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STRUCTURE OF TURBULENT SHEAR FLOWS: A NEW LOOK

by  
ANATOL ROSHKO  
California Institute of Technology  
Pasadena, California

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Anatol Roshko\*  
 Professor of Aeronautics  
 California Institute of Technology  
 Pasadena, California

### Abstract

For many years experimental research in turbulence was devoted to the measurement of various correlations and spectral functions which had evolved from the statistical theories and from engineering computing methods based on the hierarchy of Reynolds equations. A recent change in direction toward a more deterministic description of turbulent structure has been initiated by the discovery of large coherent structures in several turbulent shear flows. The new point of view suggests that with every shear flow (jet, boundary layer, mixing layer, etc.) is associated an identifiable, characteristic structure; the development of the flow is controlled by the interactions of these structures with each other. An understanding of their properties should give insight into actual physical processes in turbulent flows, such as entrainment, transport, mixing, noise production, gustiness, etc. and should lead to improved methods for analyzing and computing them. Experiments designed to study these properties are aided by recent developments in instrumentation technology such as computer-aided control of the experiments, but the venerable technique of flow visualization is still an indispensable aid.

### Introduction

The problem of turbulent flow continues to be an outstanding one in technology and in physics. Of the nine Dryden research lectures so far, four have been on some aspect of the turbulence problem. At meetings such as this one the turbulence problem is always the subject of some sessions and lurks in the background of many others; for example, separated flow, combustion, jet noise, chemical lasers, atmospheric problems, etc. It is continually the subject of conferences, workshops and reviews. In his time Hugh Dryden also wrote several reviews of turbulent flow. In reading some of them again, one statement<sup>(1)</sup> particularly relevant to the present lecture caught my attention: "--- it is necessary to separate the random processes from the non-random processes. It is not yet fully clear what the random elements are in turbulent flow." Neither is it fully clear what the non-random, orderly elements are, but some of them are beginning to be recognized and described.

Generally the picture one has had of turbulence is of chaos and disorder, implicit in the name. Although it was known that organized motion could exist, superimposed on a background of "turbulence," for example vortex shedding from a circular cylinder up to Reynolds numbers of  $10^7$ , such examples were regarded as special cases closely tied to their particular geometric origins and not characteristic of "well developed" turbulence. It was known that large structures

are important in the development of turbulent shear flows and that these ought to possess some definable features. But even when the concept of a characteristic "big eddy" was explored, it was usually in the context of a statistical quantity. The earliest and most decisive attempts to define the form of such large eddies were made by Townsend and his students.<sup>(2, 3)</sup> In recent years it has become increasingly evident that turbulent shear flows do contain structures or eddies whose description is more deterministic than had been thought, possessing identifiable characteristics, existing for significant lifetimes, and producing recognizable and important events. More accurate descriptions of their properties, how they fit into the complete description of a turbulent flow, to what extent are they central to its development, and how they can be reconciled with the apparent chaos and disorder, are problems which are becoming of interest to an increasing number of researchers. It is the purpose of this lecture to describe some of these new developments. The discussion will draw largely on experiences from our own laboratory; it is not intended to be a complete survey. Other discussions of these ideas can be found in various recent publications.<sup>(4-8)</sup>

### The Turbulent Mixing Layer

Our own ideas about turbulent flow structure were drastically altered by an investigation of plane turbulent mixing layers which we undertook,<sup>(9, 10)</sup> initially to study the effects of non-uniform density.

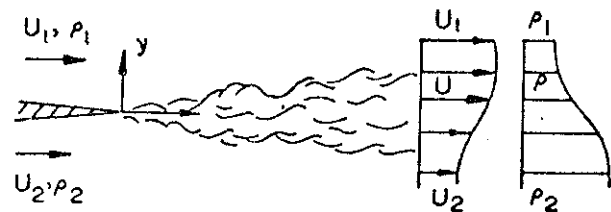


Fig. 1 Plane mixing layer between two streams with velocities  $U_1$  and  $U_2$ , densities  $\rho_1$  and  $\rho_2$ .

Fig. 1 is a schematic diagram of this class of flows, in which the uniform streams on either side of the mixing layer are characterized by their velocities  $U_1$  and  $U_2$ , etc. What is attractive about this configuration is that, of all the flows in the catalog of basic turbulent shear flows, it allows the effects of variation of various parameters such as velocity ratio  $U_2/U_1$ , density ratio  $\rho_2/\rho_1$ , Mach numbers, etc. to be explored in the simplest way (this remark applies to the concept, not to the actual realization in the laboratory!) The particular case of homogeneous flow ( $\rho_2 = \rho_1$ ) at low Mach number, along with other incompressible flows such as wakes, jets and boundary

\* Fellow AIAA.

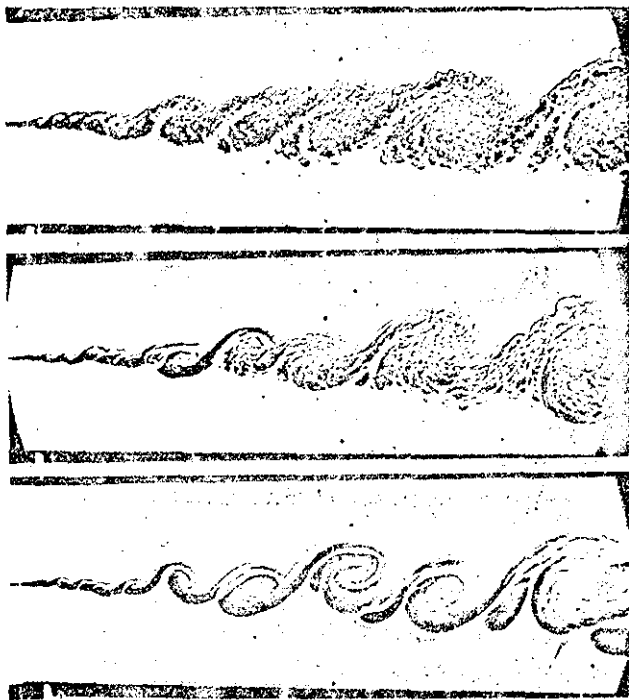


Fig. 2. Mixing layer between helium (upper) and nitrogen streams.  $U_2/U_1 = 0.38$ . Reynolds numbers based on full length of layer are 1.2, 0.6 and  $0.3 \times 10^5$  resp., from top to bottom.

layers, had long been a part of the classical literature of turbulent shear flow. No hint of other than a classical "turbulent" picture existed.

It was therefore rather astonishing when shadow pictures of the flow revealed the presence of well defined large structures (Fig. 2) which have the appearance of breaking waves or rollers or vortices, superimposed on a background of finer scaled turbulence. In this example the highest Reynolds number, based on the length of shear layer visible in the picture and on parameters on the high speed side (helium), is  $1.2 \times 10^5$  for the flow in the uppermost picture; it may be seen that fine-scale turbulence is present throughout the flow. Because of the large difference in optical indices of refraction of helium and nitrogen, these features are particularly well revealed.

That the large-scale organized phenomena are not peculiar to the large density difference is shown by Fig. 3, which is a sequence of shadow-graphs of a flow in which the density is uniform. Optical visualization was made possible by using different gases, of the same density, on the two sides of the mixing layer: nitrogen on the high speed side and a mixture of helium and argon on the low speed side. The Reynolds number is  $8.5 \times 10^5$ , comparable to the highest values in previous, well-known investigations of turbulent mixing layers.<sup>(11-13)</sup> Thus these flows, with organized large structure, correspond in every way to what have been classically called turbulent mixing layers. Their mean velocity fields, the Reynolds shear stresses calculated from them, and the corresponding growth rates<sup>(10, 14)</sup> agree

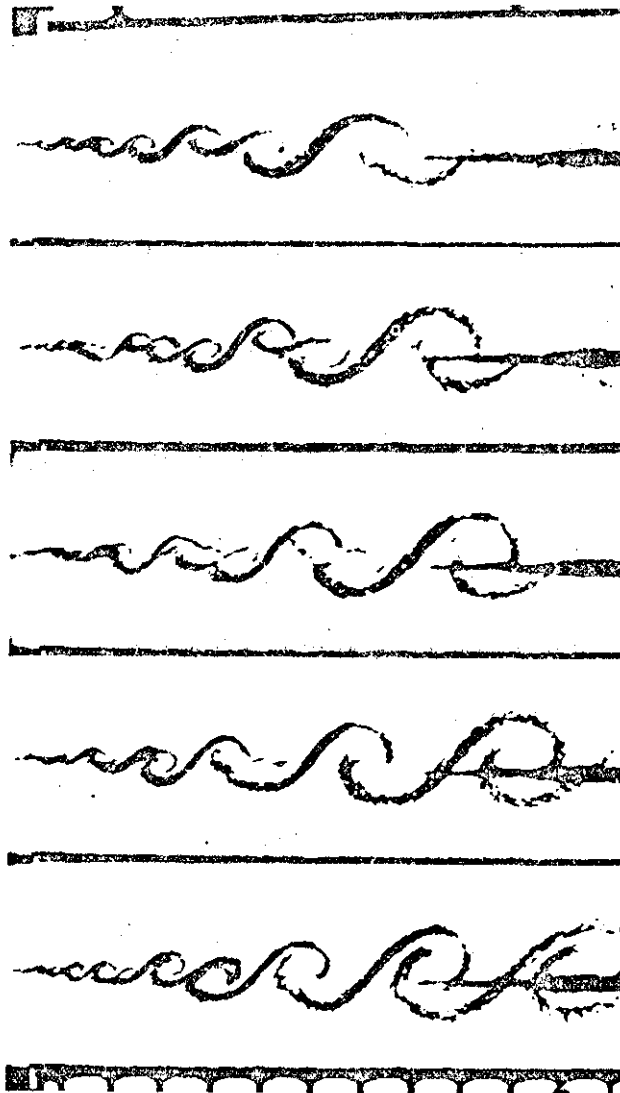


Fig. 3. Mixing layer between nitrogen (upper) and a helium-argon mixture of the same density. Reynolds number =  $8.5 \times 10^5$ . From top to bottom: frame numbers 1, 6, 9, 13 and 19 of a sequence.

with what was already known about these flows. In fact, they are the same flows, but the existence of the organized structures had not previously been recognized. Still another example obtained in our laboratory,<sup>(15)</sup> where the vortex structures were observed with dye visualization, was a mixing layer in a water channel at a Reynolds number of  $3 \times 10^6$ .

We emphasize these points because typical initial reaction to these pictures (including our own) has tended to be one of skepticism as to whether the flow is really turbulent, whether Reynolds number is high enough, etc. Although these flows had been studied for many years and the presence of the coherent structures not suspected, once the structures are known to be there it is rather easy to find them! It is also interesting that they were recognized in other investigations<sup>(5, 16)</sup> being carried on at about the same

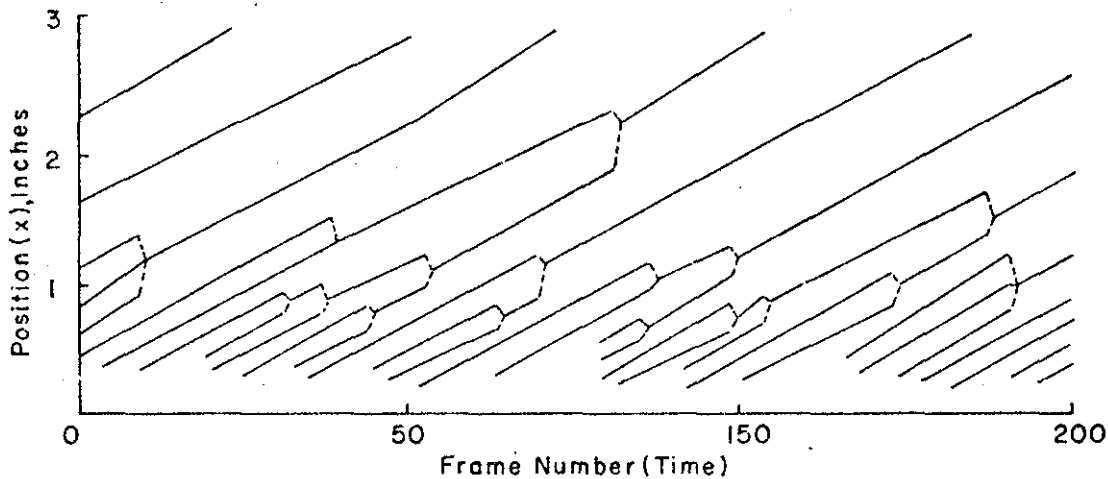


Fig. 4.  $x$ - $t$  diagram for eddy trajectories in mixing layer as in Fig. 2. Reynolds number =  $1.6 \times 10^4$  per inch.

time as our own, a not untypical occurrence in research; as mentioned earlier, there has been an increasing awareness over many years of the possibility of organized structures in turbulent shear flows.

Given that they do exist, many questions occur: how can they be described; how are they formed; how long do they exist; what do they do? Some aspects of these are discussed in the following paragraphs.

#### Coherence and Lifetimes

The vortex-like structures on the photographs in Figs. 2 and 3 can readily be identified from frame to frame on high-framing-rate motion pictures and their progress along the mixing layer can be plotted against time (frame number) on an  $x$ - $t$  diagram. A portion of such a diagram is shown in Fig. 4. Several remarkable results emerge from such observations. The trajectories of individual vortices correspond to the segments of lines (there has been some smoothing) which are nearly all parallel to each other. That is, all the vortices move at nearly constant speed, which is approximately the average of  $U_1$  and  $U_2$  (but with some variation depending on  $U_2/U_1$  and  $\rho_2/\rho_1$ ). The birth of a new vortex coincides with the demise of two or more old ones, in an interaction process which is only indicated by broken lines on the diagram. We shall presently look in more detail at this interaction; but first it will be useful to discuss some overall features.

It is expected that the scale of any feature of the flow will increase with increasing distance downstream. This follows from the general similarity property of the flow, which requires that all mean length scales be proportional to the distance from the origin. This trend can be seen in the photographs; that is, the sizes of the vortices and their spacings both tend to increase with distance downstream, and both are related

to the thickness of the mixing layer. Turning our attention to one of these, the spacing  $\lambda$ , we can say that the mean spacing  $\bar{\lambda}(x)$  must increase with  $x$ , smoothly because there is nothing special about any particular value of  $x$ , and linearly (like the thickness) because of the particular similarity property for this flow. For a mixing layer, similarity requires that  $\bar{\lambda}$  increase linearly with  $x$ . On the other hand, we can see from the  $x$ - $t$  diagram that spacings between individual vortices are fairly constant, changing only during the interaction events. Reconciliation between the two apparently contradictory features results from the fact that, in the vortex pairs passing any particular value of  $x$ , there is a distribution (Fig. 5) of the spacings about a mean value  $\bar{\lambda}$ . An example is shown in Fig. 5; values for different  $x$  have been normalized by the distance  $x$  from the (apparent) origin to the midpoint between two vortices. The mean value  $\bar{\lambda} = 0.31x$  is also close to the most probable value. Similar results were obtained by Winant and Browand in a mixing layer under quite different conditions.

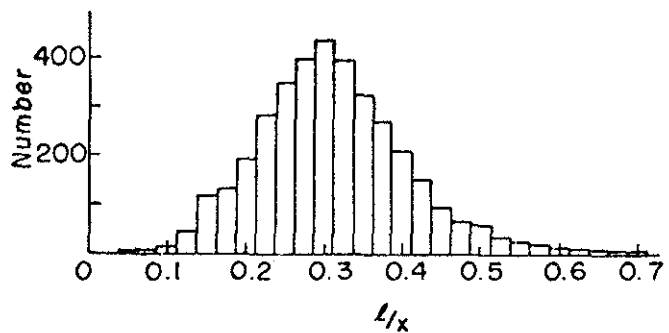


Fig. 5. Distribution of normalized eddy spacings. (Total number of measurements = 3622; total number of eddy pairs  $\cong 100$ .)

From the  $x$ - $t$  diagram it is also possible to determine the lifespan  $L$  of each vortex, i. e. the distance traveled from its motion to its absorption into a new one. At any value of  $x$  there is also a fairly broad distribution of lifespans; this will be discussed presently.

First it will be interesting and useful to digress into a comparison with correlation measurements that had earlier been used to make inferences about the form and lifetime of the large eddies. The simultaneous measurement of the velocity fluctuation  $u(t)$  at two points separated by a distance  $\xi$ , say in the streamwise direction, defines the correlation  $R(\xi) = \overline{u(x)u(x+\xi)}$ , where the bar denotes the average value. By studying such correlations for different directions and different velocity components, Townsend<sup>(2)</sup> and Grant<sup>(3)</sup> were able to make some inferences about organized, large-eddy structure in wakes. But it is very difficult to extract a sharp picture of an eddy from such measurements because the well-correlated portion of the signal resulting from the passage of any coherent structures will be degraded by other contributions to the total turbulent signal. Recently the more sophisticated method of conditional sampling<sup>(17)</sup> has helped to eliminate unwanted parts of the signal. It depends on the possibility of identifying in some way the arrival of the structure to be measured and correlating only while the structure is over the probe. Even with this, the dispersion in scales of the structures passing any point (cf. Fig. 5) will tend to blur the outlines of the structure so defined, unless additional steps are taken to discriminate for size. Variation of what might be called an early form of primitive conditional sampling was the method of space-time correlation,<sup>(18)</sup> in which the time interval as well as the space interval are varied,

$$R(\xi, \tau) = \overline{u(x, t) u(x+\xi, t+\tau)}$$

Such streamwise, time-delayed correlations gave some of the first evidence for the presence of large eddies moving at a convection velocity  $U_c$ , as follows. With  $\xi = 0$ ,  $R(\tau)$  has a (normalized) maximum value of 1 at  $\tau = 0$ , and drops off on either side as shown in Fig. 6. When one of the

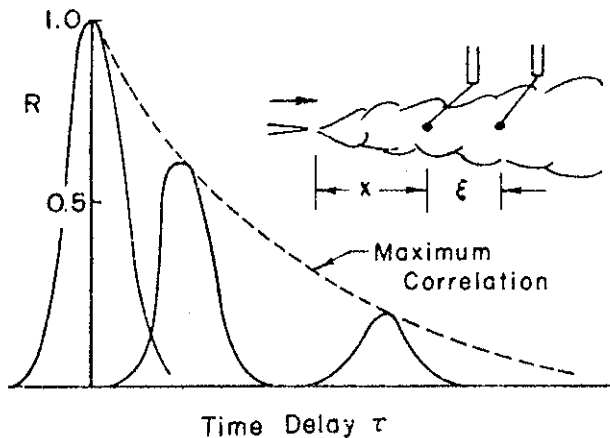


Fig. 6. Typical time correlations for various fixed values of space interval  $\xi$ . Maxima occur at values of  $\tau = \xi/U_c$ .

measuring points is shifted downstream a distance  $\xi$ , the maximum occurs at a later time  $\tau_m$  which defines a convection velocity  $U_c = \xi/\tau_m$ . Measurements of this kind in mixing layers were obtained in Refs. 19, 20 and 12.

The maximum value of the shifted correlation is found to decrease with increasing shift, as shown schematically in Fig. 6, which classically has been interpreted as due to decay of the large eddies; the envelope of the set of shifted correlations, that is, the locus of maximum correlation, thus defined the characteristic decay time. From our new point of view, there is a different interpretation; individual eddies do not decay, but their lifetimes are varied, and the correlation envelope is in fact the probability  $P(\tau)$  for an eddy to survive to an age  $\tau = \xi/U_c$ . Correspondingly, the probability function for its lifespan to exceed a value  $L = U_c \tau$  will be denoted by  $P(L)$ , or  $P(L/\bar{L})$ , where  $\bar{L}$  is the average lifespan. The measured space-time correlation envelopes for mixing layers (12, 19, 20) can be fitted by an exponential function

$$P(L) = \exp(-L/\bar{L}) = \exp(-\lambda/\bar{\lambda})$$

where  $\lambda = L/x = \tau U_c/x$  and  $\bar{\lambda}$  can now be interpreted as the average normalized lifetime (and also the most probable value). Values of  $\bar{\lambda}$  depend on how the signal was filtered before correlation; values from 0.35 to 0.5 fit the measurements of Refs. 12, 19, 20 on incompressible, homogeneous mixing layers with  $U_2 = 0$ . This implies that the average lifespan of an eddy is about 0.4x from its point of origin at x.

To compare these inferences with direct measurements of lifespans from  $x$ - $t$  diagrams we have used data reported in Ref. 10 to plot the survival distribution in Fig. 7. The data can be fitted by a normalized exponential  $P(\lambda) = \exp(-\lambda/0.43)$ , implying an average lifespan  $\bar{L} = 0.43x$ . (In Ref. 9, the average lifespan was found to be 0.39x, but x was measured to the midpoint of each lifespan, rather than to the beginning.)

Although no great significance can be placed on the numerical agreement (sample number was limited,  $U_2/U_1$  and  $\rho_2/\rho_1$  were different from those in the cited experiments), the results strongly suggest that the space-time correlation envelope is really a life-expectancy curve.

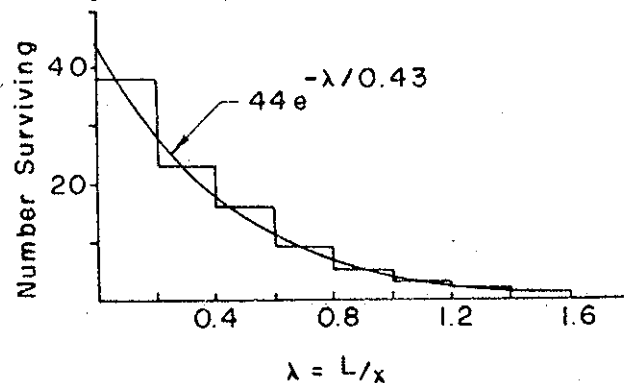


Fig. 7. Survival probability of eddies, from data corresponding to Fig. 4. Sample size = 44.

More quantitative work is needed to show the relation of the old space-time correlation decay curves to the rather different concept of survival probability of coherent structures (which do not decay during their lifetimes; they may even grow!). But it is evident that awareness of the existence of coherent structures can lead to new views of some of the statistical quantities that have been measured. Another such quantity is the energy spectrum of velocity fluctuations. That the presence of coherent vortex structure was not earlier revealed by these spectra is due, in part, to the broad dispersal (Fig. 5) of vortex spacings which produce a correspondingly broad peak in the spectrum; in addition, the velocity energy spectrum receives contributions to small wave numbers from the pairing events which may tend to overlap or submerge the broad peak. That the peaks do exist can be seen in the spectrum measurements in Refs. 12, 13 and 20. For  $U_2 = 0$  the peak is not so well separated from other low-wave-number energy than at higher values of  $U_2/U_1$  (Ref. 13). In trying to understand energy spectra, it has always been difficult to explain the sources of the energy contributions to the low wave numbers, which contain most of the energy. Clearly, important scales are furnished by eddy spacings and eddy lifespans, but even larger scales (lower wave numbers) will be introduced by the vortex coalescence events and the resulting disruptions of order along the shear layer; three-dimensional effects on all these will also play a role.

#### Shear Layer Growth by Vortex Interaction

The concepts of shear layer growth, which can correctly but imprecisely be described as due to turbulent diffusion, are also being changed by the new approach; new insight into the processes at work is obtained once the existence of the coherent structures is realized. The growth of the layer is associated with increase of all mean scales; one of these,  $\bar{l}(x)$  was discussed earlier. But from the  $x$ - $t$  diagram (Fig. 4) it is seen that the spacings between individual pairs of vortices tend to stay constant; a change of scale occurs only at the time of coalescence into larger ones, which increases the spacing between the new ones so formed. Thus the interaction which accomplishes this coalescence must be an important contributor to the growth of the mixing layer.

A detailed description of this interaction process was first given by Winant and Browand<sup>(5)</sup> who described "pairing" as the dominant mode of interaction and the principal mechanism for growth. In pairing, neighboring pairs of vortices rotate around each other and amalgamate into a larger one. Pictures showing various details of the process at rather low Reynolds number are given in Ref. 5. An example from our laboratory at much higher Reynolds number is shown in Fig. 3, where two vortices just to left of center in the top frame can be followed as they rotate around each other and merge into the single vortex which is just over the tip of the probe in the bottom frame. Following this interacting pair, a triplet of vortices is also going through an amalgamation process. What is interesting is that the Reynolds number of the uniform-density flow in Fig. 3 is comparable to the highest values

for turbulent mixing layers that have been investigated in the laboratory. (11, 13) The process of growth by vortex pairing is similar to that in the Winant-Browand experiment at a value of Reynolds number about two orders of magnitude lower.

In the flows with large density difference (Fig. 2) individual vortices can be followed over their lifespans (Fig. 4) but the mode of coalescence into larger ones is not quite clear. Pairing by orbiting is not evident. There are suggestions of other modes of amalgamation, e.g. elongation and accretion onto another one.

#### Entrainment and Mixing

A fundamental property of turbulent shear flow, related to its growth, is the phenomenon of entrainment, that is, the incorporation of nonturbulent, usually irrotational fluid into the turbulent region or, conversely, the diffusion of the turbulent region into the ambient flow. Just how this diffusion occurs and what the entrainment process is has never been quite clear.

On shadowgraphs at high Reynolds number the boundary between turbulent and nonturbulent regions is sharply marked by the presence or absence of fine scales in the structure. Implicit in the picture of a turbulent, nonturbulent interface has been the question of how it propagates into the nonturbulent fluid. An instability of the bounding surface and nibbling on the scale of the smallest eddies have been suggested as possibilities. The phenomenon of intermittency was also seen to be associated with entrainment. While elements of these are all present, a satisfactory picture was not available, largely because the definite, discrete processes associated with the coherent structures were missing.

For a plane mixing layer, at least, it now appears possible to give a more definite and satisfying description of entrainment as an engulfing action of the large coherent eddies; free-stream fluid is drawn in between vortices and ingested into the shear layer where it is digested and made turbulent by the action of the smaller eddies. There is a certain element of semantics in the question of what is turbulent and nonturbulent. Should fluid that finds itself deep within the mixing region but still undigested and irrotational be called nonturbulent? Its role in the exchange of momentum and the extraction of energy from the mean flow has already been accomplished. That aspect of the entrainment process which determines the gross, mean characteristics of the flow, the mean velocity profile, the shear stress distribution, mean transport, dissipation rate, etc., is apparently accomplished by this large scale engulfing action of the large eddies; further "turbulization" by smaller eddies is merely a stage in the dissipation of the energy that has been extracted from the mean flow. Whether there is also entrainment into the vortices by a nibbling process at the boundaries is not clear.

That there are deep incursions of ambient fluid into the mixing region is suggested by the flow pictures. An even sharper indication is provided by a point measurement from a probe which is sensitive to some passive transportable

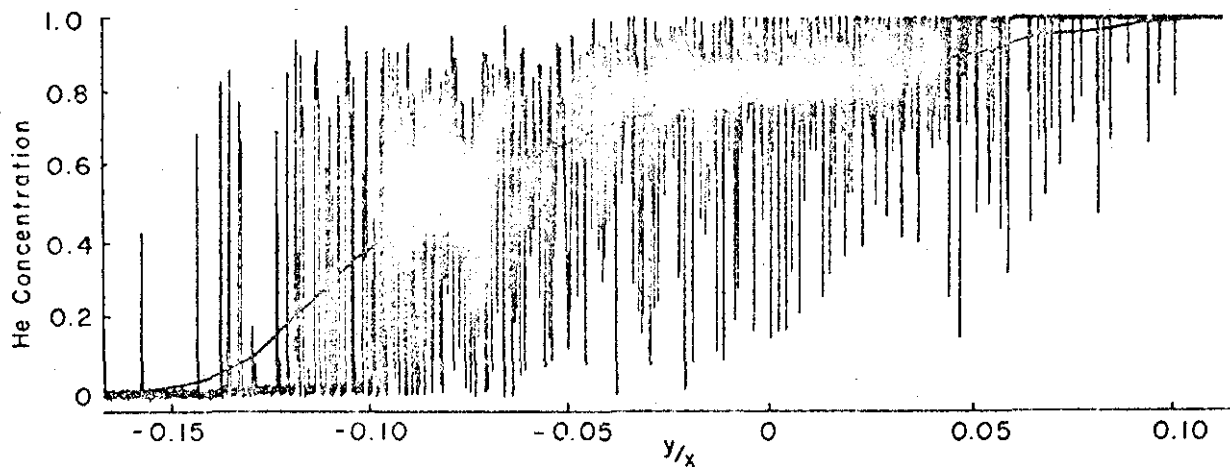


Fig. 8. Computer output of a concentration probe as it traversed a mixing layer between helium and nitrogen at  $x = 10$  cm. During time of traverse, about 150 vortices have passed by.  $\bar{C}(y)$  is shown by a heavy line.

such as temperature and not to velocity. An example of such an output is given in Fig. 8, for which an aspirating probe<sup>(21)</sup> has been traversed across a mixing layer between helium and nitrogen streams, like that in Fig. 2. The probe follows the variations in concentration ( $C = 1$  corresponds to pure helium and  $C = 0$  to pure nitrogen). It may be seen that the excursions of  $C$  are very large, and that unmixed fluid from one side penetrates deep into the other side of the mixing region. Intermittency in the flow is strikingly evident, even more so than in a fluctuating velocity signal. This is because velocity influences are long range; i. e., velocity fluctuations can be induced in the nonturbulent or irrotational parts of the fluid by those in the turbulent parts. Intermittency measuring techniques have been devised to separate the turbulent and nonturbulent parts of a velocity signal, but the discrimination occurs naturally for a passive property, which can be changed only by (short range) intermolecular diffusion.

The question of what happens to portions of fluid that have been entrained into the mixing region from the two sides of the shear layer and how well they are mixed is of interest not only for practical, technical applications but also for further insight into the turbulence structure and mechanisms. Information about this can be obtained by measuring a scalar property such as temperature, density, or species concentration. We can see from Fig. 8 that even in the middle of the mixing layer there are large variations about the local mean value of the concentration of helium in a mixture with nitrogen).

Fig. 9 shows the time variation  $C(t)$  at two points in the layer obtained from a concentration probe steadily sampling the flow at each of those positions. In Fig. 9a the sampling point is on the nitrogen side of the layer at  $y/x = -0.95$ , where  $\bar{C} = 0.45$  (cf. Fig. 8). The passage of vortices over the probe is indicated by the large pulses (but some vortices are missed). Between vortices, where there is free-stream fluid, the signal is steady at  $C = 0$ ; when a vortex passes over the probe,  $C$  increases rapidly to a higher value which has some variation in it

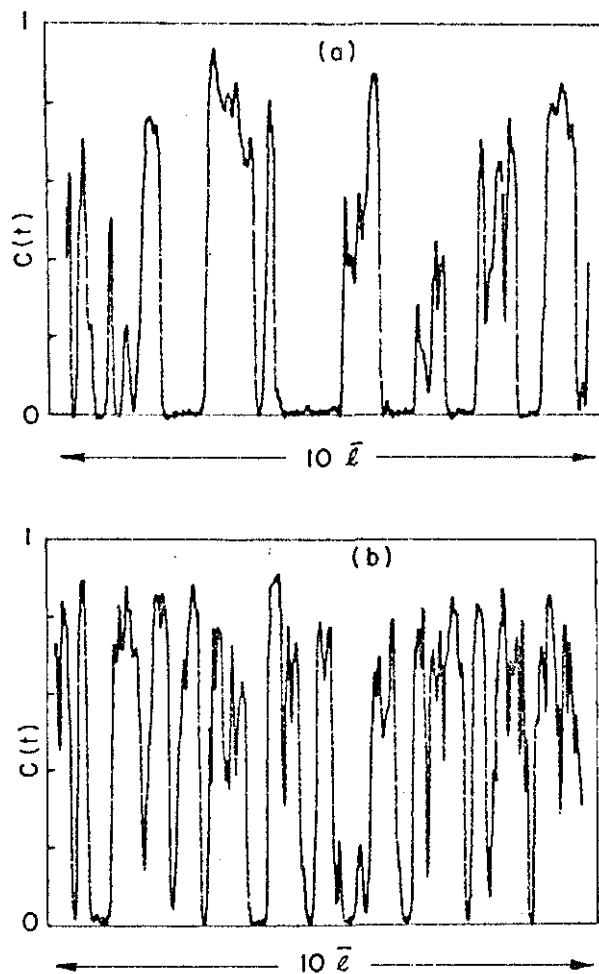


Fig. 9. Concentration fluctuations  $C(t)$ ; conditions as in Fig. 8.

(a)  $y/x = -0.095$ ;  $\bar{C} = 0.45$ ;  $UM = 0.63$

(b)  $y/x = -0.084$ ;  $\bar{C} = 0.55$ ;  $UM = 0.54$

but not of fine scale. A little closer to the middle of the mixing layer at  $y/x = -0.084$  and  $\bar{C} = 0.55$  most of the vortices are being intercepted by the probe. The finer-scale structure embedded in the large pulses corresponds, we believe, to layers which have been accreted during earlier interactions. There is remarkably little fine-scale structure (the probe is capable of resolving much finer scales); but since the structure is the result of only a few pairings, this now does not seem so surprising. By contrast, velocity fluctuation signals contain much finer structure.

Thus the picture of events in the life and interactions of a vortex might be described as follows. In the process of amalgamation, irrotational fluid is ingested and amalgamated with the coalescing vortices. The resulting composite structure consists of the two or more new coalesced vortices (plus all previous ones) and the ingested fluid. During its lifetime this structure rotates and strains. At the same time internal mixing is occurring by the action of the small scale turbulence and viscosity, and the new fluid is digested and incorporated into the structure. There may also be some small scale turbulent diffusion laterally into the free-stream fluid, thus growth of the structure before the next pairing, but evidence for this from the pictures and movies is uncertain.

To define the extent of molecular mixedness  $M$  or unmixedness  $1-M$  at any point, various definitions have been introduced. Usually the r. m. s. fluctuation is used as in the definition given by  $UM = (\overline{C-C})^2 / \overline{C}(1-\overline{C})$ , but this is difficult to measure accurately because the effects of extraneous noise are amplified at small values of  $\overline{C}$  or  $1-\overline{C}$ . An alternative definition that does not have that difficulty is the unmixedness factor

$$UM = \frac{\int_{T_1} (C-\bar{C})dt_1 + \int_{T_2} (\bar{C}-C)dt_2}{(1-\bar{C})T_1 + \bar{C}T_2}$$

where  $t_1$  corresponds to time when  $C > \bar{C}$  and  $t_2$  to time when  $C < \bar{C}$ . With this definition as with the one above,  $UM$  is zero when there is no fluctuation about the mean (i. e., the flow is completely mixed) and has a value of unity when  $C(t)$  fluctuates between values of 0 and 1 (i. e., the flow is completely unmixed). The values of  $UM$  computed in this way are listed for each case in Fig. 9. The complete distribution of  $UM$  across the mixing layer is shown in Fig. 10 for two cases: (a) is for a mixing layer between helium and nitrogen ( $\rho_2/\rho_1 = 7$ ) while (b) is the case  $\rho_2/\rho_1 = 1$  for a mixing layer between nitrogen on one side and helium/argon on the other. It will be noted that for the case of constant density the distribution of  $UM$  is nearly symmetrical while for large density difference it tends to be much better mixed on the low-density side. The peaks of unmixedness are evidently connected with the high intermittency at the edges.

Another quantity of interest is the probability distribution for the concentration at a given point. Examples for the same two cases ( $\rho_2/\rho_1 = 7$  and 1, resp.) are shown in Fig. 11 for three different points across the mixing

layer. Near the sides, due to intermittency, there is a high probability of seeing the pure gas from that side; in the middle of the layer there is a broad distribution about the peak. Again, the constant-density layer tends to show more symmetry than the one with large difference in density.

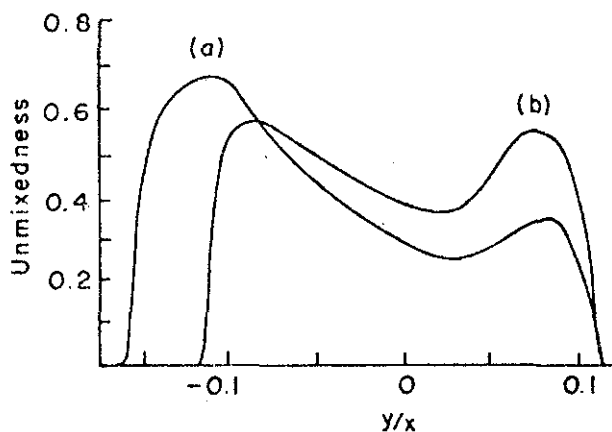
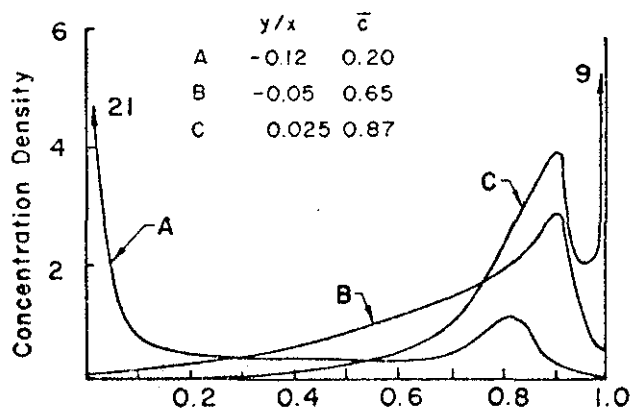
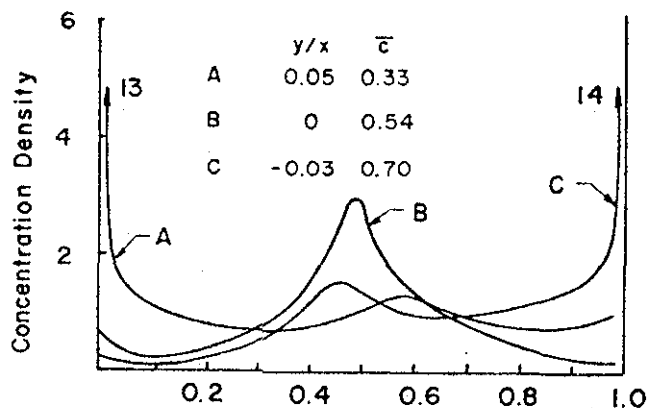


Fig. 10. Distribution of unmixedness factor  $UM$  across mixing layers.

- (a) Corresponds to Fig. 2 with  $U_1 x / \nu_1 = 4 \times 10^4$   
 (b) Corresponds to Fig. 3 with  $U_1 x / \nu_1 = 3 \times 10^5$



(a) Concentration Of He in  $N_2$



(b) Concentration Of Ar-He In  $N_2$

Fig. 11. Probability distributions of concentration at 3 points. For conditions in (a) and (b) see Fig. 10.



With increasing Reynolds number the energy-transforming and energy-dissipating scales in a turbulent flow become separated, as more and more small-scale structure becomes available and intervenes between the process of energy extraction and that of viscous dissipation. In fact, the large-scale processes and the resulting mean flow seem to be very little affected by viscosity down to surprisingly small values of Reynolds number. In the sequence of pictures of a mixing layer in Fig. 2 the unit Reynolds number is varied by a factor of 4 (by changes of pressure and velocity); the Reynolds number based on conditions on the higher velocity side and the length of layer visible in the photographs varies from 0.3 to  $1.2 \times 10^5$ . It may be seen that, while there is considerable difference in the turbulent appearance of the flows connected with the small-scale content, the large-eddy structure is similar and the lateral extent of the mixing regions the same. Measurements of the mean-flow properties<sup>(14)</sup> in these examples do not reveal any significant effects of the Reynolds number. Again, for homogeneous flows, the differences in data collected<sup>(10)</sup> from various sources seem to be due more to different experimental conditions than to Reynolds number effects. The spreading rates measured by Winant and Browand<sup>(5)</sup> for  $U_2/U_1 = 0.4$  at a Reynolds number of  $10^4$  agree well with those of Spencer and Jones<sup>(13)</sup> at a Reynolds number of  $10^6$ . On the other hand the spreading rates measured by Wygnanski and Fiedler<sup>(12)</sup> and by Liepmann and Laufer,<sup>(11)</sup> both at high Reynolds number ( $0.5$  and  $1.0 \times 10^6$  resp.), with  $U_2 = 0$ , differ by about 30%. Ratt<sup>(22)</sup> has shown that this is connected with the presence or absence of a boundary-layer trip ahead of the shear layer separation from the nozzle wall. In fact all evidence suggests that any important effects of Reynolds number appear indirectly through the initial shear layer conditions and not through direct action of viscosity on the developing turbulent structure. The thickness of the initial mixing layer just after separation and the distribution of vorticity in it depend on the particular nozzle geometry, on the thickness of the splitter plate and on the Reynolds number based on some nozzle dimension. Bradshaw showed some time ago<sup>(23)</sup> that at least 1000 initial momentum thicknesses are needed for a mixing layer to forget the initial conditions and acquire a local similarity structure. This distance seems surprisingly large in terms of initial thickness but actually corresponds to only 3 or 4 stages of pairing.<sup>(15)</sup>

Pertinent to this discussion is the role played by the transition region and its relation to the turbulent layer further downstream. When the boundary layer before separation is laminar, the separated, laminar, free shear layer is unstable and a two-dimensional instability with wave length proportional to shear layer thickness develops and rapidly amplifies downstream. It is known that even at this early stage the motion is practically independent of viscous effects; the amplification and subsequent nonlinear development can be calculated as an inviscid process, dependent only on the initial thickness and distribution of vorticity<sup>(23)</sup>. The events in such a transition region are beautifully illustrated in a photograph by Freymuth<sup>(24)</sup> reproduced in Fig. 12, where several stages in

the transition are evident. Most remarkable is the development of the wave into an S-shaped pattern, reminiscent of those in turbulent flow (Figs. 2, 3), which evolves into two vortices that subsequently rotate around each other in the pairing process described by Winant and Browand; this results in a subharmonic wave length of twice the initial spacing. At this early stage the scales are still locked into the initial instability wave length which itself is related to the initial shear layer thickness (this initial instability scale and viscous scale differ by an order of magnitude).

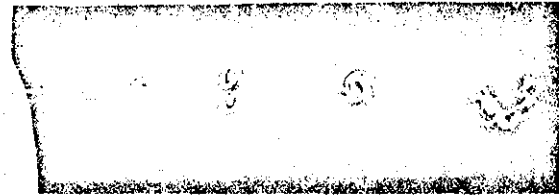


Fig. 12. Smoke picture of vortex structure development and pairing in the transition region. Courtesy of P. Freymuth.<sup>(25)</sup>

If this perfectly ordered development were to continue downstream, the mixing layer would be rather different from the one that is observed. Only integer subharmonics would be present; the energy spectrum of fluctuations would be concentrated in corresponding discrete bands. But even with only discrete scales present, a broadened spectrum and a smooth increase of mean scale along the shear layer would be possible from phase variations in the positions of pairing. Actually, as we have seen, the scales at any position  $x$  are broadly distributed around the mean value appropriate to that position. It is apparent that, after the first pairing, any small irregularities or disturbances of the vortices will alter the subsequent development and the process will increasingly depart from a deterministic sequence tied to the initial conditions. For example, any dislocations that occur in the vortex train will have upstream and downstream influence contributing to the randomization of pairing events and dispersal of scales.

What role three-dimensional disturbances play in the evolution of randomness is not certain. Three-dimensional instabilities will undoubtedly develop; but whether they are necessary and whether a strictly two-dimensional evolution would produce the same mean flow properties as a three-dimensional one is not clear. A strictly two-dimensional nonstationary flow can now be calculated in some detail on a large computer, starting from an initial distribution of vorticity. Such calculations exhibit the features of large vortex structure formation and growth by coalescence;<sup>(26, 27)</sup> it will be interesting to see how closely the mean flow properties of this calculated, two-dimensional mixing region compare with those observed in experiments.

Where transition ends and the fully developed turbulent region begins depends somewhat on the initial conditions. As a general guide, the value

of 1000 initial momentum thicknesses suggested by Bradshaw is adequate. It corresponds to 3 or 4 pairings which seems to be sufficient to forget the initial condition. The 30% discrepancy between the values of Refs. 11 and 12 for spreading rate is difficult to explain; the higher values of Ref. 12 were obtained with a boundary-layer trip and, if this caused a considerable increase in initial momentum thickness, as is plausible, the result may simply mean that the measurements were not made sufficiently far downstream for initial conditions to have been forgotten. But the measurements indicate that self-similarity had been attained. Another explanation, put forward in Ref. 15 postulates a feedback effect from downstream interactions of the large structures (and a very slowly changing growth rate in consequence). That downstream conditions can have an upstream influence is known from the example of shear-layer oscillation in flow over a cavity; elsewhere in this conference, Sarohia<sup>(28)</sup> describes experiments in which growth rate of such a shear layer was observed to change as the position of the downstream corner of the cavity was changed.

Apart from the role of three-dimensional disturbances in the development of randomness in the vortex lifespans, there remains the question of three-dimensionality *per se* in the fully developed turbulent flow. Our discussion has implied a largely two-dimensional picture of the large structures. Certainly the flows cannot be expected to be quite two-dimensional even in the large scales; if spanwise coherence exists even over two or three diameters as in vortex shedding from a circular cylinder, a quasi-two-dimensional picture will have value. What the actual three-dimensional picture is and how it might modify the ideas outlined requires investigation. Some manifestations of three-dimensionality may be seen in photographs obtained by Chandrusuda and Bradshaw,<sup>(28)</sup> in which the view is normal to the plane of the mixing layer. In one of these the pairing is seen to be a helical configuration; in another the new structure resulting from a pairing is seen to have spanwise coherence for nearly three diameters.

#### Coherent Structure in Other Flows

Although for no other flow has the picture emerged so clearly as for the mixing layer, evidence for the existence of organized structures in other turbulent shear flows has also been accumulating, preceding in some cases the discoveries in mixing layers. Of course, examples of organized structure related to particular geometric configurations are familiar. Best known is vortex shedding from cylinders, which occurs up to very high Reynolds number; the organized, periodic motion is superimposed on a background of turbulence or, perhaps more accurately, *vice versa*. Other examples are the organized large scale oscillation near the end of the potential core in a jet; and oscillation in turbulent flow over a cavity. In such cases the large-structure motion is more deterministic, because phases are still locked into an initial condition, than in fully developed turbulent flows.

In the following we shall briefly mention some of the indications of organized structure in fully developed flows and speculate about their meaning.

#### Wakes

The turbulent wake of a cylinder has, near the cylinder, a highly organized, periodic component called vortex shedding, which persists up to the highest Reynolds numbers measured. It was thought that the vortices are obliterated, in some sense, by the turbulence in about 50 diameters downstream of the cylinder.<sup>(30)</sup> Recently, using Prandtl's old technique of flow visualization, it has been found<sup>(31)</sup> that the vortices persist to much greater distances downstream but are more disorganized than closer to the cylinder. Here again is an example of the usefulness of flow visualization; again it becomes necessary to re-interpret previous results in light of the existence and survivability of coherent structures. In the experiments of Ref. 31 the vortices shed from the circular cylinder persisted to the limit of the observations 300 diameters downstream. It will be recalled that Townsend found that 500-1000 diameters are needed for the wake to achieve similarity, i. e. forget its origins. This is about the same distance in terms of momentum thickness, and compares with the 1000 momentum thicknesses needed for a mixing layer.

We must now ask how this disengagement from initial conditions is achieved, whether by coalescence of the vortices into larger structures. Very suggestive are the flow visualization experiments of Taneda<sup>(32)</sup> on the wakes of vortex shedding circular cylinders at Reynolds numbers of about 100, i. e. in the transitional range in which turbulence begins to appear in the periodic wake. Taneda observed that the primary vortex street became disorganized at some distance downstream but later reformed with larger scale; this is, of course, reminiscent of the effects of vortex pairing in the mixing layer and suggests that in the intermediate region some kind of process of amalgamation is occurring.

The discussion so far implies vortices with axes in the plane of the wake more or less parallel to the cylinder, i. e. a two-dimensional structure. Quite a different picture of coherent structure in a wake was deduced by Grant<sup>(3)</sup> from correlation measurements 533 diameters downstream of a circular cylinder at  $R_d = 1300$ . He described it as a pair of vortices, side by side and rotating in opposite directions, with axes approximately normal to the plane of the wake. This seems quite incompatible with the picture of vortices with axes parallel to the cylinder, but Grant did not say how his vortices terminated. Possibly the two views can be reconciled if in fact they are two mutually perpendicular views of a vortex loop, formed by pinching together of vortices from opposite sides of the street. This is a possibility for wakes and jets, whose underlying mean flow is basically two rows of vortices of opposite sign, and so is more complex than the mixing layer, which is one row of vortices of the sign. Such an array of vortex loops interacting with each other would appear rather chaotic, and it might be difficult to devise measurements to isolate the form of their structure and interactions.

## Turbulent Boundary Layer

Townsend<sup>(2)</sup> had inferred, from velocity correlation measurements, the presence in turbulent boundary layers of characteristic eddies of finite length with structure elongated in the flow direction. More evidence for this and new impetus to the search for coherent structure was given by the flow-visualization experiments of Kline and Rundstadler,<sup>(33)</sup> That and succeeding investigations<sup>(34, 35, 36)</sup> have gradually developed a rather complex description of a characteristic, intermittently occurring pattern which is called "bursting". It is a localized, three-dimensional pattern characterized, along with many other manifestations, by sudden and large changes of velocity near the wall. Bursting has been described as the major contributor to turbulence production near the wall; for example, it has been shown<sup>(37)</sup> that large contributions to the Reynolds stress occur during the bursting period. The cycle of events at any location is intermittent, with broad distribution about a mean period  $\bar{T}$ .

From the early observations it was thought that bursting phenomena were confined to a region near the wall, but it was later shown<sup>(38, 39)</sup> that the mean period correlates with overall boundary-layer thickness  $\delta$ , i. e.,

$$\bar{T} \approx 5\delta/U_1$$

Other measurements, for example of pressure fluctuation on the wall,<sup>(40)</sup> also pointed to the presence of large-scale structure convecting with a velocity  $U_c \approx 0.8 U_1$ . These and space-time correlations of fluctuating velocity<sup>(17)</sup> suggested decay times for the structures which correspond to a travel distance of 10-20 $\delta$ . If the interpretation of such correlations which we presented earlier is applicable to the boundary-layer structures (i. e. that the structures do not decay but have a distribution of lifespans), then 10-20 $\delta$  is in fact the lifespan of the longest lived structures.

A picture is gradually emerging<sup>(6)</sup> that the bursting phenomena are connected with the convection of characteristic, three-dimensional structures on the scale of the boundary-layer thickness. Whether a bursting event corresponds to the passage of a structure, or to its interaction with another one, or to contributions from both is not clear. The mean lifespan of 4 to 6 $\delta$  estimated above is consistent with the length scale from the mean bursting period,  $U_c \bar{T} \approx 4\delta$ . On the other hand, the mean streamwise spacing might also be comparable.

A sharp picture of a coherent structure in a turbulent boundary layer has not yet been drawn. To deduce it from measurements of fluctuating velocity or other property, even with the latest, computer-aided techniques is still very difficult. Flow-visualization techniques, which were crucial in initial identification, are less helpful in outlining the complete, three-dimensional structure.

A bold departure has been taken by Coles. Based on the observation that coherent structures were first investigated at low Reynolds numbers, down to the lowest values for which turbulence exists, he has suggested that the basic, coherent

structure is the turbulent spot discovered by Emmons,<sup>(41)</sup> which is found in the transition region of a boundary layer. To test this hypothesis, he and Barker<sup>(42)</sup> created a synthetic turbulent boundary layer by generating an array of spots in an initially laminar boundary layer. The mean velocity profile of the resulting flow fitted quite well the "standard" profile for a turbulent boundary layer with Reynolds number  $Re = 975$ . It remains to be determined whether other properties such as turbulent correlations and bursting phenomena are similar to those in natural turbulent boundary layers.

If, indeed, the turbulent spot is the basic structure in a turbulent boundary layer its form is known from the measurements of Schubauer and Klebanoff<sup>(43)</sup>. A more precise description has been obtained by Coles and Barker by using conditional-sampling techniques on single spots which were produced at a controlled rate. They found that it is a horseshoe-shaped vortex with ends on the wall. (With its mirror image below the wall, it forms a vortex loop which is sharply bent at the plane of the wall, the top and bottom of the loop being inclined downstream.) Whether this or some other form is indeed the coherent structure of a turbulent boundary layer and how it interacts with others are exciting questions to be investigated.

## Conclusion

There is little doubt that coherent structures play a central role in the development of the several turbulent shear flows that have been most extensively investigated, namely mixing layers, boundary layers and the early regions of jets and wakes. For the two-dimensional far wake a partial picture of a coherent structure has been described by Townsend and Grant. It is natural to suppose that organized structure is also present in other turbulent shear flows, in each case having a form that is characteristic for that particular flow. Large structures called puffs and slugs are found in the transition region of pipe flow;<sup>(44)</sup> one wonders whether they provide some indication of a basic, so far hidden structure in fully developed turbulent pipe flow and what the relation may be to bursting in boundary layers. A spiral vortex structure in Couette flow between two concentric cylinders has been studied by Coles.<sup>(45)</sup> Evidence for organized structure in the far region of a turbulent round jet is seen in a photograph by Higuchi<sup>(46)</sup>, reproduced in Fig. 13 on the next page.

From these examples it is evident that experimental investigators interested in turbulent flow have a wide arena. There is also ample opportunity for theoretical workers; computer solutions of nonsteady turbulent flows also can provide valuable insights<sup>(27, 47)</sup>. Understanding of the physical processes actually occurring in turbulent shear flows is indispensable for progress toward an analytical description of turbulent flow. Even before that becomes available, knowledge of these processes is helpful for understanding and coping with practical problems in which turbulent flow is prominent.

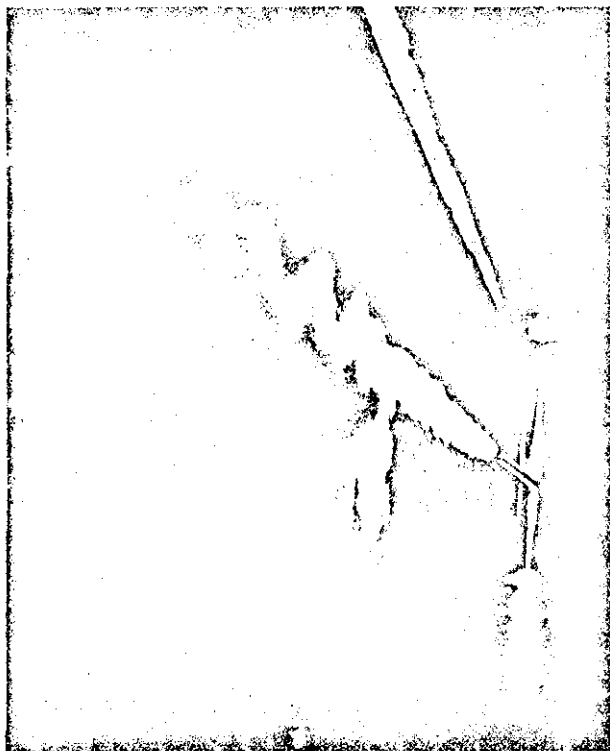


Fig. 13. View upstream in a wind tunnel of a coflowing jet with plane of symmetry illuminated by a vertical sheet of light. Jet diameter  $d = 1$  inch;  $U_j/U_\infty = 1.5$ ;  $U_j d/\nu = 3500$ ;  $x/d = 80$ .

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