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Broken Spin-Orbit Symmetry in Superfluid ^3He and B -Phase Dynamics

Mario Liu^(a) and M. C. Cross

Bell Laboratories, Murray Hill, New Jersey 07974

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It is shown that the B phase not only has a "phase dynamics" such as that in He II, but also an orbital dynamics somewhat similar to that of $^3\text{He-A}$ or a nematic liquid crystal. The rederived B -phase hydrodynamic theory, complete with nonlinear terms, establishes the equivalence between rotations in orbital and spin spaces and shows the intriguing possibilities of generating a magnetization by mechanical rotations and of substituting the magnetic field with oscillating parallel plates in "NMR" experiments.

The most exciting aspect of the two superfluid phases of ^3He is the richness of their spontaneously broken symmetries. The B phase, in particular, exhibits the rather subtle concept of spontaneously broken spin-orbit symmetry (SBSOS) introduced by Leggett, which is a broken symmetry only of *relative* spin-orbit rotations.¹ In previous studies of B -phase dynamics,^{2,3} however, the consequences of SBSOS seem indistinguishable from those of the broken rotational symmetry in spin space alone, such as that present in an antiferromagnet or a spin glass. This is indeed disturbing because it defies the general belief in the intimate relation between the spontaneously broken symmetry and the hydrodynamics of a system. The discrepancy warrants the effort to reinvestigate B -phase dynamics. One can plausibly expect a system with SBSOS, in which

the orbital variables are kept constant, to behave as if only the rotational symmetry in spin space were broken, and based on this argument, B -phase spin dynamics has been successfully investigated.² But we can also turn the argument around to conclude that keeping the spin variables constant, the system has to account for the three broken orbital symmetries (as would be present in the hypothetical biaxial nematics), and its dynamics must therefore differ from the two-fluid hydrodynamics of He II, where the only broken symmetry is the gauge invariance. So qualitatively speaking, B -phase dynamics will be given by the combination of the spin-glass dynamics of Halperin and Saslow⁴ and superfluid biaxial nematic dynamics with a number of elastic coefficients vanishing to reflect the symmetry of the Balian-Werthamer state, which is invariant un-

der a simultaneous rotation of both spaces. As we shall see below, this will give rise to some intriguing effects, unsuspected hitherto, such as the generation of a magnetization by mechanical rotations, enabling one to perform "NMR" experiments without a rf field to excite the resonance.

In what follows, B -phase dynamics will be re-derived along the line of the above arguments, with all the terms of second order either in the wave vector or in the small variables retained. This has the same accuracy as Landau's two-fluid hydrodynamics⁵ or the nonlinear A -phase dynamics.⁶ We shall employ the phase Φ and the *relative* infinitesimal rotation vector $d\vec{\eta}$ to characterize the fluctuations from the equilibrium state. This yields four independent equations of motion for the order parameter and represents the least redundant hydrodynamic description. Because of the lack of complex constraints, it is also the simplest one. We define⁷

$$d\eta_i \equiv d\theta_i^o - R_{i\alpha} d\theta_\alpha^s, \quad (1)$$

where $d\theta_i^o$ and $d\theta_\alpha^s$ are components of the infinitesimal rotation vectors in orbital and spin space, respectively; $R_{i\alpha}$ is the order parameter, a spin-orbit rotation matrix. Actually, $d\eta_i$ is globally ill defined, but since only its time and space derivatives enter the hydrodynamic equations, this forms no objection of substance (cf. the analogous discussions on $^3\text{He-A}$ by Brinkman and Cross).² However, as in the case of the phase variable there, one has to pay attention to a commutation rule closely connected to the Mermin-Ho relation⁸:

$$(\delta\nabla - \nabla\delta)\vec{\eta} = \delta\vec{\eta} \times \nabla\vec{\eta}, \quad (2)$$

where δ and ∇ stand for any first-order differential operator such as $\partial/\partial t$ or ∇_k . Equation (2) can be conveniently proved for any rotation angle by defining a set of three orthogonal axes. The fact that $d\vec{\eta}$ is a relative rotation angle turns out to be irrelevant here.

Without the dipole interaction, the energy ϵ can only depend on $\nabla_i \eta_j$. Since it is impossible to construct a pure orbital tensor from $R_{i\alpha}$,³ the textural part of ϵ takes a truly isotropic form^{2,9}:

$$\begin{aligned} & M_1 \nabla_i \eta_m \nabla_i \eta_m + 2M_2 \nabla_i \eta_i \nabla_m \eta_m \\ & + 2M_3 \nabla_m \eta_i \nabla_i \eta_m. \end{aligned} \quad (3)$$

For bulk calculations, M_2 and M_3 appear only in the combination $M_2 + M_3$. The coefficients in Eq. (3) are chosen to be consistent with the definition of Ref. 2. One can also arrive at Eq. (3) by tak-

ing ϵ as a function of $\nabla_i R_{j\alpha}$ only,³ and then re-writing ϵ employing the relation

$$dR_{i\alpha} = -R_{j\alpha} \epsilon_{ijm} d\eta_m. \quad (4)$$

In the B phase the energy is generally given as

$$\begin{aligned} d\epsilon = & T ds + \mu d\rho + v_i^n dg_i + \omega_\alpha dS_\alpha \\ & + j_i^s dv_i^s + \psi_{ij} d\nabla_j \eta_i + \varphi_i d\eta_i, \end{aligned} \quad (5)$$

where s , ρ , g_i , and s_α are the densities of the conserved quantities, entropy, mass, momentum, and spin, respectively. $\vec{v}^s = (\hbar/2m)\nabla\Phi$ and $d\vec{\eta}$ are the symmetry variables. The conjugate variables are defined by Eq. (5), and we have, in particular,

$$\vec{j}^s = \rho^s (\vec{v}^s - \vec{v}^n) \text{ and } \omega_\alpha = (\gamma^2/\chi)S_\alpha - \gamma H_\alpha. \quad (6)$$

The magnetic field is denoted by H_α , the gyromagnetic ratio by γ , and the susceptibility by χ . The tensor ψ_{ij} is given by the derivative of Eq. (3), and $\varphi_i = -\varphi n_i$ where φ is the usual dipole torque ($\partial V_D/\partial\theta$). Here θ and \vec{n} denote the rotation angle and axis of the matrix $R_{i\alpha}$, with $d\theta = -\vec{n} \cdot d\vec{\eta} = \vec{n} \cdot (d\vec{\theta}^s - d\vec{\theta}^o)$. The equilibrium texture is given by minimizing the energy, leading to

$$\nabla_i j_i^s = 0 \quad (7)$$

and

$$\Psi_i \equiv \nabla_j \psi_{ij} - \epsilon_{ikl} \nabla_j \eta_k \psi_{lj} = 0, \quad (8)$$

where explicitly

$$\begin{aligned} \psi_i = & 2M_1 \nabla^2 \eta_i + 4(M_2 + M_3) \\ & \times [\nabla_i (\nabla \cdot \vec{\eta}) + (\nabla \cdot \vec{\eta}) (\nabla \times \vec{\eta})_i] + \varphi n_i. \end{aligned} \quad (9)$$

The second term in ψ_i , Eq. (8), arises from using Eq. (2) in the variational derivative of the energy, with δ now the variational operator and ∇ the spatial derivative. A similar term arises in the phase dynamics of the A phase.²

We now proceed to investigate the dynamics. Neglecting the dissipative terms for the moment, we can write

$$\dot{\theta}_\alpha^s = \partial\epsilon/\partial S_\alpha \equiv \omega_\alpha, \quad (10)$$

$$\dot{\theta}_i^o = \frac{1}{2} (\nabla \times \vec{v}^n)_i \equiv \Omega_i. \quad (11)$$

The first equation is well known: It can be derived by arguing along the line of Leggett¹ that the total-spin operator is the generator of infinitesimal rotations in spin space. The second equation is slightly more subtle, but also quite familiar from the hydrodynamics of nematics^{10,11} and $^3\text{He-A}$.^{12,13} It is usually derived by expansion of $\dot{\theta}_i^o$ in the thermodynamic force, $\nabla_i v_j^{(n)}$, where

an ensuing solid-body rotation gives the coefficient $\frac{1}{2}$. However, aside from possible ambiguities in more complicated systems, this method obscures the important fact that the presence of the vorticity Ω_i in Eq. (11) is due solely to the spontaneously broken orbital symmetry. A more physical approach is given by the complete analogy between S_α and the intrinsic-orbital-angular-momentum density τ_i :

$$\dot{\theta}_i^o = \partial\epsilon/\partial\tau_i. \quad (12)$$

Although τ_i is not a conserved quantity, the energy depends on it through the relation¹⁴

$$\vec{g} = \vec{p} + \frac{1}{2}\nabla \times \vec{\tau}, \quad (13)$$

where \vec{p} is the linear part¹⁵ of the momentum density \vec{g} . A partial integration leads readily to $\partial\epsilon/\partial\vec{\tau} = \vec{\Omega}$. A third method, actually equivalent, consists of studying the explicit equations¹¹ of motion for \vec{p} and $\vec{\tau}$, which show that $\partial\epsilon/\partial\vec{\tau}$ relaxes to $\vec{\Omega}$ in a wave-number-independent time. The equation of motion for the relevant quantity $d\vec{\eta}$ is given by combining Eqs. (10) and (11):

$$\dot{\eta}_m = \Omega_m - R_{m\alpha}\omega_\alpha. \quad (14)$$

It must be stressed that since the energy [Eq. (5)] does not depend on the quantity $d\theta_i^o + R_{i\alpha}d\theta_\alpha^s$, its time derivative cannot couple to any of the hydrodynamic equations and need not be considered. Therefore we may expect only three additional "spin-orbit waves" rather than six.² With these preliminary discussions, it is straightforward to write down the complete set of hydrodynamic equations by employing the well-known standard procedure^{5,16} and the symmetry of the B -phase order parameter:

$$\dot{\rho} + \nabla \cdot \vec{g} = 0, \quad \dot{\vec{v}}^s + \nabla(\vec{v}^s \cdot \vec{v}^n + \mu + Z^D) = 0, \quad (15)$$

$$\dot{s} + \nabla_i(sv_i^n + \alpha\epsilon_{ijk}R_{j\alpha}\nabla_k\omega_\alpha + f_i^D) = R/T, \quad (16)$$

$$\dot{g}_i + \nabla_i P - \frac{1}{2}(\nabla \times \vec{\Psi})_i + \nabla_k(\Pi_{ik}^e + \Pi_{ik}^D) = 0, \quad (17)$$

$$\dot{\eta}_i + (\vec{v}^n \cdot \nabla)\eta_i + R_{i\alpha}\omega_\alpha - \frac{1}{2}(\nabla \times v^n)_i + Y_i^D = 0, \quad (18)$$

$$\begin{aligned} \dot{S}_\alpha - \gamma(\vec{S} \times \vec{H})_\alpha \\ + \nabla_i(S_\alpha v_i^n + \psi_{ji}R_{j\alpha} + \alpha\epsilon_{ijk}R_{j\alpha}\nabla_k T + \Sigma_{\alpha i}^D) \\ = R_{k\alpha}\varphi_k. \end{aligned} \quad (19)$$

The dissipative terms are denoted by superscript D and will be discussed together later. Aside from the usual convective term $s\vec{v}^n$, the entropy current in Eq. (16) contains a nonlinear reactive flux preceded by α , which is reminiscent of the Ho term in the A -phase dynamics.⁶ With its corresponding term in Eq. (19) it forms an addition-

al coupling between the spin and orbital dynamics. In Eq. (17), P denotes the pressure, $\Pi_{ik}^a \equiv -\frac{1}{2} \times \epsilon_{ikm}\Psi_m$ is characteristic of the broken orbital symmetry, and Π_{ik}^e represents the generalized Erikson tensor:

$$\Pi_{ik}^e = g_i v_k^n + \psi_{jk}\nabla_j \eta_i + j_k^s v_i^s. \quad (20)$$

Since only $\nabla_k(\Pi_{ik}^a + \Pi_{ik}^e)$ is defined, we can substitute $\Pi_{ik}^a + \Pi_{ik}^e$ with $\frac{1}{2}(\Pi_{ik}^s + \epsilon_{ikm}\phi_m)$, where

$$\Pi_{ik}^s = (\Pi_{ik}^e + \epsilon_{mji}\nabla_j \psi_{mk}) + (ik - ki). \quad (21)$$

The dipole torque $\frac{1}{2}\epsilon_{ikm}\phi_m$ with its corresponding term on the right-hand side of Eq. (19) guarantees the conservation of total angular momentum $\vec{r} \times \vec{g} + \vec{S}$. The entropy production is given by

$$\begin{aligned} -R = f_k^D \nabla_k T + Z^D \nabla_k j_k^s + \Pi_{ik}^D \nabla_k v_i^n \\ + Y_k^D \Psi_k + \Sigma_{\alpha i}^D \nabla_i \omega_\alpha, \end{aligned} \quad (22)$$

where f_k^D , Z^D , and Π_{ik}^D are, as pointed out by Graham and Pleiner,³ unchanged from their counterparts in the hydrodynamics⁵ of He II. To include nonlinear terms, Y_k^D and $\Sigma_{\alpha i}^D$ have to be slightly generalized from their form as given in Ref. 3:

$$Y_k^D = -\nu\psi_k, \quad (23)$$

$$\begin{aligned} \Sigma_{\alpha i}^D = -\mu_1 \nabla_i \omega_\alpha \\ - (\mu_2 R_{i\alpha} R_{j\beta} + \mu_3 R_{j\alpha} R_{i\beta}) \nabla_j \omega_\beta. \end{aligned} \quad (24)$$

We shall now employ Eqs. (15)–(19), which represent a concise nonlinear hydrodynamic description of the B phase, to investigate a few simple and experimentally relevant situations.

For the uniformly rotating case, the linearized hydrodynamic equations derived above reduce to

$$\dot{S}_\alpha = -n_\alpha \varphi + \gamma(\vec{S} \times \vec{H})_\alpha \quad (25)$$

and

$$\dot{\theta} = \vec{n} \cdot (\vec{\omega} - \vec{\Omega}) - \nu\varphi. \quad (26)$$

In a stack of parallel plates, immersed in ³He- B , if the viscous penetration depth is larger than the plate spacing (so that \vec{v}^n will follow the movement of the plates), and if the magnetic field $\vec{H} = \vec{n}H e^{-i\omega t}$ and the angular velocity $\vec{\Omega} = \vec{n}\Omega e^{-i\omega t}$ of the plates are perpendicular to the plates, the magnetization is given as

$$\begin{aligned} \gamma \vec{S} = (-\Omega_L^2 \chi / \gamma)(\gamma H + \Omega)(\omega^2 - \Omega_L^2 + i\omega\Gamma)^{-1} \\ \times \exp(-i\omega t) \vec{n}, \end{aligned} \quad (27)$$

where Ω_L denotes the longitudinal resonance frequency and $\Gamma = \Omega_L^2 \chi \nu / \gamma^2$ is the relaxation rate. Note the complete equivalence between rotations

in spin and orbital space, given by γH and Ω , respectively. Thus a longitudinal resonance can actually be excited by mechanical oscillations alone. This is also true for a transverse resonance, if \vec{n} is not aligned along the static magnetic field.¹⁷ The static magnetization given by Eq. (27) is $\chi(H + \Omega/\gamma)$; this latter result has also been observed in solids with strong spin-orbit coupling and is known as the Barnett effect. There, however, the magnetization usually relaxes and does not oscillate.

There is, in principle, also the reverse effect where an ac magnetic field $\vec{n} H e^{-i\omega t}$ results in tiny oscillations of the helium liquid and the plates. Because in this case Ω_i is the fluctuating variable rather than an external perturbation, we have to include its equation of motion

$$\dot{\Omega}_i = (V/I)m_i \varphi, \quad (28)$$

where I denotes the orbital part of the moment of inertia and V the volume of ${}^3\text{He}$. The oscillation can be calculated from Eqs. (25), (26), and (28) and is given by

$$\vec{\Omega} = \alpha \Omega_L^2 \gamma H [\omega^2 - \Omega_L^2(1 + \alpha) + i\omega\Gamma] \times \exp(-i\omega t) \vec{n}. \quad (29)$$

Although, strictly speaking, the resonance frequency is $\Omega_L(1 + \alpha)^{1/2}$ here, the ratio of moments of inertia in spin and orbit space, $\alpha \equiv \chi V/\gamma^2 I$, is much too small a quantity to give any measurable deviation.

We summarize our results briefly: By employing a "B-phase Mermin-Ho relation" the nonlinear dynamics is derived and given in a simple and concise form, where the significance of the vorticity as a broken-symmetry term is emphasized and the equivalence of rotations in spin and orbital space is demonstrated. Some of the more intriguing consequences of the modified dynamics are the possibility of producing a magnetization by mechanical oscillations and the coupling of the spin dynamics to the shear viscosity, which may help us to understand the puzzling spin-relaxation behavior. The spectrum of the hydrodynamic modes, however, remains the same as given by

previous studies.^{2,3}

We would like to thank D. Osheroff for telling us about the Barnett effect.

^(a)On leave from the Technical University of Munich, Munich, West Germany.

¹For reviews, please see A. J. Leggett, *Rev. Mod. Phys.* **47**, 331 (1975); and P. W. Anderson and W. F. Brinkman, in *The Helium Liquids*, edited by J. G. M. Armitage and I. E. Farghar (Academic, London, 1975). Because the spin and orbital symmetries are independently broken in the A phase, SBSOS is far less subtle and indeed a somewhat trivial consequence there.

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⁹The reader is reminded that only for linear calculations is $\nabla_i \eta_j$ the derivative of a vector field " $\vec{\eta}(\mathbf{r})$." For example, in general, $\nabla_i \nabla_k \eta_j - \nabla_k \nabla_i \eta_j \neq 0$, but instead is given by Eq. (2).

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