and so some concern has been expressed as to the efficacy of least-squares-based phase-recovery techniques in the presence of such data. Indeed, whenever least-squares fitting to the bispectrum phases has been attempted, either the fitting has been effected in the complex plane[3] in order to avoid dealing with the modulo 2π nature of the data explicitly or the bispectrum phases have been unwrapped[4] before the fitting procedure.

In this paper an argument is made that the case for the unwrapping of the bispectrum phase as presented by Marron *et al.*[4] does not reflect any intrinsic indeterminacy arising from the modulo 2π nature of the measured bispectrum phase but instead derives from their adoption of a linear-algebraic formulation of the phase-retrieval problem. In particular, the need for phase unwrapping identified by Ref. 4 can be seen to be a direct consequence of the use of a linear least-squares algorithm that assumes that both the bispectrum and Fourier phases are known in the interval $\pm\infty$.

In Section 2 of this paper an alternative nonlinear least-squares method that explicitly takes account of the modulo 2π nature of the bispectral phases is introduced. Based on the minimization of an objective function measuring the misfit between the observed data and a model, it incorporates the advantages of the complex fitting approach of Gorham *et al.*[3] without requiring the use of complex phasors. Numerical simulations presented in Section 3 indicate that this new phase-retrieval technique is both rapid and reliable. Furthermore, a comparison with the method of Ref. 4 has exposed a number of significant deficiencies in its unwrapping algorithm and clearly demonstrates the versatility of an objective-function-based approach. Conclusions are summarized in Section 4.

2. BISPECTRAL PHASE INVERSION

In order to clarify the notation somewhat and to facilitate comparison with results in the literature, it is simplest to consider a real discrete one-dimensional example (the extension to two dimensions is straightforward) in which an image $I(x)$ comprising $2N$ pixels is to be recovered from measurements of its complex bispectrum $\tilde{I}^3(u, v)$. This is defined as

$$\begin{aligned}\tilde{I}^3(u, v) &= \tilde{I}(u)\tilde{I}(v)\tilde{I}^*(u + v)\\ &= |\tilde{I}^3(u, v)|\exp[i\psi(u, v)]. \end{aligned} \qquad (1)$$

Although the bispectrum contains information concerning both the amplitude and phase of the image Fourier transform, its major role in most image-processing applications is to provide a basis for Fourier phase recovery: Hereafter this particular usage of the bispectrum will be concentrated on, and it will be assumed that the source Fourier amplitudes have already been successfully determined.

As is evident from Eq. (1), the bispectrum phase ψ can be expressed as the sum of three separate Fourier phases ϕ by

$$\psi(u, v) = \phi(u) + \phi(v) - \phi(u + v) \qquad (\mathrm{mod}\ 2\pi) \qquad (2)$$

or in discrete notation by

$$\psi_{i,j} = \phi_i + \phi_j - \phi_{i+j} \qquad (\mathrm{mod}\ 2\pi), \qquad (3)$$

where ϕ_i is the phase of the ith Fourier component of the image and $\psi_{i,j}$ represents the phase of the bispectrum element that incorporates the ith, the jth, and the $(i + j)$th

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Fourier phase. The reader is reminded that, in these expressions and throughout this paper, the term Fourier phase will be taken to mean the principal (i.e., modulo 2π) argument of the complex visibility function. Note that, because the bispectrum is defined as the product of three complex numbers, its argument is by definition a modulo 2π quantity. Consequently, it cannot be correctly represented as a simple linear combination of the Fourier phases. In the subsequent discussion, the corresponding true linear combinations of Fourier phases obtained by algebraic summation will be referred to as the unwrapped bispectrum phases

$$\psi_{i,j}' = \phi_i + \phi_j - \phi_{i+j}, \tag{4}$$

and the procedure for mapping $\psi \rightarrow \psi'$ will be termed unwrapping.

As a final point of introduction, it is necessary to specify an appropriate set of bispectrum elements from which the source Fourier phases are to be determined. Hereafter, this set will be denoted by $\mathcal{P}$. Because of the highly symmetric nature of the bispectrum, arising both from its symmetric definition and the Hermitian properties of the Fourier transform of a real image, it is possible to gain access to the full information content of the bispectrum by using a small nondegenerate subset of the whole. Here we use the same representation employed by Ref. 4 and define $\mathcal{P}$ to be the set of bispectrum points generated by all pairs of indices i and j such that $1 \leq i \leq j \leq N$ and $i + j \leq N$. For a $2N$-pixel image there will be N independent Fourier components and therefore, according to this prescription, a total of $N^2/4$ ($= M$) independent bispectrum points.

One starting point for the phase-retrieval process has been to identify the bispectrum phases with linear combinations of Fourier phases. If this were so, the determination of the source Fourier phases could be described as the solution of a set of M linear equations in N unknowns ($M \gg N$). However it is evident that the correct identification of linear combinations of the Fourier phases is with the unwrapped bispectrum phases of Eq. (4) and not the bispectrum phases themselves. Therefore, if phase-retrieval algorithms based on this assumption are to be successful, the unwrapped bispectrum phases must somehow be derived from the bispectrum phases. The complete set of relations linking the Fourier phases to the unwrapped bispectrum phases can be expressed as the matrix equation

$$\mathbf{A\Phi} = \mathbf{\Psi}', \tag{5}$$

where $\mathbf{\Phi}$ is a vector whose ith component is ϕ_i, $\mathbf{\Psi}'$ is the M-component vector of unwrapped bispectrum phases, and $\mathbf{A}$ is the $M \times N$ bispectral matrix in which the nonzero elements in each row indicate the Fourier phases that are combined in any particular element of $\mathbf{\Psi}'$. Reference 4 presents an elegant algorithm, in which the structure of the bispectral matrix is exploited to enable a set of multiples of 2π to be determined, which can be added to the bispectrum phases ψ in order to obtain their corresponding unwrapped values ψ'. The method is not unique, but it can be shown that the multiplicity of possible solutions introduces an ambiguity concerning only the absolute position of the reconstructed image, which is always the case even if the unwrapped bispectrum phases can be observed directly. Once the unwrapped bispectrum phases have been computed, the overdetermined set of linear equations contained in Eq. (5) can be solved straightforwardly, in a least-squares sense, by using standard linear-algebraic methods.

An alternative methodology for phase retrieval can be developed by considering a quantity $\mathcal{F}_1$, which is a function of the N Fourier phases ϕ_i:

$$\mathcal{F}_1 = \sum_{i,j \in \mathcal{P}}^{N} \left[\frac{\psi_{i,j}' - (\hat{\phi}_i + \hat{\phi}_j - \hat{\phi}_{i+j})}{w_{i,j}}\right]^2, \tag{6}$$

where $w_{i,j}$ is a measure of the error on the unwrapped bispectrum phase $\psi_{i,j}'$ and the circumflexes denote the quantities that are to be varied in order to minimize $\mathcal{F}_1$. In this functionally equivalent formulation of the phase-retrieval problem, which describes the conventional weighted least-squares approach, the coordinates of the global minimum of $\mathcal{F}_1$ in the N-dimensional space of Fourier phases correspond to the components of the least-squares solution vector of Eq. (5). Unfortunately, this procedure shares a common feature with the previous method in that knowledge of ψ' is assumed, and so it still requires the preliminary step whereby the bispectrum phases are unwrapped.

A suggested solution to this problem[3] has been to minimize a modified objective function $\mathcal{F}_2$ defined as

$$\mathcal{F}_2 = \sum_{i,j \in \mathcal{P}}^{N} \left[\frac{\mathrm{Re}(\Delta_{i,j})}{w_{i,j}}\right]^2 + \left[\frac{\mathrm{Im}(\Delta_{i,j})}{w_{i,j}}\right]^2. \tag{7}$$

In this notation the quantity $\Delta_{i,j}$ measures the misfit between a unit-amplitude bispectrum phasor (with an argument equal to the observed bispectrum phase) and the corresponding model bispectrum phasor

$$\Delta_{i,j} = \exp(i\psi_{i,j}) - \exp[i(\hat{\phi}_i + \hat{\phi}_j - \hat{\phi}_{i+j})]. \tag{8}$$

This misfit measure takes advantage of the periodicity of the complex exponential function to eliminate the need to unwrap the observed bispectrum phases at the expense of increasing the number of terms in the objective function summation by a factor of 2. In graphical terms (Fig. 1) each summand in the objective function corresponds to the weighted chord length between the tips of a measured unit bispectrum phasor and the corresponding model unit bispectrum phasor.

Although this type of algorithm has been used successfully with real astronomical data at both optical[3] and infrared[5] wavelengths, an even simpler scheme that requires neither the unwrapping of the bispectrum phase nor the minimization in the complex plane is investigated here. Rather than minimize $\mathcal{F}_2$ we have an objective function $\mathcal{F}_3$, given by

$$\mathcal{F}_3 = \sum_{i,j \in \mathcal{P}}^{N} \left\{\frac{\mathrm{Mod}[\psi_{i,j} - (\hat{\phi}_i + \hat{\phi}_j - \hat{\phi}_{i+j})]}{w_{i,j}}\right\}^2 = \sum_{i,j \in \mathcal{P}}^{N} \left[\frac{\mathrm{Mod}(\Theta_{i,j})}{w_{i,j}}\right]^2, \tag{9}$$

where the function $\mathrm{Mod}(x)$ returns the value of x wrapped into the primary interval $\pm\pi$. This is virtually identical to the expression for $\mathcal{F}_1$, given above, but the reduction of $\Theta_{i,j}$ to its modulo 2π value ensures that the bispectrum phases, and not their unwrapped counterparts, can be utilized in this procedure.

In this scheme the graphical representation of each term

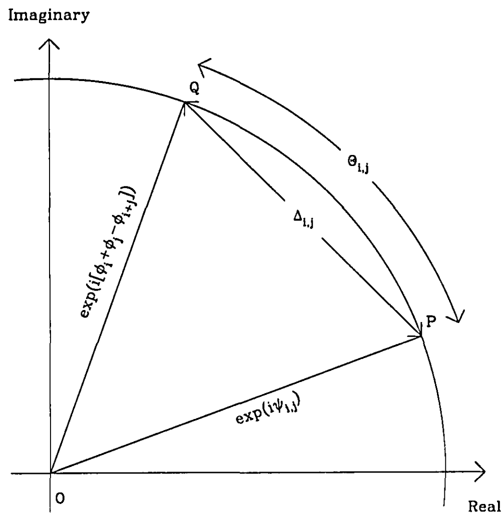

Fig. 1. Argand diagram representations of the misfit criteria $\Delta_{i,j}$ and $\Theta_{i,j}$ corresponding to the objective function $\mathcal{F}_2$ and $\mathcal{F}_3$ described in the text. The two vectors $\overline{OP}$ and $\overline{OQ}$, ending on the unit circle, represent a pair of measured and model unit bivectors, respectively.

in the summation is the weighted arclength between the tips of the measured and the model unit bispectrum phasors. Because this arclength is just the angular difference between the arguments of the measured and the model bispectrum phasors, the weights in Eq. (9) are simply the standard deviations of the measured bispectrum phases. In speckle interferometry it is usual to average a large number of separate measurements of any single bispectrum value and to extract the bispectrum phase as the argument of the mean bispectrum phasor. A useful and consistent estimator for the variance in the phase of such a mean phasor can be obtained from[6]

$$\mathrm{Var}[\psi_{i,j}] = w_{i,j}{}^2 = \frac{V_\perp[\tilde{I}_{i,j}{}^3]}{\mathbf{N}\tilde{I}_{i,j}{}^3}, \tag{10}$$

where $V_\perp[\tilde{I}_{i,j}{}^3]$ is the variance of the complex bispectrum in the direction perpendicular to the direction of the mean bispectrum phasor $\tilde{I}_{i,j}{}^3$ and $\mathbf{N}$ is the number of independent measurements averaged to compute the mean bispectrum phasor.

The weights denoted in Eq. (9) take explicit account of the directional dependence of the variance of the complex bispectrum and can be evaluated straightforwardly as the bispectrum estimates are being accumulated. We have found that the notion of the bispectrum variance in the direction perpendicular to its length has been particularly useful when working with real astronomical data: in many instances one sees that the variance of the mean bispectrum phasor differs considerably in the directions perpendicular and parallel to its length, so that the phase of a particular bispectrum element can be well defined despite the fact that its modulus is known with relatively poor precision. This observation, which has apparently been overlooked by most other researchers, is clearly of importance if any weighted least-squares method is incorporated into the phase-retrieval procedure.

Examination of the two objective functions $\mathcal{F}_2$ and $\mathcal{F}_3$ reveals that they are closely related and differ only in the quantification of the misfit between the measured and the model unit bispectrum phasors. With data of high or moderate signal-to-noise ratio (SNR $\gtrsim 2.5$) the two misfit measures are essentially identical and the only relevant distinction between the two functions is the computational simplicity of $\mathcal{F}_3$. At lower SNR's the arclength measure of $\mathcal{F}_3$ tends to penalize errant bispectrum phasors slightly more than the chord-length measure of $\mathcal{F}_2$, but in the regime in which the magnitude of this effect becomes significant (SNR $\lesssim 0.5$) our simulations indicate that the minimization of neither $\mathcal{F}_2$ nor $\mathcal{F}_3$ converges to a reliable solution, and so this distinction need not concern us here. However the simulations here have indicated that the complex formulation of Gorham *et al.* has a slightly better SNR threshold (SNR $\gtrsim 0.6$) than our simplified objective-function method (SNR $\gtrsim 0.8$), so that at the lowest SNR levels it may be preferable to minimize $\mathcal{F}_2$ rather than $\mathcal{F}_3$.

Because the objective function given in Eq. (9) has been constructed to eliminate the difficulties associated with the modulo 2π nature of the bispectrum phases, its minimization can be accomplished with a wide range of standard software packages. However, as it is not a smooth function, owing to the Mod(x) operation, a number of implementations of Newton-type, quasi-Newton, and conjugate-gradient algorithms were tested in order to ascertain whether discontinuities in the function cause problems while searching for its minimum. It was found that all the routines tested provide satisfactory solutions for the Fourier phases when presented with bispectrum phases of reasonable SNR, and so it appears that the use of the Mod(x) operator does not introduce pathological discontinuities in this application.

3. NUMERICAL SIMULATIONS AND DISCUSSION

In order to assess the reliability of the proposed bispectral inversion method, a number of image reconstructions that utilized simulated noisy data were performed and the results were compared with those obtained by using the phase-unwrapping and linear-algebraic method of Ref. 4 and the complex-phasor-based scheme of Gorham *et al.*[3] In order to ensure a fair comparison with results in the literature, the trials were modeled on the simulations of Ref. 4, in which the test source was a real two-component one-dimensional image comprising 128 pixels. For the results presented here the source was not identical to that employed in Ref. 4 because the components were not quite so far apart and they were smoothed slightly in order to reduce the effects of Gibbs' phenomenon. Nevertheless, tests conducted with a variety of models produced equivalent results, and it is believed that the conclusions presented here are independent of the exact source morphology.

The test source was Fourier transformed, and its 64 independent Fourier phases were extracted and used to generate a nondegenerate set of bispectrum phases. This gave a total of 1024 bispectrum phases, by using the prescription given in Section 2. These values were then corrupted with noise and subsequently used as the raw data for each phase-recovery scheme. The bispectrum phase errors were drawn from the distribution expected for the measurement of a complex phasor in the presence of circular Gaussian noise,[7] and so their magnitude is characterized by a SNR parameter γ. This is the ratio of the magnitude of the signal phasor to the standard deviation of the real (or the imaginary) component of the noise phasor, so that smaller values of γ correspond to noisier data. For $\gamma \gtrsim 1$, this distribution is well approximated by a Gaussian with standard deviation $\simeq\gamma^{-1}$ rad, but at lower signal levels it tends to have flatter wings than the corresponding Gaussian distribution.

The implementation of the phase-unwrapping method was identical to that described in Ref. 4 and employed the same Lanczos-based weighted least-squares algorithm,[8] while for the minimizations of $\mathcal{F}_2$ and $\mathcal{F}_3$ the conjugate-gradient routine FRPRMN of Ref. 9 was utilized. Neither routine requires storage of the bispectral matrix **A**, and so they are ideally suited to this type of problem in which **A** is both large and sparse. For the $\mathcal{F}_2$ and $\mathcal{F}_3$ minimizations, random phases distributed as N(0°, 20°) were employed as the initial solution estimates, and for both the reconstruction of Ref. 4 and the objective-function reconstruction identical weightings were used for the bispectrum phases when performing the least-squares minimizations. The success of the phase recovery was judged, following Ref. 4, by examination of the hybrid image obtained by inverse Fourier transforming the set of unperturbed Fourier amplitudes and the recovered Fourier phases.

Figures 2 and 3 show the hybrid images obtained by using the phase-unwrapping algorithm and the weighted least-squares solution of Ref. 4 (Fig. 2) and by using the minimization of the objective function $\mathcal{F}_3$ (Fig. 3). For all the examples discussed here, the images recovered by using the complex-phasor minimization were similar to those obtained by using the $\mathcal{F}_3$ minimization and so have not been displayed. In each case, and in all subsequent plots, the original test image is displayed as well, and, because of the inability of the bispectrum to distinguish between identical images at different locations in the field, the reconstructions have been shifted to facilitate comparison with the original object.

At these moderate SNR levels ($\gamma = 2.5$) both methods were equally successful in recovering the object Fourier phases, and the resulting image reconstructions displayed only minimal residual noise at a level of approximately 5% of the peak intensity. However, at increasing noise levels the images recovered from the unwrapped bispectrum phases of Ref. 4 were always of lower quality than those recovered by using the objective-function minimization. Examples of this behavior are given in Figs. 4 and 5, which show the images recovered from noisier data ($\gamma = 1.25$) by using the unwrapping and the objective-function method, respectively. Although both images correctly reproduced the double structure in the test object, spurious peaks in the reconstruction of Ref. 4 were present at a level of more than 20% of the peak intensity. On the other hand, the objective-function reconstruction was considerably cleaner with less high-frequency noise and residual artifacts at a much reduced level. Similar results obtained at even higher noise levels ($\gamma = 1.0$) are presented in Figs. 6 and 7. In this case the spurious components in the image restored with the method of Ref. 4 (Fig. 6) had reached an unacceptable level, and it was difficult to assess whether the secondary component was real (thus confirming the noise limit established by Ref. 4), while the image recovered by using the technique described here remained reliable down to the 10% level. With noisier data

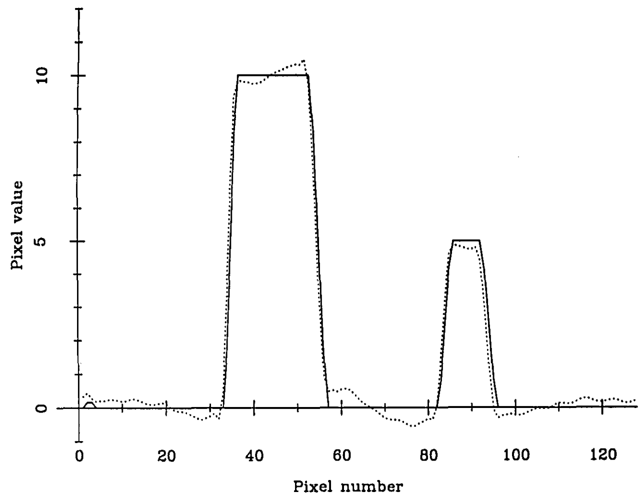

Fig. 2. Reconstructed hybrid image derived from slightly noisy bispectrum phases ($\gamma = 2.5$) by using the unwrapping and weighted least-squares method of Ref. 4. In this, and all subsequent plots, the solid curve shows the original test source and the dotted curve the reconstructed image, shifted so as to facilitate comparison with the original.

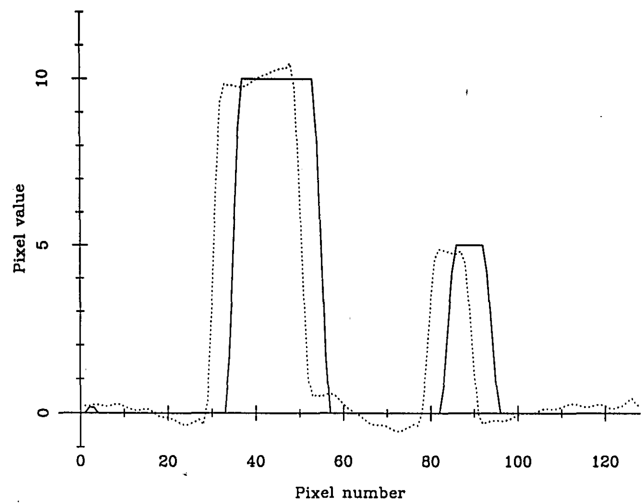

Fig. 3. Image reconstruction from the same data set as Fig. 2 but using the weighted least-squares minimization of the objective function $\mathcal{F}_3$.

($\gamma < 1$) phase retrieval from the unwrapped bispectrum phases of Ref. 4 was inevitably unsuccessful, but satisfactory reconstructions could still be obtained by using the objective-function minimization methods.

While these simulations are revealing, they do not correspond closely to the situations encountered in most imaging experiments. In practice the bispectrum phases will not be corrupted by a uniform noise level because those that incorporate Fourier phases at spatial frequencies for which the source is resolved will always be less well determined than those constructed from triplets of low spatial frequencies. In addition, in some instances, a number of them may not be known at all. In order to model this scenario, the simulations described above were repeated but with an additional fraction of the bispectrum phases corrupted with phase er-

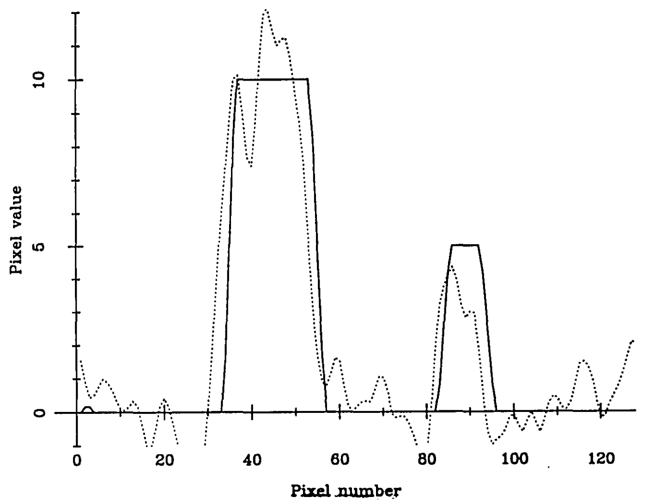

Fig. 4. Unwrapped hybrid image of Ref. 4 derived from bispectral phases of low SNR ($\gamma = 1.25$).

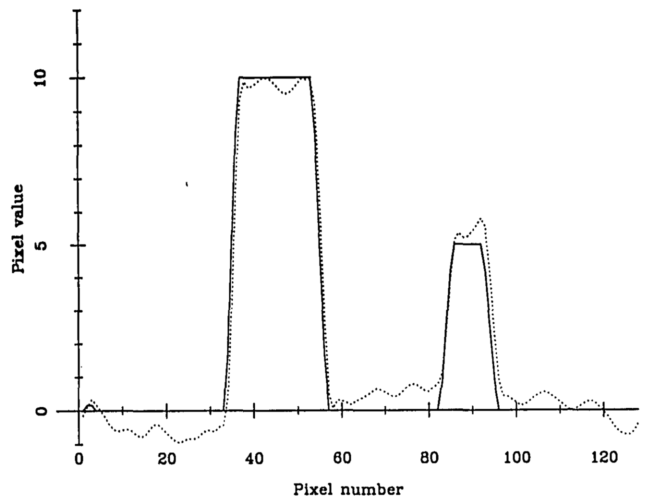

Fig. 5. $\mathcal{F}_3$ image obtained by using the same data set as that of Fig. 4.

rors distributed uniformly between $\pm\pi$ rad in order to mimic a subset of low quality, or perhaps missing, data.

The hybrid images obtained from a data set in which γ was equal to 2.5 and in which 20% of the bispectrum phases were completely corrupted are displayed in Figs. 8 and 9. Even though the fraction of unreliable data was small and the remaining measurements were of relatively high SNR, the reconstruction derived from the unwrapped bispectrum phases of Ref. 4 (Fig. 8) was of poor quality and exhibited high-frequency noise of considerable amplitude. This was despite the fact that the less-well-known values were weighted down appropriately in the final least-squares fitting, thus revealing the magnitude of the errors that missing or poorly defined bispectrum phases can introduce during the unwrapping procedure. In contrast, the image recovered by using the $\mathcal{F}_3$ minimization (Fig. 9) was excellent and essentially identical to that obtained from the uncorrupted data set, demonstrating the immunity of this objective-function minimization method to a small subset of aberrant data values. In order to highlight further the difficulties of the unwrapping-based method of Ref. 4, the corresponding hybrid images recovered from an identical data set to that described above but with 75% of the bispectrum phases perturbed with errors distributed uniformly on $\pm\pi$ are reproduced in Figs. 10 and 11. Once again, the results were encouraging with only a small deterioration of the image quality visible in the $\mathcal{F}_3$ image (Fig. 11) but complete failure in the case of the phase-unwrapped reconstruction (Fig. 10).

The previous simulations highlight the two most important deficiencies of the unwrapping method of Ref. 4. The first of these is simply the nature of the method itself: Because this algorithm derives the mapping of $\psi \rightarrow \psi'$ from linear combinations of the bispectrum phases themselves, any errors in the raw bispectrum phases are rapidly propagated throughout the whole data set. The magnitude of this problem can be appreciated by realizing that an individual

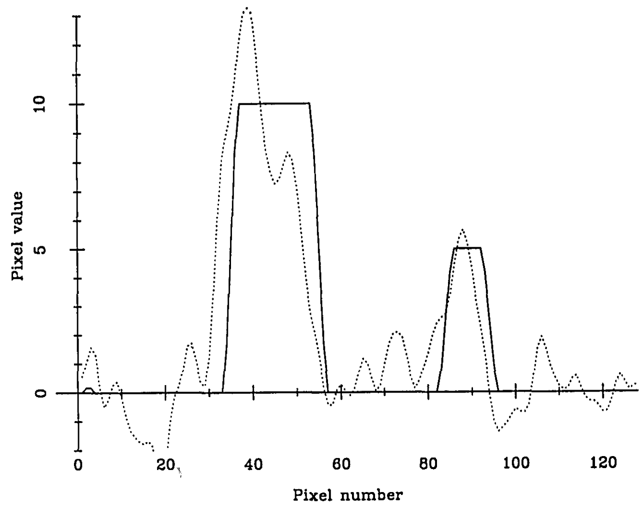

Fig. 6. Unwrapped hybrid image of Ref. 4 derived from bispectral phases of low SNR ($\gamma = 1.0$). At these noise levels the unwrapping procedure of Ref. 4 has become unreliable, and so accurate image restoration is no longer possible.

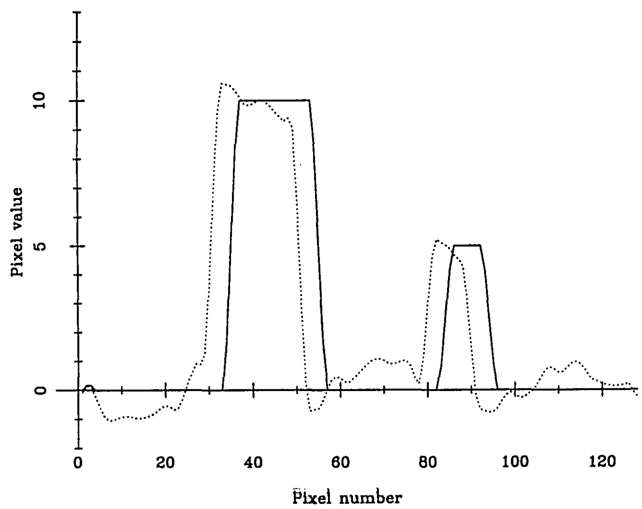

Fig. 7. $\mathcal{F}_3$ image obtained by using the same data set as that of Fig. 6. Note the considerable improvement in image quality.

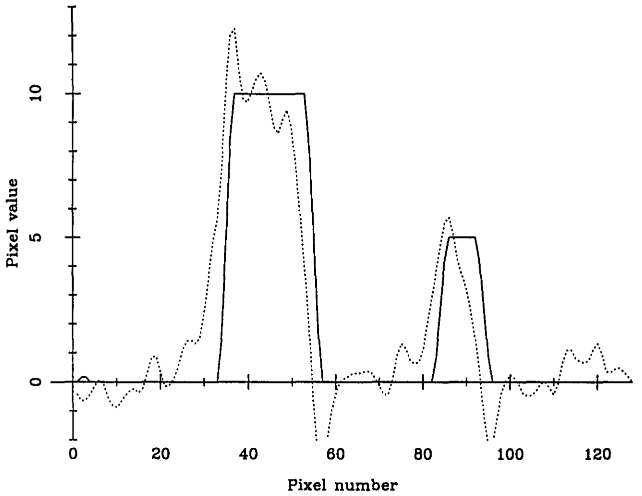

Fig. 8. Hybrid image reconstruction obtained by using the unwrapping algorithm of Ref. 4. In this example 20% of the bispectrum phases have been corrupted with errors uniformly distributed on $\pm\pi$. The remainder are characterized by a γ value of 2.5.

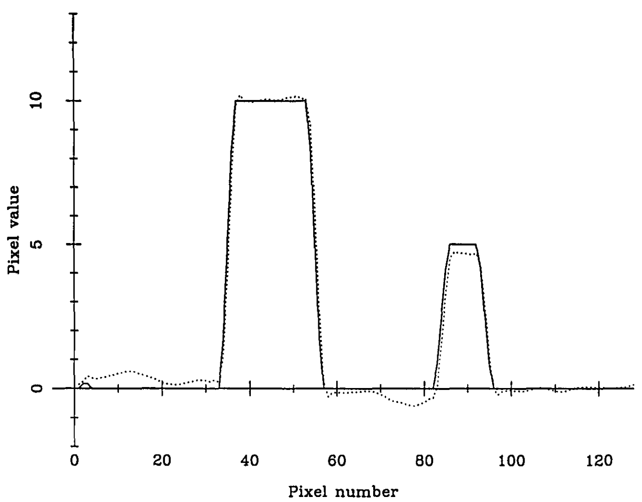

Fig. 9. $\mathcal{F}_3$ hybrid image recovered from the same data set used to produce Fig. 8.

bispectrum phase will typically be employed in the determination of four unwrapped bispectrum phases. Not surprisingly, any phase-recovery scheme that utilizes the unwrapped phases will always be disadvantaged by the higher noise levels of these data as compared with schemes that use the raw measurements. More importantly though, the unwrapping algorithm of Ref. 4, which exploits the linear dependencies of the rows of the bispectral matrix **A** to great advantage, relies on the complete set of bispectrum phases being available and on their all having equal weight. It is not at all clear that the procedure of Ref. 4 for computing a set of multiples of 2π, with which to correct the measured bispectrum phases, can be generalized to the realistic case in which the bispectrum is only sparsely sampled and the data are of unequal reliability.

Conversely, any method that solves for the unknown Fourier phases by minimizing an objective function can compensate for variations in the SNR of the bispectrum measurements explicitly and, furthermore, can remain useful in the absence of a large fraction of the data: This is a particularly valuable feature in the case of two-dimensional imaging experiments for which it may be impracticable to compute anything more than a small fraction of the nondegenerate portion of the bispectrum. In the formalism of least-squares fitting, this robustness, of course, simply reflects the much larger number of data constraints (i.e., bispectrum phases) than unknowns (i.e., Fourier phases).

In a wider context it seems appropriate to clarify the points raised by Ref. 4 and this paper concerning the need to unwrap the bispectrum phases. Undoubtedly, the unwrapping of the bispectrum phase is crucial if a phase-recovery scheme based on the proposition that the bispectrum phases are linear combinations of the Fourier phases is adopted, as exemplified by the linear least-squares approach of Ref. 4. However the evident success of recursive methods for phase retrieval from modulo 2π bispectrum phases, and the present commonplace use of modulo 2π closure phases in

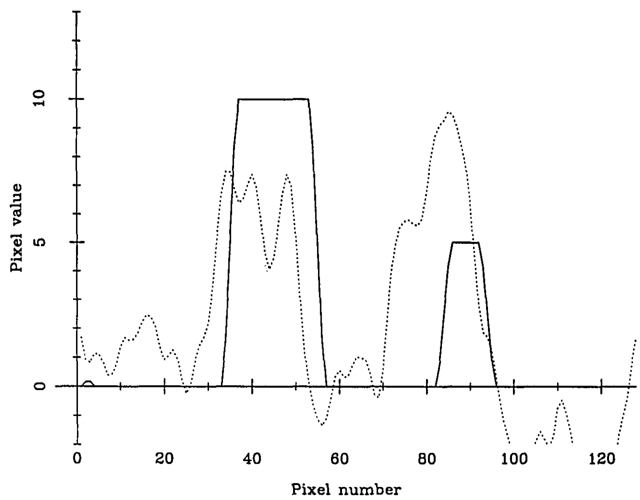

Fig. 10. Hybrid image of Ref. 4 recovered from a $\gamma = 2.5$ data set, in which 75% of bispectrum phases have been corrupted with errors uniformly distributed on $\pm\pi$.

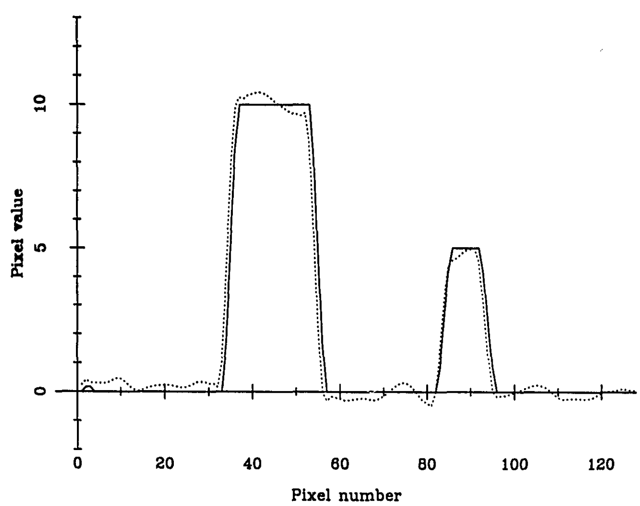

Fig. 11. $\mathcal{F}_3$ hybrid image recovered from the same data set used to produce Fig. 10.

radio astronomical imaging, indicate that alternative approaches that utilize wrapped bispectrum phases can be equally valuable and that the information contained in the modulo 2π bispectrum phases is more than adequate to permit complete recovery of the source Fourier phases. Thus the choice of whether to unwrap depends solely on the type of phase-retrieval algorithm being considered, although it is asserted here that SNR considerations alone should be sufficient to discourage any attempt at unwrapping.

4. CONCLUSIONS

The recovery of Fourier phases from measurements of the complex bispectrum forms a vital step in many imaging experiments. Contrary to current opinion, trials performed with simulated data have revealed that it is possible to implement a robust least-squares method for such phase recovery despite the modulo 2π nature of the bispectrum phases, so that the case for unwrapping of the bispectrum phases, as presented by Marron *et al.*,[4] is valid for only the particular class of linear least-squares algorithm that they considered. An alternative approach to Fourier phase retrieval, based on the minimization of an objective function measuring the misfit between the bispectrum phases and those predicted by a set of model source Fourier phases, has been developed and tested with samples of data of varying SNR's. These results indicate that there is a large region of parameter space for which attempts to unwrap the bispectrum phases can fail, so that schemes for phase retrieval from the unwrapped phases will be unsuccessful too, but for which least-squares Fourier phase recovery remains possible. This can occur, for example, when the bispectral data are of low quality or when the bispectrum measurements are incomplete.

Our alternative phase-retrieval method is complementary to the complex minimization of Gorham *et al.*[3] and produces comparable results at all but the lowest SNR levels but is somewhat simpler to implement and converges slightly more rapidly. Compared with schemes that require bispectrum phase unwrapping, the primary advantages of such objective-function-based methods are the proper treatment of noise and considerably improved performance in the presence of large amounts of noise and sparse bispectral coverage. Since these are not atypical features of real bispectral data, it is asserted here that procedures for unwrapping the modulo 2π bispectrum phase should not be implemented and that Fourier phase retrieval should be performed by using the modulo 2π measurements themselves.

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