positive mass. By analogy we assign a positive charge to the solitary hole, and consistency then requires its effective mass to be negative.[4] This in turn implies that the Coulomb repulsion between two holes gives rise to an attraction. In order to demonstrate the quasiparticle behavior of the solitary electron hole, we modified the exciting pulse so that two holes close to each other and having almost equal velocity were excited; see Fig. 5. Following their trajectory we observed an attraction and a subsequent coalescence that prevailed throughout the entire plasma column. The collision thus appears to be inelastic. Such a coalescence has already been observed in numerical investigations.[4] If, however, the initial velocity difference is large, two holes are observed to pass through each other. We may thus conclude that the observed properties of the solitary electron holes are indeed compatible with those expected for quasiparticles, at least within the limits of space and time in our experimental setup.

We are aware that the solitary holes described in this Letter may have some relevance for describing one-dimensional strong Langmuir turbulence.[11]

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[1] K. Saeki, J. Phys. Soc. Jpn. 35, 251 (1973).
[2] H. Ikezi, P. J. Barett, R. B. White, and A. Y. Wong, Phys. Fluids 14, 1997 (1971).
[3] I. B. Bernstein, J. M. Green, and M. D. Kruskal, Phys. Rev. 108, 546 (1957).
[4] H. L. Berk, C. E. Nielsen, and K. V. Roberts, Phys. Fluids 13, 980 (1970).
[5] M. Kako, T. Taniuti, and T. Watanabe, J. Phys. Soc. Jpn. 31, 1820 (1971).
[6] A. W. Trivelpiece and R. W. Gould, J. Appl. Phys. 30, 1784 (1959).
[7] P. H. Sakanaka, Phys. Fluids 15, 1323 (1972).
[8] F. Hohl, Phys. Fluids 12, 230 (1969).
[9] H. W. Bloomberg, Phys. Fluids 17, 1404 (1974).
[10] R. N. Franklin, S. M. Hamberger, G. Lampis, and G. J. Smith, United Kingdom Atomic Energy Association Culham Laboratory Report No. CLM-R 131 (unpublished).
[11] T. H. Dupree, Bull. Am. Phys. Soc. 23, 869 (1978).

Phase Transitions in Two-Dimensional Superfluid ^{3}He

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A discussion of phase transitions in two-dimensional ^{3}He is presented. A new type of phase transition is proposed, in which "islands of reversed $\vec{l}$" spontaneously nucleate, leading to an Ising-like transition. It is shown that such transitions may need to be taken into account, possibly competing with the well-know Kosterlitz-Thouless transition. Consequences of both types of transition are discussed.

Phase transitions in two-dimensional systems have attracted considerable attention over the past few years since the discovery of Kosterlitz and Thouless[1] that a new kind of ordering ("topological ordering") may exist in these systems when conventional order is absent. This new type of ordering is destroyed upon the unbinding of vortex-antivortex pairs (or their analogs) in the system under consideration. The phase diagram then exhibits a line of critical points below T_c with continuously variable critical exponents.[2] Renormalization-group analyses based on these ideas have recently been applied to two-dimensional superfluid ^{4}He and X-Y ferromagnets,[2] two-dimensional solids,[3] and two-dimensional liquid crystals.[4,5]

Experimental attention is just now beginning to be focused on two-dimensional superfluid ^{3}He films. Although a p-wave superfluid is efficiently depaired by *incoherent* scattering, it is thought that on a "specularly reflecting" surface films of superfluid ^{3}He can exist with the thicknesses down to and even less than the coherence length ξ_0 of the the Cooper pair. (The possible p-wave states, however, are limited to those with orbital motion in the plane of the surface—for the sake of argument we will take the state to be the bulk-stable state known as the A phase.) In such films two-dimensional behavior may be expected to be important. It is the purpose of this Letter to suggest some possibilities for such transitions in ^{3}He.

The obvious choice for such a transition is a Kosterlitz-Thouless phase transition in which vortices appear in the phase of the order parameter, to be described below. However, because of the richness of the structure of the order parameter in the superfluid ^{3}He phases, another possibility at least should be considered, in contrast to those systems previously studied; as will be shown, this is a two-dimensional Ising-like phase transition in the ^{3}He order parameter.

The situation in two-dimensional ^{3}He is somewhat complicated due to the tensorial nature of the order parameter which may be written

$$d_{\alpha i} = d_i(\vec{\Delta}_1 + i\vec{\Delta}_2)_\alpha, \tag{1}$$

where $\vec{d}$ is a unit vector defining the spin state, $|\vec{\Delta}_1| = |\vec{\Delta}_2| = \Delta$, and the direction $\vec{l}$ of the angular momentum is given by $\vec{\Delta}_1 \times \vec{\Delta}_2$. In small geometries, it is usually a good approximation to treat the spin and orbital parts of the order parameter independently, and we will do this here, concentrating on the orbital part. It is interesting to note that a vector order parameter may exhibit no phase transition in two dimensions[1] (c.f. a Heisenberg ferromagnet) and so for two-dimensional ^{3}He, long-range order in $\vec{d}$ in the low-temperature phase must arise from spin-orbit coupling to an ordered $\vec{l}$. At a surface, $\vec{l}$ must be perpendicular to the surface,[6] so that the order parameter for ^{3}He on a surface is $\mathrm{SO}(2)\times Z_2$, where SO(2) describes rotations of the $\vec{\Delta}$'s in the plane of the surface, and Z_2 indicates that $\vec{l}$ may be parallel or antiparallel to a given surface normal. A Toulouse-Kléman[7] analysis then indicates that two topologically stable defects may occur: point vortices in the $\vec{\Delta}$'s which add isomorphically to the integers, and the border between two regions of antiparallel $\vec{l}$, which is its own "antiparticle."[8]

Consider now a two-dimensional layer of ^{3}He with the orthonormal triad order parameter described above. To obtain as simple a description as possible, a square lattice of spacing a, of the order of the Cooper-pair coherence length, will be utilized with the order parameter given at each point on the lattice. The gradient energies for spatially varying order parameters have been worked out in the three-dimensional case.[9,10] These results assume a slowly varying order parameter, which limits their applicability to the present situation, where two adjacent $\vec{l}$ vectors may be antiparallel. In order to determine, to a first approximation at least, the gradient energies involved in each type of transition considered, so that their transition temperatures may be compared, it is necessary to examine the order-parameter singularities involved in more detail.

First, suppose that $\vec{l}$ is everywhere uniform over the plane so that the free energy in the continuum approximation may be written

$$F \approx \tfrac{1}{2}K_\varphi \int (\nabla\varphi)^2 d^2x, \tag{2}$$

where φ is a "phase" variable which describes the orientation of the $\vec{\Delta}_1, \vec{\Delta}_2$ axes with respect to some standard orientation, and K_φ is an elastic constant related to the superfluid density: $K_\varphi = \sigma_\perp{}^{(s)}\hbar^2/4m^2$, where m is the mass of a ^{3}He atom and $\sigma_\perp{}^{(s)}$ is the superfluid (areal) density transverse to $\vec{l}$. For a not too thin film (thickness d) and leaving out possible renormalizations $\sigma_\perp{}^{(s)}$ may be related to the bulk superfluid density tensor $\sigma_\perp{}^{(s)} = d\rho_\perp{}^{(s)} = 2d\rho^{(s)}$, where $\rho^{(s)}$ is the longitudinal component and the latter equality only holds for temperatures near the three-dimensional transition temperature. The situation here is reminiscent of ^{4}He, and the phase transition occurring will be of the Kosterlitz-Thouless variety where vortex-antivortex pairs in the phase dissociate. The analysis in Nelson and Kosterlitz[2] can then be carried through to yield

$$\lim_{T\to T_c^-} \frac{\sigma_\perp{}^{(s)}}{T} = \frac{8m^2 k_B}{\pi\hbar^2}, \tag{3}$$

where the additional factor of 4 comes from the pairing.

Now consider the case where $\vec{l}$ is free to vary. Here we arrive at the possibility of a new type of phase transition in which the spontaneous appearance of islands of reversed $\vec{l}$ dominate the high-temperature phase, destroying the "ferromagnetic" order of the $\vec{l}$ vectors in the low-temperature phase. These islands may be classified into two types: In the first kind the $\vec{\Delta}^{(1)}$ and $\vec{\Delta}^{(2)}$ vectors rotate as the island is circumnavigated, leading to a net circulation in the superfluid velocity. (Examples of such islands are described by Mermin.[11]) These islands therefore look like vortices at large distances and will have long-range (logarithmic) interactions. The second kind of island has no surrounding circulation, and corresponds to a simple inversion of $\vec{l}$. An example may be easily generated by rotating the orbital triad through π about a *fixed* direction in the plane as the border is crossed going outwards in any direction. In this case one need only consider the variation of $\vec{l}$ as one moves across the border, and the free energy for the discretized model may be simply written

$$F = -K_l \sum_{i,\delta} (\vec{l}_i \cdot \vec{l}_{i+\delta}), \tag{4}$$

where K_l is the appropriate coupling constant to be determined, and $\vec{i}, \vec{\delta}$ are two-dimensional vectors, with $\vec{i}$ running over all lattice sites, and with $\vec{\delta}$ a unit vector to be summed over all nearest-neighbor sites.

This type of island is of interest because its interactions are short ranged, the energy and entropy scale similarly, and the problem reduces to a two-dimensional Ising model, which has been worked out exactly.[12] Because of the second consequence in particular, the nucleation of such islands becomes a possible candidate for a phase transition in two-dimensional superfluid ^{3}He. These ideas will now be made more explicit.

If the coupling parameters K_l and K_φ are of the same order of magnitude (in fact they are, as will be shown), this Ising type of transition may successfully compete with the Kosterlitz-Thouless phase transition described in Eq. (3). To see why such a transition may be important, we follow an argument due to Peierls (see Ziman's book[13]). From Eq. (4), the energy of a border of length L lattice spacings is $E \approx 2LK_l$. Furthermore, at any point on a square lattice the border may continue in any of three directions, so $S \approx k_B \ln 3^L$. Hence the free energy of the border is given by

$$F = E - TS \approx L(2K_l - k_B T \ln 3), \tag{5}$$

yielding a transition temperature

$$T_c^{\,l} \approx \frac{2K_l}{k_B \ln 3}. \tag{6}$$

For comparison, the dissociation of a vortex-antivortex pair in the phase occurs at the temperature given by Eq. (3):

$$T_c^{\,\varphi} \approx \frac{\pi K_\varphi}{2k_B}, \tag{7}$$

and so if K_l and K_φ are of the same order of magnitude $T_c^{\,\varphi}$ and $T_c^{\,l}$ will also be near each other.

An estimate of K_φ has already been given. In order to estimate K_l it is helpful to think of a thicker film, where d, the sample thickness, is greater than ξ_0 and the "small bending limit" expressions[10] may be used. Using as a model the configuration in Fig. 1, K_l is found by estimating the energy per unit length of the pair of line singularities on the surface with a cutoff of order d. The result is

$$\frac{K_l}{a} \lesssim \frac{5\pi}{4} \rho_s \left(\frac{\hbar}{2m}\right)^2 \ln\left(\frac{\alpha d}{\xi_0}\right), \tag{8}$$

with α a number of order unity and $a \sim \xi_0$ the lattice spacing used. Since $\ln(d/\xi_0)$ is $O(1)$ Eqs. (6) and (8) yield

$$T_c^{\,(l)} \simeq \frac{5\pi\rho_s a}{2k_B \ln 3}\left(\frac{\hbar}{2m}\right)^2 \tag{9}$$

compared with $T_c^{\,(\varphi)} \simeq (\pi\rho_s d/k_B)(\hbar/2m)^2$ in the same limit.

It is clear that for $d \sim a \sim \xi_0$ the two transition temperatures are of the same order of magnitude and may in fact be rather close. Of course, the numerical factor in Eq. (9) is only an order-of-magnitude estimate—for thin films, a more microscopic calculation is needed. Our purpose here is only to suggest the necessity of thinking about these "Ising" transitions when considering possible phase transitions in two-dimensional superfluid films.

The Ising model in two dimensions has been solved exactly, and a large literature exists on the subject. Only two points of interest will be

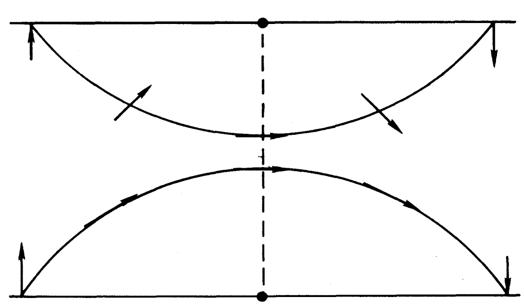

FIG. 1. The circular disgyration pattern used as a model to estimate K_l. The arrows represent the $\vec{l}$ vector.

made here. First, the exact value of the transition temperature for a two-dimensional Ising model on a square lattice with coupling K_I is given by[12,13]

$$T_c^{\,I} = \frac{2K_I}{k_B \ln(1+\sqrt{2})}. \tag{10}$$

Second, values of critical exponents for this model are given in Wilson and Kogut.[14] In particular, the falloff of the correlation function $\langle l(\vec{r}) l(\vec{0})\rangle$ for large $|\vec{r}|$ is given by the exponent $\eta = \frac{1}{4}$, which is identical to the Kosterlitz-Thouless case. In general, however, the critical exponents of the Ising and Kosterlitz-Thouless X-Y model are different, suggesting one possible experimental means to decide which of the transitions is actually occurring. Critical exponents for the X-Y model have been given by Kosterlitz.[15] Among the more interesting features of a two-dimensional Ising-model transition is the logarithmic singularity of the specific heat as T_c, although this may be difficult to detect. Perhaps more important is the prediction by Nelson and Kosterlitz[2] that a Kosterlitz-Thouless transition is accompanied by a finite jump in the renormalized coupling constant at T_c. For ^{3}He this leads to a jump in the superfluid density at $T_c^{\,\varphi}$, which, though small, should be observable. If the transition is Ising-like, however, no such jump is expected for islands with phase coherence maintained across their boundaries; the scheme of Fig. 1 provides one example of such a border. This suggests that a useful means for distinguishing between the two transitions consists in the observation of the superfluid density at the transition temperature, which may, for example, be measured by the response of the film to rotating the substrate. In addition other possibilities may occur, the most interesting case being the nucleation of islands with net circulation, which would interact with other such islands as well as with point vortices. There is also the intriguing possibility that $T_c^{\,\varphi} < T_c^{\,I}$, leading to an intermediate temperature phase without quasi-long-range superfluid (phase) order, but with long-range "ferromagnetic" ordering of $\vec{I}$ which ultimately disappears at $T_c^{\,I}$ as discussed here. In any case it appears plausible that the simple circulation-free islands may have an important role to play in phase transitions in thin films of ^{3}He.

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[1] J. M. Kosterlitz and D. J. Thouless, J. Phys. C 6, 1181 (1973).
[2] D. R. Nelson and J. M. Kosterlitz, Phys. Rev. Lett. 39, 1201 (1977), also Refs. 1–6 therein.
[3] B. I. Halperin and D. R. Nelson, Phys. Rev. Lett. 41, 121 (1978).
[4] D. R. Nelson and R. A. Pelcovits, Phys. Rev. B 16, 2191 (1977).
[5] D. L. Stein, Phys. Rev. B 18, 2397 (1978).
[6] A. J. Leggett, Rev. Mod. Phys. 47, 331 (1975).
[7] G. Toulouse and M. Kléman, J. Phys. (Paris), Lett. 37, L149 (1976).
[8] D. L. Stein, R. D. Pisarski, and P. W. Anderson, Phys. Rev. Lett. 40, 1669 (1978).
[9] V. Ambegaokar, P. G. de Gennes, and D. Rainer, Phys. Rev. A 9, 2676 (1974).
[10] M. C. Cross, J. Low Temp. Phys. 21, 525 (1975).
[11] See N. D. Mermin, in *Quantum Fluids and Solids*, edited by S. B. Trickey, E. D. Adams, and J. W. Duffy (Plenum, New York, 1977).
[12] L. Onsager, Phys. Rev. 65, 117 (1944).
[13] J. M. Ziman, *Principles of the Theory of Solids* (Cambridge Univ. Press, New York, 1972), 2nd ed.
[14] K. G. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974).
[15] J. M. Kosterlitz, J. Phys. C 7, 1046 (1974).