Supplemental material to "Narrowing the filter cavity bandwidth via optomechanical interaction"

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In this supplemental material, we will show additional details and derivations for the optomechanical dynamics of the optical-dilution scheme shown in Fig. 2 of the main text.

Hamiltonian and equation of motion

The Hamiltonian of the system can be written as:

$$H = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}) + \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_m^2\hat{x}^2 + \hbar\omega_s(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger) + \hbar G_0\hat{x}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}) + H_{ext}^{opt} + H_{ext}^{m}. \tag{A.1}$$

Here, $\hat{a}, \hat{b}$ are annihilation operators for cavity modes in left and right sub-cavity (with resonant frequency ω_c) respectively. $\hat{x}, \hat{p}$ are the position and momentum operators of the vibrating mirror. ω_s is the coupling constant for $\hat{a}$ and $\hat{b}$ and G_0 is defined to be ω_0/L. $H_{ext}^{opt} = i\hbar\sqrt{2\gamma_f}(\hat{a}^\dagger\hat{a}_{\rm in} - {\rm h.c}) + i\hbar\sqrt{2\gamma_\epsilon}(\hat{b}^\dagger\hat{b}_{\rm in} - {\rm h.c})$ and H_{ext}^{m} correspond to the coupling of the system to the environment.

The Heisenberg equations of motion in the rotating frame of the trapping beam at frequency ω_0' can be derived as:

$$\dot{\hat{a}} = i\Delta_t\hat{a} - \gamma_f\hat{a} - i\omega_s\hat{b} - iG_0\hat{x}\hat{a} + \sqrt{2\gamma_f}\hat{a}_{\rm in}, \tag{A.2a}$$

$$\dot{\hat{b}} = i\Delta_t\hat{b} - \gamma_\epsilon\hat{b} - i\omega_s\hat{a} + iG_0\hat{x}\hat{b} + \sqrt{2\gamma_\epsilon}\hat{b}_{\rm in}, \tag{A.2b}$$

$$\dot{\hat{p}} = -m\omega_m^2\hat{x} - \gamma_m\hat{p} - \hbar G_0(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}) + F_{\rm th}, \tag{A.2c}$$

$$\dot{\hat{x}} = \hat{p}/m. \tag{A.2d}$$

Here, $\gamma_f = cT_f/4L$ and $\gamma_\epsilon = c\epsilon/4L$, T_f and ϵ are the transmissivity of the front mirror and the loss of the system through the end mirror, $\Delta_t = \omega_0' - \omega_0$ is the detuning of the pumping laser field with respect to the half-cavity resonance. Suppose we pump the cavity by injecting a laser field through the front mirror (*single-side pumping*), then $\bar{b}_{\rm in} = 0$. These equations can be solved perturbatively. The zeroth order terms give us the classical amplitude of the intra-cavity mode in both sub-cavities and the first order terms carry information about the mirror vibration along with quantum noise due to the non-zero transmissivity of the cavity end mirror.

From the above Heisenberg equations of motion, we have the steady state fields in the two sub-cavities:

$$\bar{a} = \frac{(i\Delta_t - \gamma_\epsilon)\sqrt{2\gamma_f}\bar{a}_{\rm in}}{\Delta_t^2 - \omega_s^2 - \gamma_f\gamma_\epsilon + i\Delta_t(\gamma_f + \gamma_\epsilon)} \tag{A.3a}$$

$$\bar{b} = \frac{i\sqrt{2\gamma_f}\omega_s\bar{a}_{\rm in}}{\Delta_t^2 - \omega_s^2 - \gamma_f\gamma_\epsilon + i\Delta_t(\gamma_f + \gamma_\epsilon)}, \tag{A.3b}$$

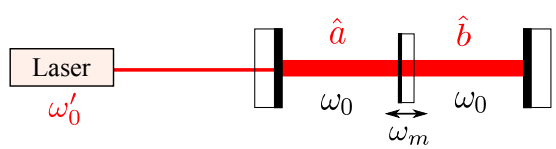

FIG. A.1: Basic configuration of the proposed scheme: a vibrating mirror trapped in a Fabry-Pérot cavity. $\hat{a}$ and $\hat{b}$ are the light field operators in the left and right subcavities, respectively.

As we can see from above equations, when we set the detuning of the trapping beam to be $\Delta_t = \omega_s$ and set $\gamma_\epsilon/\gamma_f \ll 1$, the intracavity field amplitude is strong: $\bar{a} = \bar{b} = \sqrt{2/\gamma_f}\bar{a}_{\rm in}$ with $\bar{a}_{\rm in} = \sqrt{P_{\rm trap}/\hbar\omega_0'}$. The fluctuating field consists of mechanical modulation and quantum fluctuations as:

$$\hat{a}(\omega) = \frac{-G_b\omega_s\hat{x} + i\omega_s\sqrt{2\gamma_\epsilon}\hat{b}_{\rm in} + [i(\omega+\Delta_t) - \gamma_\epsilon][-iG_a\hat{x} + \sqrt{2\gamma_f}\hat{a}_{\rm in}]}{(\omega+\Delta_t)^2 - \omega_s^2 - \gamma_\epsilon\gamma_f + i(\omega+\Delta_t)(\gamma_f+\gamma_\epsilon)} \tag{A.4a}$$

$$\hat{b}(\omega) = \frac{G_a\omega_s\hat{x} + i\sqrt{2\gamma_f}\omega_s\hat{a}_{\rm in} + [i(\omega+\Delta_t) - \gamma_f][iG_b\hat{x} + \sqrt{2\gamma_\epsilon}\hat{b}_{\rm in}]}{(\omega+\Delta_t)^2 - \omega_s^2 - \gamma_\epsilon\gamma_f + i(\omega+\Delta_t)(\gamma_f+\gamma_\epsilon)}\,, \tag{A.4b}$$

with $G_a \equiv G_0\bar{a}$ and $G_b \equiv G_0\bar{b}$ (notice that in case of $\Delta_t = \omega_s$, we have $G_a = G_b$). The radiation pressure force acting on the trapped mirror is given by

$$\hat{F}_{\rm rad}(\omega) = \hbar[G_a^*\hat{a}(\omega) + G_a\hat{a}^\dagger(-\omega) - G_b^*\hat{b}(\omega) - G_b\hat{b}^\dagger(-\omega)], \tag{A.5}$$

which can be split into two parts:

$$\hat{F}_{\rm rad}(\omega) = -K_{\rm opt}(\omega)\hat{x}(\omega) + \hat{F}_{\rm BA}(\omega) \tag{A.6}$$

The first and second term represent the pondermotive modification of the mechanical dynamics and the back-action quantum radiation pressure noise respectively. The $K_{\rm opt}(\omega)$ here is the optomechanical rigidity which can be expanded in terms of ω if the typical frequency of mechanical motion is smaller than the other frequency scale in the trapping system:

$$K_{\rm opt}(\omega) \approx K_{\rm opt}(0) + \frac{\partial K_{\rm opt}}{\partial\omega}\omega + \frac{1}{2}\frac{\partial^2 K_{\rm opt}}{\partial\omega^2}\omega^2 \equiv m\omega_{\rm opt}^2 - im\Gamma\omega - m_{\rm opt}\omega^2. \tag{A.7}$$

The first term in (A.7) gives the trapping frequency and the second and third terms give the velocity and acceleration response of the trapped mirror which are optical (anti-)damping Γ and optomechanical inertia $m_{\rm opt}$, respectively.

Substituting (A.3) and (A.4) into (A.5) and taking the expansion with respect to detection frequency ω, we can get analytical expressions of the optical rigidity and radiation pressure noise. However, they are too cumbersome to show. In the following, we show approximate results in the interesting parameter region of $\Delta_t \sim \omega_s$ and $\gamma_\epsilon \ll \gamma_f$ in which the back-action noise can be coherently canceled.

Dynamics and back-action

The optical spring frequency is given by:

$$\omega_{\rm opt}^2 = \frac{\hbar G_a^2}{m\omega_s} + \mathcal{O}(\eta). \tag{A.8}$$

Substitute $\bar{a}_{\rm in}$, G_a, ω_s and γ_f in, and we have Eq.(16) in the main text. The $\mathcal{O}(\eta)$ here describes all the high order terms with $\eta \sim (\Delta_t - \omega_s)/\omega_s, \gamma_\epsilon/\gamma_f$. Notice that this optical spring can be treated effectively as a quadratic trap of the vibrating mirror on the anti-node of our trapping beam as shown in [1]

The optical (anti)-damping factor Γ is given by (to $1st$ order of ω):

$$\Gamma = \frac{16\hbar G_a^2}{m\gamma_f\omega_s}\left(\frac{\Delta_t - \omega_s}{\omega_s}\right) - \frac{8\hbar G_a^2\gamma_f}{m\omega_s^3}\left(\frac{\gamma_\epsilon}{\gamma_f}\right) + \mathcal{O}(\eta^2) \tag{A.9}$$

It is clear from this formula that in the ideal case when $\Delta_t = \omega_s$ and $\gamma_\epsilon = 0$, the optical damping is completely cancelled. Therefore by carefully choosing the system parameters, we can achieve a small positive damping when the end mirror is not perfectly reflective.

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[1] H. Miao, S. Danilishin, T. Corbitt, and Y. Chen, Phys. Rev. Lett. **103**, 100402 (2009)

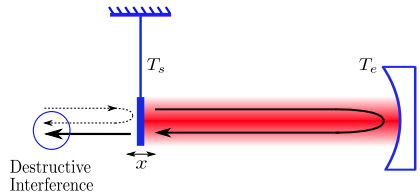

FIG. A.2: Back-action evasion: destructive interference between the field directly reflected from the oscillating mirror and the field transmitted out of the cavity.

The main contribution to the optomechanical inertia is at zeroth order of ϵ:

$$m_{\rm opt} = -\frac{\hbar G_a^2}{\omega_s^3} + \mathcal{O}(\eta) \tag{A.10}$$

which is extremely small as we have shown in the main text.

Finally, the back-action radiation pressure force noise spectrum is given by:

$$S_{\rm FF}^{\rm rad} = \frac{2\hbar^2 G_a^2 \gamma_f}{\omega_s^2}\left(\frac{\gamma_\epsilon}{\gamma_f}\right) + \mathcal{O}(\eta^2) \tag{A.11}$$

Notice that the back-action force spectrum is zero when the end mirror is perfectly reflective ($\gamma_\epsilon = 0$).

The physical explanation of this back-action evasion phenomenon is shown in Fig.A.2. The part of the outgoing fields which contains the displacement signal can be written as (*suppose the end mirror is perfectly reflective*):

$$\hat{a}_{\rm out}^m = -2iG_a\hat{x} + 2i\frac{G_a\omega_s^2}{\Delta_t^2}\hat{x} \tag{A.12}$$

The first term on the right hand-side is the field directly reflected from the trapped mirror while the second term is the field transmitted out of the cavity. We can see that they cancel when $\Delta_t = \omega_s$. Therefore in this case the output field does not contain the x-information.

Given the parameters listed in Tab.I of the main text, we use (A.8)-(A.11) to calculate the modification of the mechanical dynamics by the trapping beam, and list the effective parameters in Tab.II of the main text. We can see that velocity response is a mechanical damping factor Γ which will not cause instability and is too small to affect the OMIT effective cavity bandwidth $\gamma_{\rm opt}$. The negative inertia $m_{\rm opt}$ is also too small to be comparable to the mass of the mechanical oscillator.

EFFECTIVE TEMPERATURE OF THE MIRROR-ENDOWED OSCILLATOR

Here we estimate the temperature of the mirror-endowed oscillator due to the additional heating caused by optical absorption. We assume the oscillator to be a silicon cantilever mirror with thickness h, Young's modulus Y and density ρ. Under the assumption that $l > b, h$, the fundamental frequency of the cantilever is given by [2]

$$\omega_{m0} = 1.875^2\sqrt{\frac{YI}{\rho S l^4}}, \tag{A.13}$$

where the $S = bh$ and l are the cross-sectional area and length of the cantilever. Then $I = bh^3/12$ is the moment of inertia of the beam cross-section. Using the parameters given in Tab. I, the resonant frequency has the value about 180Hz.

We also assume that the suspended mirror inside the cavity has thermal conductivity $\kappa(T) = \kappa_0 T^n$. The cantilever is illuminated by the trapping field with intra-cavity power $P_{\rm trap}^c$. As a simple 1-D heat transport problem, Fourier's law says that the heat power passing through the cross-section $S = bh$ of the mirror material at distance z from its

[2] L. Meirovitch *Analytical Methods in Vibrations.* Prentice Hall (1967)

TABLE I: Sampling parameters for the trapped oscillator

ρ	mass density	2329 kg/m^3
Y	Young's modulus	130GPa
l	cantilever beam length	1.5mm
h	cantilever thickness	0.37μm
b	cantilever width	0.3mm

center equals to $P_{\rm cond} = -S\kappa T'(z)$. Integrating the heat transport equation from the illuminated spot center with temperature T_0 to the boundary with temperature T, we have the relation between the T_0 and the absorbed power for a rectangular shape mirror $P_{\rm abs}$ as:

$$P_{\rm abs} = \frac{2S\kappa_0}{l(n+1)}(T_0^{n+1} - T^{n+1}) \tag{A.14}$$

Typically we have $n \sim 2$ at cryogenic temperature[3]. Using the sample parameters 10ppm, $P_c \sim 15$W and the conductivity of the material[3] $\kappa_0 = 10\text{W}/(\text{m.K}^n)$, we have $T_0 = 6$K from Eq.(A.14).

How this absorption-induced 6K temperature around the hot spot and its nonuniform distribution across the beam cantilever influences the thermal noise is not entirely clear and needs further study. The loss of the cantilever motion can be classified as surface loss and body loss. The body loss is mainly through the clamping point where the temperature is around the cryogenic environment temperature. The surface loss, on the other hand, influences the cantilever motion through the coupling of the material surface motion with the local thermal bath, which has a raised temperature from the trapping beam heating. Whether the thermal noise due to surface loss degrades the squeezed light or not depends on detailed design of the experiment and needs a more sophisticated study. Moreover, the trapping beam does not illuminate the cantilever beam uniformly thereby a heat flux will be built up across the cantilever beam with temperature gradient about $\nabla T \sim 3 \times 10^3$K/m. The non-equilibrium thermal noise associated with this heat flux is also unclear and needs to be addressed in future research.

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[3] C. J. Glassbrenner and G. A. Slack, Phy. Rev. 134, A1058 (1964)