Semiclassical Theory of Noise in Semiconductor Lasers—Part II

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Abstract—**A model of semiconductor laser noise is presented which includes the carrier density as a dynamical variable and the carrier density dependence of the refractive index. The Van der Pol laser noise model is shown to be a special case of this treatment. Expressions are calculated for all laser spectra and compared with their Van der Pol counterparts. The power fluctuations spectrum and the frequency fluctuations spectrum exhibit a resonance corresponding to the relaxation resonance and the field spectrum contains fine structure, similar to sidebands which result from harmonic frequency modulation of a carrier signal. The role of carrier noise in determining the field spectrum linewidth is also considered.**

I. Introduction

In "Semiclassical Theory of Noise in Semiconductor Lasers—Part I," [1] we presented a Van der Pol model of laser noise. This analysis showed that an intensity dependent refractive index causes broadening of the field spectrum linewidth beyond that predicted by the modified Schawlow-Townes expression. The essential feature of the Van der Pol model is the analytic form used to relate the polarization to the field. The underlying assumption in such a description is that the field intensity is instantaneously in equilibrium or quasiequilibrium with the carriers so that the gain and refractive index can be treated as instantaneous functions of intensity (i.e., adiabatic elimination of the population variable). This essentially bypasses any role the carriers might have in the overall noise process. A more complete description of semiconductor laser noise requires inclusion of the carrier density as a dynamical variable. In this paper, we accomplish this by assuming the complex susceptibility to be an instantaneous function of carrier density rather than field intensity. By so doing, saturation time constants omitted in the Van der Pol analysis are included, thus accounting for the effects of the relaxation resonance between the carriers and the field intensity. Furthermore, the granular nature of the carriers means that a resultant carrier noise is present. This noise can now take its natural place in the analysis.

To clarify the two frequency regimes referred to throughout this paper, we define the low-frequency regime as the frequency band $|\omega - \omega_m| \leqslant 1/\tau_R$ where τ_R is the relaxation oscillation damping time and ω_m is the lasing frequency, and the high-frequency regime as all frequencies outside this band. The Van der Pol model is valid in the low-frequency regime since the characteristic fluctuation times in this band are longer than τ_R, thereby justifying the equilibrium assumption. The results of Part I can be recovered as low-frequency limits of the results of this paper (although in some cases, new terms appear due to carrier noise contributions). For example, the $1 + \alpha^2$ broadening enhancement of the field spectrum linewidth will be recovered in terms of new quantities.

In the next section, we develop a set of equations governing fluctuations of the field and the carrier density, then use these equations to calculate correlations and spectral densities. We also demonstrate the relationship between the Van der Pol noise equations and this more general set of noise equations. As in Part I, fluctuations will be driven by Langevin forces. These forces are normalized in Section III. In Section IV, the three laser spectra described in Part I will be calculated and compared qualitatively to the corresponding results of Part I. The prediction of new spectral features due to relaxation resonance phenomena distinguishes the results of the current paper from those of Part I. The manifestation of this resonance in the power fluctuations spectrum of semiconductor lasers has been intensely studied, both theoretically and experimentally. Only recently, however, has its presence been observed in the frequency fluctuations spectrum [2] and field spectrum [3] of semiconductor lasers. The present analysis is, to our knowledge, the first theoretical treatment to consider the effect of this resonance on these noise spectra. In the remainder of the paper, we will discuss the effect of carrier noise on the field spectrum.

Finally, we mention that fluctuations induced by mechanisms, such as diffusion (i.e., nonuniform carrier density) and temperature fluctuations, are not treated by this analysis.

II. Noise Equations

In the analysis of Part I, the polarization was represented as a nonlinear function of the field. The nonlinearity arises from gain and refractive index saturation terms in the complex susceptibility. Such a relation is an equilibrium equation of state implying that the carriers are instantaneously in equilibrium with the field intensity. This is only true on a time scale long compared to the relaxation oscillation damping time. The reciprocal of this time is a measure of the frequency band centered on the lasing frequency for which the polarization and the field are in quasi-equilibrium. Outside this frequency range, equilibrium no longer exists and delays intrinsic to the gain saturation process become important. For these frequencies, it must be recognized that the gain and refractive index

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are actually functions of the carrier density, thus making the analytic form for the polarization linear in the field and nonlinear in the carrier density

$$P_n = \epsilon_o \Gamma \chi(n) E_n \tag{1}$$

instead of the form $P_n = \epsilon_o(\chi^{(1)} + \chi^{(3)}|E_n|^2) E_n$ used in Part I. P_n and E_n are the projections of the polarization and the field onto the nth spatial mode and Γ is a filling factor resulting from incomplete spatial overlap of the field and polarization. We will assume that the carrier density is uniform over the active region and the laser oscillates in a single mode throughout this analysis. The essential difference between this representation for the polarization and that used in Part I is that the complex susceptibility is now an instantaneous function of carrier density rather than field intensity. This fundamental change allows gain saturation dynamics to be incorporated into the noise model. It should be noted that inherent in (1) is an adiabatic approximation. By assuming gain and refractive index are specified by the carrier density, we assume, on the time scales of interest, the occupation of states in the conduction and valence bands is accurately described by appropriate quasi-Fermi distribution functions. Since intraband thermalization occurs on a picosecond time scale, whereas noise phenomena of interest will occur on a time scale more than 100X larger than this, the validity conditions of this approximation will always be satisfied.

Since carrier density is a dynamical variable, an equation for the carrier density is required. In the limit discussed above, the carrier density n is described by the standard semiconductor laser rate equation

$$\frac{dn}{dt} = -g(n)\,p - \frac{n}{\tau_s} + E + \vartheta \tag{2}$$

where n is the carrier density, $g(n)$ is gain, p is photon density, τ_s is the spontaneous lifetime, E is the pumping rate, and ϑ is a Langevin noise force associated with the carriers. It is straightforward to show that $g(n)$ is related to the susceptibility $\chi_i(n)$ as follows [4]:

$$g(n) = \frac{\omega_m \chi_i(n)}{\mu^2} \tag{3}$$

where ω_m is the lasing frequency and μ is the nonresonant index. Using (3) and the following expression for the average photon density:

$$p = \frac{\epsilon_o \mu^2}{2\hbar\omega_m} |E_n|^2 \tag{4}$$

the carrier rate equation can be reexpressed as follows:

$$\frac{dn}{dt} = -\frac{\epsilon_o}{2\hbar}\chi_i(n)\,|E_n|^2 - \frac{n}{\tau_s} + E + \vartheta. \tag{5}$$

Aside from the new form for the active medium polarization and the addition of a rate equation for the carrier density, the derivation of the noise equations parallels the derivation given in Part I. The essential steps of the derivation will be repeated. The starting point is the above carrier rate equation and the field equation derived in Part I:

$$\ddot{E}_n + \frac{1}{\tau_p}\dot{E}_n + \omega_n^2 E_n = -\frac{1}{\epsilon_o \mu^2}\ddot{P}_n + \Delta e^{i\omega_m t} \tag{6}$$

where τ_p is the photon lifetime, ω_n is the resonant frequency of the nth mode (i.e., nth solution to the homogeneous wave equation without loss), ω_m is the lasing frequency, and Δ is the slowly varying complex amplitude of the Langevin force term. Substitution of (1) for P_n into (6) and subsequent rearrangement of terms leads to

$$\frac{d^2}{dt^2}\left[\left(1 + \frac{\Gamma\chi(x)}{\mu^2}\right)E_n\right] + \frac{1}{\tau_p}\frac{d}{dt}E_n + \omega_n^2 E_n = \Delta e^{i\omega_m t}. \tag{7}$$

Equations (5) and (7) are linearized by expanding variables in small quantities as follows:

$$\begin{aligned}
E_n &= [A_o + \delta(t)]\, e^{i[\omega_m t + \varphi(t)]}\\
|E_n|^2 &\approx A_o^2 + 2A_o\delta\\
n &\to n_o + n\\
\chi_i(n) &\approx \chi_i(n_o) + \xi_i n\\
\chi_r(n) &\approx \chi_r(n_o) + \xi_r n
\end{aligned} \tag{8}$$

where ξ_i and ξ_r are the first-order Taylor coefficients in expansions of $\chi_i(n)$ and $\chi_r(n)\,[\chi(n) \equiv \chi_r(n) + i\chi_i(n)]$ about the operating point carrier density n_o. These forms are substituted into (5) and (7). Terms in $\ddot{\delta}$, $\ddot{\varphi}$, and $\ddot{n}$ are neglected in (7), as their variation is slow in comparison to the lasing frequency. Products of small quantities are neglected in both (5) and (7). The resulting linearized equations are

$$\begin{aligned}
&2i\omega_m(\dot{\delta} + iA_o\dot{\varphi}) + \frac{2i\omega_m \Gamma\xi A_o}{\mu^2}\dot{n} - \frac{\omega_m^2 \Gamma \xi A_o}{\mu^2} n\\
&\quad + \left(\omega_n^2 - \omega_m^2 + i\frac{\omega_m}{\tau_p} - \frac{\omega_m^2}{\mu^2}\Gamma\chi(n_o)\right) A_o = \Delta e^{-i\varphi}
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\dot{n} + \frac{\epsilon_o \xi_i A_o^2}{2\hbar} n + \frac{\epsilon_o \chi_i(n_o) A_o}{\hbar}\delta + \frac{n}{\tau_s}\\
&\quad + \frac{\epsilon_o \chi_i(n_o) A_o^2}{2\hbar} + \frac{n_o}{\tau_s} - E = \vartheta
\end{aligned} \tag{10}$$

where $1/\omega_m \tau_p$ and $\chi(n_o)/\mu^2$ have been assumed negligible in comparison to unity, and where $\xi \equiv \xi_r + i\xi_i$. As discussed in Part I, the $e^{-i\varphi}$ coefficient of Δ in (9) can be neglected. The perturbations to field amplitude, field phase, and carrier density, as well as the Langevin forces are assumed to have zero mean values. Consequently, time averaging (9) and (10) results in the following set of equations which establish the operating point:

$$\chi_i(n_o) = \frac{\mu^2}{\Gamma\omega_m \tau_p} \tag{11}$$

$$\omega_m^2 = \frac{\omega_n^2}{1 + \dfrac{\Gamma\chi_r(n_o)}{\mu^2}} \tag{12}$$

$$p_o = \Gamma\tau_p\left(E - \frac{n_o}{\tau_s}\right). \tag{13}$$

In principle, these equations can be solved to determine the operating point carrier density (n_o), the lasing frequency (ω_m), and the photon density (p_o). The residual fluctuation terms in (9) and (10) make up the equations which relate the Langevin noise terms to the field and carrier fluctuations. Extracting these terms and separating (9) into its real and imaginary parts yields

$$\dot{\rho} + \frac{\Gamma\xi_r}{\mu^2}\dot{n} - \frac{\omega_m\Gamma\xi_i}{2\mu^2}n = \frac{\Delta_i}{2\omega_m A_o} \tag{14}$$

$$\dot{\varphi} + \frac{\Gamma\xi_i}{\mu^2}\dot{n} + \frac{\omega_m\Gamma\xi_r}{2\mu^2}n = -\frac{\Delta_r}{2\omega_m A_o} \tag{15}$$

$$\dot{n} + \frac{1}{\tau_R}n + \frac{2\mu^2\omega_R^2}{\omega_m\Gamma\xi_i}\rho = \vartheta \tag{16}$$

where, for convenience, the following definitions have been made:

$$\frac{1}{\tau_R} \equiv \frac{1}{\tau_s} + \frac{\epsilon_o A_o^2\xi_i}{2\hbar} = \frac{1}{\tau_s} + g'(n_o)p_o \tag{17}$$

$$\omega_R^2 \equiv \frac{\epsilon_o A_o^2\omega_m\Gamma\xi_i\chi_i(n_o)}{2\hbar\mu^2} = \Gamma g(n_o)g'(n_o)p_o \tag{18}$$

$$\rho \equiv \frac{\delta}{A_o}. \tag{19}$$

The equalities in (17) and (18) follow from (3) and (4) and give more familiar forms of ω_R and τ_R: the relaxation resonant frequency and the damping time associated with it [5].

Recall from Part I that gain saturation manifests itself near the lasing frequency as a "viscous drag" force acting only on the amplitude fluctuation. This restraining action causes the amplitude fluctuation contribution to field linewidth to be negligible in comparison to that of the phase fluctuation. The noise equations which resulted from the Van der Pol treatment in Part I are

$$\left(1 + \frac{2A_o^2\chi_r^{(3)}}{\mu^2}\right)\dot{\rho} - \frac{A_o^2\omega_m\chi_i^{(3)}}{\mu^2}\rho = \frac{\Delta_i}{2\omega_m A_o} \tag{20}$$

$$\dot{\varphi} + \frac{2\chi_i^{(3)}A_o^2}{\mu^2}\dot{\rho} + \frac{\omega_m\chi_r^{(3)}A_o^2}{\mu^2}\rho = -\frac{\Delta_r}{2\omega_m A_o}. \tag{21}$$

The viscous drag force is the term proportional to ρ in (20). It can be seen that (21) has no drag force terms (i.e., no terms which are proportional to φ). The counterpart of (20) in this analysis, (14), appears to contain no viscous drag terms. Closer inspection of both (14) and (16), however, reveals that the instantaneous negative feedback, provided in (20) by the term involving ρ, is replaced by negative feedback which propagates first through the carrier rate equation before providing restraining action in (14). There is thus a phase delay in the saturation process due to the term $\dot{n}$ in (16). For low-frequency fluctuations, the $\dot{n}$ term can be neglected and (16) reduces to an equilibrium relation between the carrier density fluctuation and the amplitude fluctuation causing it:

$$n = -\frac{2\mu^2\omega_R^2\tau_R}{\omega_m\Gamma\xi_i}\rho \tag{22}$$

where ϑ has been omitted. Using (22) to substitute for n and $\dot{n}$ in (14) and (15) yields

$$\left(1 - \frac{2\omega_R^2\tau_R\xi_r}{\omega_m\xi_i}\right)\dot{\rho} + \omega_R^2\tau_R\rho = \frac{\Delta_i}{2\omega_m A_o} \tag{23}$$

$$\dot{\varphi} - \frac{2\omega_R^2\tau_R}{\omega_m}\dot{\rho} - \frac{\omega_R^2\tau_R\xi_r}{\xi_i}\rho = -\frac{\Delta_r}{2\omega_m A_o} \tag{24}$$

which are exactly the noise equations of Part I cast in terms of the quantities appearing in the present analysis. Comparing these equations with (20) and (21), the following equivalences between quantities in Part I and quantities in Part II can be made:

$$\alpha = \frac{\chi_r^{(3)}}{\chi_i^{(3)}} = \frac{\xi_r}{\xi_i} \tag{25}$$

$$\chi_i^{(3)} = -\frac{\omega_R^2\mu^2\tau_R}{A_o^2\omega_m} = -\frac{\tau_R\epsilon_o\mu^4\Gamma g'(n_o)g(n_o)}{2\hbar\omega_m^2} \tag{26}$$

$$\chi_r^{(3)} = -\frac{\omega_R^2\mu^2\tau_R\xi_r}{A_o^2\omega_m\xi_i} = -\frac{\alpha\tau_R\epsilon_o\mu^4\Gamma g'(n_o)g(n_o)}{2\hbar\omega_m^2} \tag{27}$$

where α is the linewidth broadening term discussed in Part I and where the second equalities in (26) and (27) follow from (3) and (4). Equation (25) for α in terms of ξ_r and ξ_i can be recognized (not surprisingly) as (47) of Part I. Notice that since the ratio ξ_r/ξ_i is simply α ($\alpha \approx 1$-10 [6]), the second term in both (14) and (15) can be neglected in comparison to the third term of each equation. Summarizing, the shortcomings of the Van der Pol noise analysis are first, that it treats the gain saturation process as instantaneous [i.e., neglects phase delays brought about by the term $\dot{n}$ in (16)], and second, that it neglects the carrier noise force (i.e., fluctuation in the pumping rate) which is represented by ϑ in (16). A direct consequence of the former assumption is the Van der Pol model's failure to predict the observed spiking resonance in the various noise spectra.

In the remainder of this section, we will use (14), (15), and (16) to calculate the autocorrelations $\langle\rho(t+\tau)\rho(t)\rangle$, $\langle\varphi(t+\tau)\varphi(t)\rangle$, and $\langle n(t+\tau)n(t)\rangle$. The approach used in Part I for similar calculations will be followed here. Equations (14) and (16) are solved first. Laplace transform techniques yield

$$\begin{aligned}\rho(t) = &\frac{1}{2\omega_m A_o}\int_0^t \Delta_i(\tau)\exp\left[-\frac{1}{2\tau_R}(t-\tau)\right]\cos\beta(t-\tau)\,d\tau \\ &+ \frac{1}{\beta}\int_0^t\left(\frac{\Delta_i(\tau)}{4\omega_m A_o\tau_R} + \frac{\omega_m\Gamma\xi_i\vartheta(\tau)}{2\mu^2}\right) \\ &\cdot\exp\left[-\frac{1}{2\tau_R}(t-\tau)\right]\sin\beta(t-\tau)\,d\tau\end{aligned} \tag{28}$$

$$\begin{aligned}n(t) = &\int_0^t \vartheta(\tau)\exp\left[-\frac{1}{2\tau_R}(t-\tau)\right]\cos\beta(t-\tau)\,d\tau \\ &- \frac{1}{\beta}\int_0^t\left(\frac{\omega_R^2\mu^2\Delta_i(\tau)}{\omega_m^2 A_o\Gamma\xi_i} + \frac{\vartheta(\tau)}{2\tau_R}\right) \\ &\cdot\exp\left[-\frac{1}{2\tau_R}(t-\tau)\right]\sin\beta(t-\tau)\,d\tau\end{aligned} \tag{29}$$

$$\beta \equiv \left(\omega_R^2 - \frac{1}{4\tau_R^2}\right)^{1/2} \tag{30}$$

where $\rho(0)=0$ and $n(0)=0$ have been assumed. Equation (29) is substituted into (15) and a single time integration is performed to find $\varphi(t)$. The products $\rho(t_2)\,\rho(t_1)$, $\varphi(t_2)\,\varphi(t_1)$, and $n(t_2)\,n(t_1)$ are formed using these solutions. These products are then ensemble averaged and the integral expressions for them are simplified using the following Langevin force correlation forms:

$$\langle \Delta_i(t+\tau)\,\Delta_i(t)\rangle = \langle \Delta_r(t+\tau)\,\Delta_r(t)\rangle = WD(\tau) \tag{31}$$

$$\langle \Delta_i(t+\tau)\,\Delta_r(t)\rangle = 0 \tag{32}$$

$$\langle \Delta_i(t+\tau)\,\vartheta(t)\rangle = W_1 D(\tau) \tag{33}$$

$$\langle \Delta_r(t+\tau)\,\vartheta(t)\rangle = 0 \tag{34}$$

$$\langle \vartheta(t+\tau)\,\vartheta(t)\rangle = W_2 D(\tau) \tag{35}$$

where $D(\tau)$ is the delta function. In the next section, the normalization coefficients W, W_1, and W_2 will be calculated and the zero correlation assumptions in (32) and (34) will be justified.

The simplified expressions for $\langle\rho(t_2)\,\rho(t_1)\rangle$, $\langle\varphi(t_2)\,\varphi(t_1)\rangle$, and $\langle n(t_2)\,n(t_1)\rangle$ contain exponentially decaying terms in t_2, t_1, and t_2+t_1 with decay time $2\tau_R$. These nonstationary terms result from our specification of initial conditions for the system. For large t_2 and t_1, the system's history is forgotten and these terms become negligible. The resulting expressions are

$$\begin{aligned}\langle\rho(t+\tau)\,\rho(t)\rangle = \Bigg[&\left(\frac{\tau_R W}{8\omega_m^2 A_o^2}\left(1+\frac{1}{\omega_R^2\tau_R^2}\right) + \frac{\Gamma\xi_i W_1}{4\mu^2\omega_R^2 A_o} + \frac{\tau_R\omega_m^2\Gamma^2\xi_i^2 W_2}{8\omega_R^2\mu^2}\right)\cos\beta\tau \\ &+\left(\frac{-W}{16\omega_m^2 A_o^2\beta}\left(1-\frac{1}{\omega_R^2\tau_R^2}\right) + \frac{\Gamma\xi_i W_1}{8\mu^2 A_o\beta\omega_R^2\tau_R} + \frac{\omega_m^2\Gamma^2\xi_i^2 W_2}{16\mu^2\beta\omega_R^2}\right) \\ &\cdot\sin\beta|\tau|\Bigg]\,e^{-(|\tau|/2\tau_R)}\end{aligned} \tag{36}$$

$$\begin{aligned}\langle n(t+\tau)\,n(t)\rangle = \Bigg[&\left(\frac{\tau_R W}{2}\left(\frac{\omega_R\mu^2}{A_o\Gamma\xi_i\omega_m^2}\right)^2 + \frac{\tau_R W_2}{2}\right)\cos\beta\tau \\ &+\left(\frac{W}{4\beta}\left(\frac{\omega_R\mu^2}{A_o\Gamma\xi_i\omega_m^2}\right)^2 - \frac{W_2}{4\beta}\right) \\ &\cdot\sin\beta|\tau|\Bigg]\,e^{-(|\tau|/2\tau_R)}\end{aligned} \tag{37}$$

$$\begin{aligned}\langle\varphi(t_1)\,\varphi(t_2)\rangle = &\frac{W}{4\omega_m^2 A_o^2}(1+\alpha^2)\begin{cases} t_1 & t_1<t_2 \\ t_2 & t_2<t_1\end{cases} \\ &+\left[\frac{\alpha^2\tau_R W}{8\omega_m^2 A_o^2}\left(1-\frac{1}{\omega_R^2\tau_R^2}\right) + \frac{\Gamma^2\xi_i^2\tau_R\alpha^2\omega_m^2 W_2}{8\mu^4\omega_R^2}\right]e^{-(|\tau|/2\tau_R)}\cos\beta\tau \\ &+\left[\frac{\alpha^2 W}{16\omega_m^2\beta A_o^2}\left(3-\frac{1}{\omega_R^2\tau_R^2}\right) + \frac{\Gamma^2\xi_i^2\alpha^2\omega_m^2 W_2}{16\mu^4\omega_R^2\beta}\right]e^{-(|\tau|/2\tau_R)}\sin\beta|\tau|\end{aligned} \tag{38}$$

where $\tau\equiv t_2-t_1$ in (38). As demonstrated in Part I, $\langle\varphi(t_2)\,\varphi(t_1)\rangle$ can be used to calculate the instantaneous frequency deviation autocorrelation function $\langle\Delta\omega(t_2)\,\Delta\omega(t_1)\rangle \equiv \langle\dot\varphi(t_2)\,\dot\varphi(t_1)\rangle$. As was found in Part I, this function is stationary. For later reference, its spectral density and the spectral density of the other stationary functions (i.e., ρ and n) are calculated using the Wiener-Khintchine relation:

$$W_\rho(\Omega) = \frac{\dfrac{W}{4\omega_m^2 A_o^2}\left(\Omega^2+\dfrac{1}{\tau_R^2}\right) + \dfrac{\Gamma\xi_i W_1}{2\mu^2\tau_R A_o} + \dfrac{\omega_m^2\Gamma^2\xi_i^2 W_2}{4\mu^4}}{(\Omega^2-\omega_R^2)^2 + \dfrac{\Omega^2}{\tau_R^2}} \tag{39}$$

$$W_n(\Omega) = \frac{\left(\dfrac{\omega_R^2\mu^2}{A_o\Gamma\xi_i\omega_m^2}\right)^2 W + \Omega^2 W_2}{(\Omega^2-\omega_R^2)^2+\dfrac{\Omega^2}{\tau_R^2}} \tag{40}$$

$$W_{\Delta\omega}(\Omega) = \frac{W}{4\omega_m^2 A_o^2} + \frac{\left(\dfrac{\alpha^2\omega_R^4 W}{4\omega_m^2 A_o^2} + \dfrac{\omega_m^2\Gamma^2\xi_r^2 W_2}{4\mu^4}\right)\Omega^2}{(\Omega^2-\omega_R^2)^2+\dfrac{\Omega^2}{\tau_R^2}}. \tag{41}$$

III. Langevin Force Normalization

In the Van der Pol analysis, a fluctuation term was inserted into the field equation and treated as a random component of polarization causing spontaneous emission into the lasing mode. The Langevin force associated with this fluctuation was normalized by equating the known spontaneous energy in the mode with that calculated in terms of the Langevin force autocorrelation. This treatment emphasized the continuous or wave-like aspects of the Langevin force. In the present analysis, there are three Langevin force terms (counting $\Delta\equiv\Delta_r+\iota\Delta_i$ as two terms), and we will deal with them in a manner that emphasizes their granular or shot-like nature. Most of the results we obtain have been derived elsewhere using similar approaches [7]. To our knowledge, however, the zero correlation result of (34) has not been treated, thus necessitating some explanation.

Consider a bath of electrons interacting with other particle baths so as to establish an equilibrium or mean number of electrons n_o in the bath. If the interaction is assumed to be instantaneous (i.e., short compared to times of interest), then the time derivative of the total particle count is a sequence of delta functions of unit strength (i.e., the Langevin force). If the random variables ρ, φ, and n are treated as continuous, then we argue that interactions can be taken into account by driving the time derivatives of these quantities with delta impulses.

To first order the photon number fluctuation in terms of ρ is

$$p = \frac{\epsilon V A_o^2}{\hbar\omega_m}\rho \tag{42}$$

where V is the mode volume [1]. Using this expression, the Langevin forces Δ_i and ϑ appearing in (14) and (16) are

$$\Delta_i(t) = \frac{2\hbar\omega_m^2}{\epsilon V A_o}\sum_n a_n\delta(t-t_n) \tag{43}$$

$$\vartheta(t) = \frac{1}{V_c} \sum_n b_n \delta(t - t_n) \tag{44}$$

where a_n (b_n) is +1 or −1, depending on the event being an emission or absorption (absorption or emission) and where V_c is the volume occupied by the carriers. Considering events occurring during the time interval $[O, T]$, it is clear from the randomness of the t_n that

$$\int_0^T \vartheta(t+\tau)\,\vartheta(t)\,dt = \frac{R_2 T}{V_c^2}\,\delta(\tau) \tag{45}$$

where R_2 is the average rate of events during time T. Time averaging yields

$$\langle \vartheta(t+\tau)\,\vartheta(t)\rangle = \frac{R_2}{V_c^2}\,\delta(\tau). \tag{46}$$

Similarly,

$$\langle \Delta_i(t+\tau)\,\Delta_i(t)\rangle = \left(\frac{2\hbar\omega_m^2}{\epsilon V A_o}\right)^2 R\delta(\tau) \tag{47}$$

$$\langle \Delta_i(t+\tau)\,\vartheta(t)\rangle = -\frac{2\hbar\omega_m^2}{\epsilon V_c V A_o} R_1 \delta(\tau) \tag{48}$$

where R is the average rate of events which change photon number and R_1 is the average rate of events which change photon number and carrier number simultaneously. Using event rates given in [7], we find

$$W = \frac{4\hbar\omega_m^3 E_{cv}}{\epsilon V} \tag{49}$$

$$W_1 = -\frac{\omega_m A_o}{V_c}(E_{cv} + E_{vc}) \tag{50}$$

$$W_2 = \frac{p_o V}{V_c^2}(E_{cv} + E_{vc}) + \frac{n_o}{V_c \tau_s} \tag{51}$$

where $E_{cv}p_o$ and $E_{vc}p_o$ are the stimulated emission and stimulated absorption rates per unit volume. Rate terms due to spontaneous emission into the lasing mode have been omitted from (49) and (50) as their relative contribution is small for operating points reasonably well above threshold. It should be noted that W found by this method is identical to the W derived in Part I.

The Langevin force Δ_r drives the phase fluctuation and must be modeled somewhat differently than Δ_i. Since φ's amplitude is continuous, Δ_r is given by

$$\Delta_r(t) = 2\omega_m A_o \sum_n \Delta\varphi(t_n)\,\delta(t - t_n) \tag{52}$$

where $\Delta\varphi(t_n)$ is the phase change caused by the nth event. Using the model discussed by Henry [6] in which the field phasor is buffeted about by small unit vectors representing spontaneous events, the mean square $\Delta\varphi$ after time T is

$$\sum_{n=1}^{E_{cv}T} \Delta\varphi_n^2 = \frac{E_{cv}T}{2p_o V}. \tag{53}$$

This expression, (52), and (4) yield

$$\langle \Delta_r(t+\tau)\,\Delta_r(t)\rangle = \frac{4\hbar\omega_m^3 E_{cv}}{\epsilon V}\,\delta(\tau) = W\delta(\tau). \tag{54}$$

Finally, consider the correlations $\langle \Delta_i(t+\tau)\,\Delta_r(t)\rangle$ and $\langle \Delta_r(t+\tau)\,\vartheta(t)\rangle$. Only spontaneous events alter phase, therefore $a_n = +1$ and $b_n = -1$ in (43) and (44). $\Delta\varphi(t_n)$ in (52) is equally likely to be positive or negative, however. Therefore, the events are correlated in time, but have totally uncorrelated amplitudes. The net effect must be zero correlation, as indicated in (32) and (34).

IV. Discussion

In this section, we will first compare the laser spectra calculated in Part I with the laser spectra of the present analysis. We will then consider the role of carrier fluctuations in determining the field spectrum. This has relevance to the recent measurements by Welford and Mooradian of a power-independent component of linewidth in the field spectrum of semiconductor lasers [8]. For convenience, the laser spectra [power fluctuations spectrum—$W_{\Delta P}(\Omega)$, frequency fluctuations spectrum—$W_{\Delta\omega}(\Omega)$, and field spectrum—$W_\epsilon(\Omega)$] derived using the Van der Pol analysis appear below:

$$W_{\Delta P}(\Omega) = \frac{\hbar\omega_m \epsilon V A_o^2 \gamma^2 E_{cv}}{\Omega^2 + \left(\dfrac{\chi_i^{(3)} A_o^2}{\mu^2}\,\omega_m\right)^2} \tag{55}$$

$$W_{\Delta\omega}(\Omega) = \frac{\hbar\omega_m E_{cv} v_g \ln(1/R)}{4P_o L}(1+\alpha^2) \tag{56}$$

$$W_\epsilon(\omega) = \frac{A_o^2 \Delta\omega}{(\omega - \omega_m)^2 + \left(\dfrac{\Delta\omega}{2}\right)^2} \tag{57}$$

$$\Delta\omega \equiv \frac{\hbar\omega_m E_{cv} v_g \ln(1/R)}{4P_o L}(1+\alpha^2). \tag{58}$$

Using (39) and the results of Section III, the power fluctuations spectrum is given by

$$\begin{aligned} W_{\Delta P}(\Omega) &= A_o^4 \epsilon^2 V^2 \gamma^2 W_p(\Omega) \\ &= 4p_o^2\hbar^2\omega_m^2 V^2\gamma^2 \frac{\dfrac{E_{cv}}{2p_o V}\left(\Omega^2 + \dfrac{1}{\tau_R^2}\right) + \dfrac{(E_{cv}+E_{vc})\Gamma g'}{2V_c}\left(\dfrac{\Gamma V g' p_o}{2V_c} - \dfrac{1}{\tau_R}\right) + \dfrac{\Gamma^2 g'^2 n_o}{4\tau_s V_c}}{(\Omega^2 - \omega_R^2)^2 + \dfrac{\Omega^2}{\tau_R^2}} \end{aligned} \tag{59}$$

where (3) and (4) have been used to express A_o^2 and ξ_i in terms of p_o and g', and where γ is the facet loss rate. This result can also be derived using the simple rate equations for pho-

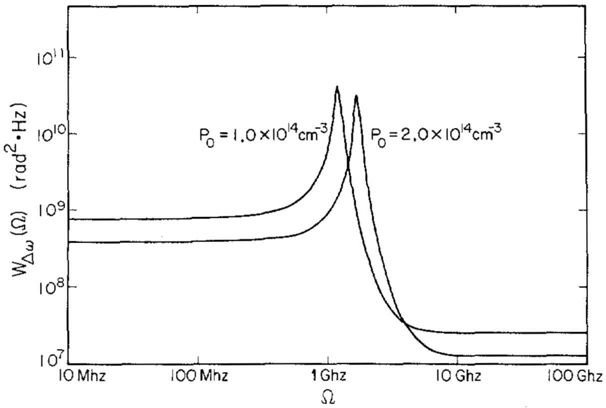

Fig. 1. Frequency fluctuations spectrum $W_{\Delta\omega}(\Omega)$ at two photon densities.

ton density and carrier density [9]. Comparing this expression with that derived in Part I (55), both the low-frequency and the high-frequency behavior is different. Carrier fluctuation causes the dc value of $W_{\Delta P}$ to be larger in the present case, although using equivalence relation (26), it can be shown that the dc spontaneous noise term in (59) is identical to the dc value of (55). The other obvious difference is the resonance behavior of (59) versus the simple pole behavior of (55). This is, as discussed in Section II, a consequence of the dynamic interaction between the carriers and the field.

Consider next the frequency fluctuations spectrum $W_{\Delta\omega}(\Omega)$. Equation (41) and the results of Section III yield

$$W_{\Delta\omega}(\Omega) = \frac{E_{cv}}{2p_o V} + \frac{\dfrac{\alpha^2 \omega_R^4 E_{cv}}{2p_o V}}{(\Omega^2 - \omega_R^2)^2 + \dfrac{\Omega^2}{\tau_R^2}} + \frac{\dfrac{\alpha^2 \Gamma^2 g'^2}{4V_c}\left(\dfrac{p_o V}{V_c}(E_{cv} + E_{vc}) + \dfrac{n_o}{\tau_s}\right)\Omega^2}{(\Omega^2 - \omega_R^2)^2 + \dfrac{\Omega^2}{\tau_R^2}}. \quad (60)$$

Again, the high-frequency behavior, exhibiting the distinguishing resonance, differs from the Van der Pol result (56). Surprisingly, however, the low-frequency value of (60) is identical to that of (56), which means that in the present model, carrier noise due to pump fluctuations ϑ makes no contribution to the dc frequency fluctuation (i.e., frequencies at or near the lasing frequency). This can be understood to result from nearly perfect carrier density clamping (i.e., gain clamping) at low frequencies. $W_{\Delta\omega}(\Omega)$ is shown at two photon densities in Fig. 1. The following laser parameters have been assumed: $\alpha^2 = 30$, $g' = 10^{-6}$ $\mathrm{cm}^3 \cdot \mathrm{s}^{-1}$, $\tau_s = 1$ ns, $V_c = V = 3 \times 10^{-10}$ cm^3, $g = 0.5 \times 10^{12}$ s^{-1}, $n_o = 10^{18}$ cm^{-3}, $\omega_m = 2.2 \times 10^{15}$ $\mathrm{rad} \cdot \mathrm{s}^{-1}$, $\Gamma = 0.8$, and $E_{cv} + E_{vc} = 2\Gamma g'(1 - \frac{1}{2}\zeta)$. ζ is defined as $E_{cv}/(E_{cv} - E_{vc})$ and is typically 2.6 [6]. The value for E_{cv} was chosen to given a linewidth of 100 MHz at 1 mW output power per facet using (67) (or (41) in Part I). The resonance peaks in Fig. 1 are, perhaps, somewhat exaggerated since the present model does not account for relaxation resonance damping mechanisms such as diffusion. We emphasize that the resonance does not cause the field spectrum linewidth broadening $1 + \alpha^2$ discussed in Part I. This broadening arises from $W_{\Delta\omega}(\Omega)$ contributions near dc where $W_{\Delta\omega}(\Omega)$ is essentially "white" in comparison to the lasing linewidth. Instead, the resonance causes fine structure to appear in the field spectrum, as discussed below.

The field spectrum is calculated as in Part I. We first use $\langle\varphi(t_2)\,\varphi(t_1)\rangle$ to calculate the field autocorrelation and then apply the Wiener-Khintchine relation as the final step. The field autocorrelation is given by

$$\begin{aligned}\langle E(t+\tau)\,E^*(t)\rangle &= A_o^2\,\langle \exp i[\varphi(t+\tau) - \varphi(t)]\,\rangle \\ &= A_o^2\, e^{-(1/2)\langle[\Delta\varphi]^2\rangle} \\ &= A_o^2 \exp\left[-\frac{W}{8\omega_m^2 A_o^2}(1+\alpha^2)\,|\tau| \right. \\ &\qquad \left. + \exp -\frac{|\tau|}{2\tau_R}(\kappa \cos\beta\tau + \lambda \sin\beta|\tau|) - \kappa\right]\end{aligned} \quad (61)$$

where

$$\kappa \equiv \frac{\alpha^2 \tau_R W}{8\omega_m^2 A_o^2}\left(1 - \frac{1}{\omega_R^2 \tau_R^2}\right) + \frac{\Gamma^2 \xi_i^2 \tau_R \alpha^2 \omega_m^2 W_2}{8\mu^4 \omega_R^2} \quad (62)$$

$$\lambda \equiv \frac{\alpha^2 W}{16\omega_m^2 \beta A_o^2}\left(3 - \frac{1}{\omega_R^2 \tau_R^2}\right) + \frac{\Gamma^2 \xi_i^2 \alpha^2 \omega_m^2 W_2}{16\mu^4 \omega_R^2 \beta}. \quad (63)$$

Equation (61) neglects amplitude fluctuations of the field. In Part I, this approximation was justified since concern was focused on frequencies near the lasing frequency where gain saturation is active. The proceeding calculation, however, will encompass frequencies at which this approximation is no longer valid. We will discuss inaccuracies that arise at these frequencies due to the approximation. The basic qualitative features of the field spectrum are not affected by it, however. The field spectrum is the Fourier transform of (61). Unfortunately, (61) is not easily transformed analytically. To demonstrate basic features of the field spectrum, we will proceed on the assumption $2\tau_R > 8\omega_m^2 A_o^2 / W(1 + \alpha^2)$ and $2\tau_R \gg 1/\beta$. This amounts to assuming a weakly damped relaxation resonance. In this regime, λ can be neglected in comparison to κ and the decaying exponential can be approximated as unity so that (61) reduces to

$$\langle E(t+\tau)\,E^*(t)\rangle = A_o^2 \exp\left[-\frac{W}{8\omega_m^2 A_o^2}(1+\alpha^2)\,|\tau| + \kappa \cos\omega_R \tau - \kappa\right].$$

Using the modified Bessel function generating function, (64) can be expressed as

$$\langle E(t+\tau)\,E^*(t)\rangle = A_o^2 \exp\left[-\frac{W}{8\omega_m^2 A_o^2}(1+\alpha^2)\,|\tau| - \kappa\right] \cdot \left(I_o(\kappa) + 2\sum_{n=1}^{+\infty} I_n(\kappa) \cos n\omega_R \tau\right) \quad (65)$$

where $I_n(\kappa)$ is the modified Bessel function of order n. Fourier transform of (65) yields the field spectrum

$$W_\epsilon(\omega) = A_o^2 \Delta\omega \sum_{n=-\infty}^{+\infty} \frac{e^{-\kappa} I_n(\kappa)}{(\omega - \omega_m - n\omega_R)^2 + \left(\frac{\Delta\omega}{2}\right)^2} \tag{66}$$

$$\Delta\omega = \frac{W}{4\omega_m^2 A_o^2}(1+\alpha^2) = \frac{\hbar\omega_m E_{cv} v_g \ln(1/R)}{4P_o L}(1+\alpha^2) \tag{67}$$

where $\omega \equiv \omega_m + \Omega$, v_g is the group velocity, R is the facet reflectivity, L is the cavity length, and P_o is the output power per facet. Within the weakly damped resonance regime, the field spectrum consists of a series of Lorentzians. The linewidth of each Lorentzian is precisely the field spectrum linewidth calculated in the Van der Pol analysis. The form of $W_\epsilon(\omega)$ is similar to the spectrum produced when a carrier frequency is harmonically frequency modulated. Such modulation scatters energy into an infinite series of sidebands spaced at intervals of the modulation frequency and with the nth sideband amplitude determined by the regular Bessel function J_n. The relaxation resonance has an analagous effect. The ensuing phase modulation scatters energy at the lasing frequency into sidebands with relative amplitudes given by the modified Bessel functions I_n and spaced at intervals of approximately ω_R.

Fig. 2 contains field spectra calculated numerically using the exact expression (61) for the field autocorrelation. Laser parameters used in Fig. 1 were also used here. In Fig. 2(a), (b), and (c), photon densities are $p_o = 0.5$, 1.0, 2.0×10^{14} cm^{-3} which corresponds to approximately 0.5, 1.0, and 2.0 mW output power per facet. The form given in (17) has been used for τ_R. Field spectrum fine structure has been observed by Vahala, Harder, and Yariv [3]. In addition to the characteristics discussed above, the fine structure exhibited assymmetrical amplitudes. This asymmetry is shown to result from amplitude phase coupling at the relaxation resonance, and as such would not appear in the preceding analysis due to the form of (61). There are two ways to include this effect within the present formalism: one method, as shown in [3], involves a small angle approximation of $e^{i\varphi}$ and is only valid in the high-frequency regime and for strongly damped relaxation oscillations; a second method would be to calculate $\langle E(t+\tau)\, E^*(t)\rangle$ using the distribution function $f[\delta(t), \delta(t+\tau), \Delta\varphi(\tau)]$. f could be taken as Gaussian, allowing it to be completely specified by $\langle \delta(t+\tau)\, \delta(t)\rangle$, $\langle \delta(t+\tau)\, \Delta\varphi(\tau)\rangle$, and $\langle (\Delta\varphi(\tau))^2 \rangle$. The calculation would be rather tedious, however.

An interesting feature of semiconductor laser noise, recently measured by Welford and Mooradian [8], is the power-independent component of field spectrum linewidth. Welford and Mooradian attribute this residual linewidth to carrier induced fluctuations of the refractive index. By assuming the mean square electron number fluctuation is equal to the total number of electrons present in the active region, they have calculated an rms frequency deviation in agreement with their linewidth data. The expression they have used for electron number fluctuation describes fluctuations of a nondegenerate system of electrons which is in thermodynamic equilibrium with a bath of electrons of prescribed chemical potential [10].

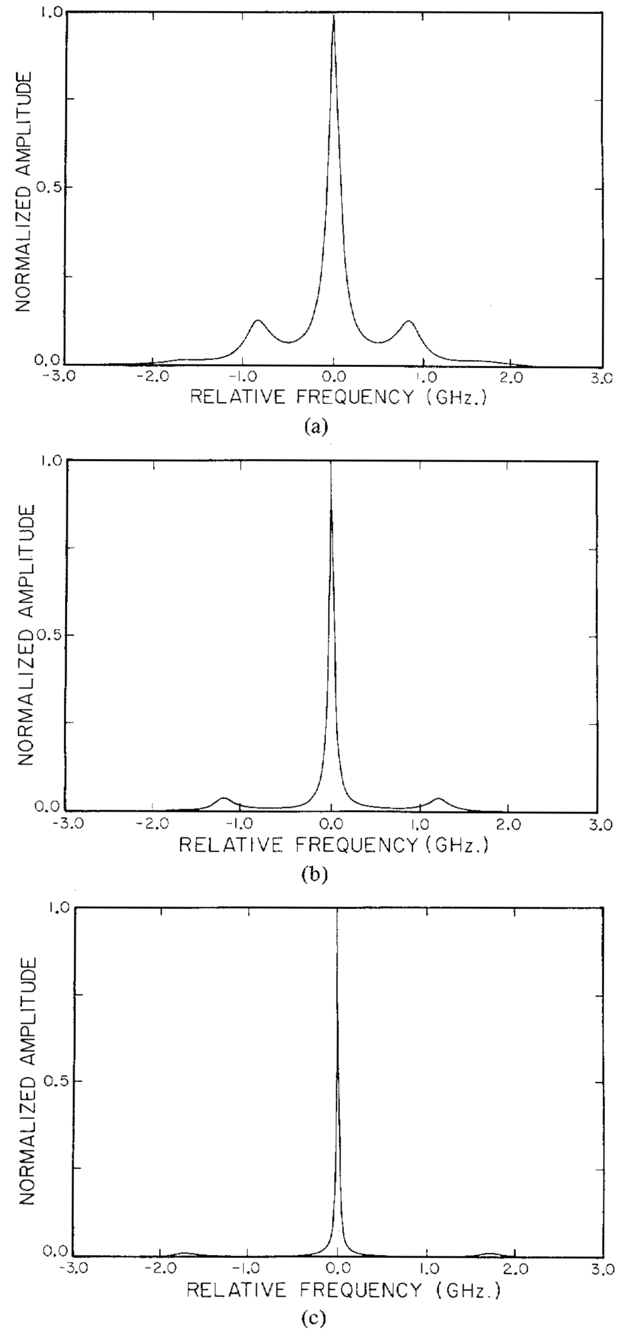

Fig. 2. Normalized field spectrum $W_\epsilon(\omega)$ at output powers of (a) 0.5 mW, (b) 1.0 mW, and (c) 2.0 mW per facet.

As discussed in Section III, whenever a system of particles interacts with other systems via particle fluxes into and out of the system, there are particle number fluctuations associated with the interaction. The system of electrons in the active region of a semiconductor laser interacts with holes and photons through emission and absorption and with electrons outside the active region through thermal particle fluxes. The latter interaction causes the number fluctuation discussed by Welford and Mooradian. The combined interactions, however, determine the total number fluctuation. Corresponding particle rates determine R_2 in (45) and in turn the Langevin force normalization W_2 in (51) [although we have explicitly included only the emission/absorption rates in (51)]. We note, however, that W_2 does not enter into the field spectrum linewidth given by (67) or by more exact calculations we have made using (61). Gain clamping channels the carrier density (or carrier number) fluctuations associated with W_2 into fine structure frequencies. Consequently, the observed power-independent linewidth must result from mechanisms other than carrier number fluctuation.

V. Conclusion

We have presented a semiclassical analysis of semiconductor laser noise which includes the carrier density as a dynamical variable and the carrier density dependence of the refractive index. The Van der Pol treatment given in Part I is a special case of this analysis, and we have demonstrated that for fluctuations which occur near the lasing frequency, the noise equations of the present analysis reduce to the Van der Pol noise equations. The transformation is accomplished at the expense of carrier noise, however, resulting in certain inaccuracies in the Van der Pol results. Low-frequency intensity noise, for instance, is somewhat larger in the present case due to carrier noise contributions.

All laser spectra calculated exhibit resonance behavior at frequencies where time delays in the gain saturation process are important. The Van der Pol model fails to predict this behavior since it assumes that saturation occurs instantaneously. This resonance has been studied both theoretically and experimentally in the power fluctuations spectrum, but only recently has it been observed in the frequency fluctuations spectrum [2] and the field spectrum [3]. This analysis is the first to consider the resonance in these spectra.

Finally, we have considered the role of carrier number fluctuations in the field spectrum. These fluctuations result from interactions of the carriers in the active region with other systems of particles. This analysis does not predict field spectrum linewidth broadening due to carrier number fluctuation in the active region. The dynamics of the gain saturation process cause these fluctuations to contribute only to fine structure components of the field spectrum. We believe two other possible, and as yet unexplored, explanations of the observed broadening are temperature fluctuations of the active region and fluctuations of electronic state occupancy due to intraband scattering. We will investigate these mechanisms in a future publication.

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Kerry Vahala, photograph and biography not available at the time of publication.

Amnon Yariv (S'56–M'59–F'70), for a photograph and biography, see this issue, p. 1101.