Optimal Branch Exchange for Distribution System Reconfiguration

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Abstract—The feeder reconfiguration problem chooses the on/off status of the switches in a distribution network in order to minimize a certain cost such as power loss. It is a mixed integer nonlinear program and hence hard to solve. A popular heuristic search consists of repeated application of branch exchange, where some loads are transferred from one feeder to another feeder while maintaining the radial structure of the network, until no load transfer can further reduce the cost. Optimizing each branch exchange step is itself a mixed integer nonlinear program. In this paper we propose an efficient algorithm for optimizing a branch exchange step. It uses an AC power flow model and is based on the recently developed convex relaxation of optimal power flow. We provide a bound on the gap between the optimal cost and that of our solution. We prove that our algorithm is optimal when the voltage magnitudes are the same at all buses. We illustrate the effectiveness of our algorithm through the simulation of real-world distribution feeders.

Index Terms—Distribution System, Feeder Reconfiguration, Nonconvex Optimization

I. INTRODUCTION

A. Motivation

PRIMARY distribution systems have sectionalizing switches that connect line sections and tie switches that connect two primary feeders, two substation buses, or loop-type laterals. In normal operation these switches are configured such that a distribution network is acyclic and every load is connected to exactly one substation. The topology of the network can be reconfigured by changing the on/off status of these switches, for the purpose of load balancing, loss minimization, or service restoration, e.g., [1]–[4]. See also an survey in [5] for many early papers and references to some recent work in [6].

For instance when a single feeder is overloaded, a currently open tie switch can be closed to connect the feeder to another substation. Since this will create a loop or connect some loads to two substations, a currently closed sectionalizing switch will be opened to maintain a radial topology in which every load is connected to a single substation. Following [3] we call this a “branch exchange” where the goal is to select the pair of switches for closing/opening that achieves the best load balancing. More generally one can optimize a certain objective over the topology of the entire distribution network by choosing the on/off status of all the switches, effectively selecting a best spanning tree among all possible spanning trees of the network topology. Even though the problem of minimum spanning tree has been well studied [7], the problem here is different. Unlike the standard minimum spanning tree problem where the link costs are fixed and the minimization is only over the topology, in our case, the link costs result from an optimal power flow (OPF) problem that must be solved for each candidate spanning tree. This is therefore a mixed integer nonlinear program and can generally be NP-hard. As a result the large majority of proposed solutions are heuristic in nature [5]; see also references in [6]. A heuristic search method is proposed in [2], [3] which we discuss in more detail below. The problem is formulated as a multi-objective mixed integer constrained optimization in [3] and solved using a simulated annealing technique. Ordinal optimization is proposed in [6] to reduce the computational burden through order comparison and goal softening. Unlike these heuristic methods, an interesting exhaustive search method is proposed in [8] to compute a globally optimal solution under the assumption that loads are constant-current, instead of constant-power as often assumed in load flow analysis. Starting from an initial spanning tree, the proposed method applies the branch exchange step in a clever way to generate all spanning trees exactly once and efficiently compute the power loss for each tree recursively in order to find a tree with the minimum loss. A constant-current load model is also used in [9] where the optimization problem becomes a mixed integer linear program. A global optimality condition is derived and an algorithm is provided that attains global optimality under certain conditions. Recently sparse recovery techniques have been applied to this problem in [10], [11] where the network is assumed to be unbalanced and the optimization is formulated in terms of currents. Even though the Kirchhoff current law at each node is linear, nonconvexity arises due to the binary variables that represent the status of the switches and the quadratic (bilinear) relation between the power injection (representing load or generation) and the current injection at each bus. To deal with the former nonconvexity, [10] removes the binary variable and adds a regularization term that encourages group sparsity, i.e., at optimality, either a branch current is zero for all phases (corresponding to opening the line switch) or nonzero for all phases (closing the line switch). The latter nonconvexity is removed by approximating the quadratic relation by a linear relation. The resulting approximate problem is a (convex) second-order cone problem and can be solved efficiently. The formulation in [11] adds a chance constraint that the probability of loss of load is less than a threshold. Chance constraints are generally
intractable and the paper proposes to replace it with scenario-based approximation which is convex.

In this paper we study a single branch exchange step proposed in [2]. Each step transfers some loads from one feeder to another if it reduces the overall cost. An efficient solution for a single branch exchange step is important because, as suggested in [3], a heuristic approach for optimal network reconfiguration consists of repeated application of branch exchanges until no load transfer between two feeders can further decrease the cost. This simple greedy algorithm yields a local optimal. The key challenge is to estimate the cost reduction for each load transfer. Specifically once a currently open tie switch has been selected for closing, the issue is to determine which one of several currently closed sectionalizing switches should be opened that will provide the largest cost reduction. Each candidate sectionalizing switch (together with the given tie switch) transform the existing spanning tree into a new spanning tree. A naive approach will solve an OPF for each of the candidate spanning tree and choose one that has the smallest cost. This may be prohibitive both because the number of candidate spanning trees can be large and because OPF is itself a nonconvex problem and therefore hard to solve. The focus of [2], [3], [12] is to develop much more efficient ways to approximately evaluate the cost reduction by each candidate tree without solving the full power flow problem. The objective of [2] is to minimize loss and it derives a closed-form expression for approximate loss reduction of a candidate tree. This avoids load flow calculation altogether. A new branch flow model for distribution systems is introduced in [2] that allows a recursive computation of cost reduction by a candidate tree. This model is extended to unbalanced systems in [12].

B. Summary

We make two contributions to the solution of branch exchange. First we propose a new algorithm to determine the sectionalizing switch whose opening will yield the largest cost reduction, once a tie switch has been selected for closing. We use the full AC power flow model introduced in [13], [14] for radial system and solve them through the method of convex relaxation developed recently in [15]–[17]. Instead of assuming constant real/reactive power for each load bus as prior works, we consider the scenario where the real/reactive power can also be control variables. Moreover the algorithm requires solving at most three OPF problems regardless of the number of candidate spanning trees. Second we bound the gap between the cost of our algorithm and the optimal cost. We prove that when the voltage magnitude of each bus is the same our algorithm is optimal. We illustrate our algorithm on two Southern California Edition (SCE) distribution feeders, and in both cases, our algorithm has found the optimal branch exchange.

The rest of the paper is organized as follows. We formulate the optimal feeder reconfiguration problem in Section II and propose an algorithm to solve it in Section III. The performance of the algorithm is analyzed in Section IV. The simulation results on SCE distribution circuits are given in Section V. We conclude in Section VI.
define \( x, y \) be its complex impedance and \( S \) complex branch current from buses \( i \). Power flow from buses \( i \)
\[ \begin{align*}
& p_i = -\sum_{(k,i) \in E} (P_{ki} - \ell_{ki} F_{ki}) + \sum_{(i,j) \in E} P_{ij}, \quad i \in \mathcal{N} \\
& q_i = -\sum_{(k,i) \in E} (Q_{ki} - \ell_{ki} 2Q_{ki}) + \sum_{(i,j) \in E} Q_{ij}, \quad i \in \mathcal{N} \\
& v_j = v_i - 2(r_{ij} P_{ij} + x_{ij} Q_{ij}) + \ell_{ij} |z_{ij}|^2, \quad (i, j) \in E \\
& \ell_{ij} v_i = P_{ij}^2 + Q_{ij}^2, \quad (i, j) \in E
\end{align*} \]

Given a vector \( x \) that satisfies (1)-(4), the phase angles of the voltages and currents can be uniquely determined for a radial network, and therefore this relaxed model (1)-(4) is equivalent to the full AC power flow model for a radial network; See section III-A for details.

In addition \( x \) must also satisfy the following operational constraints:

- **Power injection constraints:** for each bus \( i \in \mathcal{N} \)
  \[ p_i \leq p_i \leq P_i \quad \text{and} \quad q_i \leq q_i \leq Q_i \]  

- **Voltage magnitude constraints:** for each bus \( i \in \mathcal{N} \)
  \[ v_i \leq v_i \leq \bar{v}_i \]  

- **Line flow constraints:** for each line \( (i, j) \in E \)
  \[ \ell_{ij} v_i \leq \tilde{\ell}_{ij} \]  

C. Problem formulation

As described in section II-A we focus on reconfiguring a network path where two feeders are served by two different substations as shown in Fig. 2a. Consider a (connected) tree network \( \mathcal{G}(\mathcal{N}, \mathcal{E}) \). \( \mathcal{N} := \{0, 1, \ldots, n, 0'\} \) denote the set of buses, where the two substations are indexed by \( 0 \) and \( 0' \)

Since \( \mathcal{G} \) is a tree there is a unique path between any two buses in \( \mathcal{G} \). For every pair of buses \( i, j \in \mathcal{N} \), let \( \mathcal{E}(i, j) \subseteq \mathcal{E} \) be the collection of edges on the unique path between \( i \) and \( j \). Given any subgraph \( \mathcal{G}' \) of \( \mathcal{G} \) let \( x' := (p', q', P', Q', \ell', v') \) denote the set of variables defined on \( \mathcal{G}' \) with appropriate dimensions. For notational simplicity we often ignore the superscript \( \mathcal{G}' \) and write \( x := (p, q, P, Q, \ell, v) \) instead when the meaning is clear from the context. Given any subgraph \( \mathcal{G}' \) of \( \mathcal{G} \), let \( \mathcal{X}(\mathcal{G}') := \{x' | x' satisfies (1)-(7)\} \) be the feasible set of variables \( x \) defined on \( \mathcal{G}' \). In particular, \( \mathcal{X}(\mathcal{G}) \) is the feasible set for the entire distribution network represented by \( \mathcal{G} \).

Each load bus connects to one substation if and only if one of the switch on a line in path \( \mathcal{E}(0, 0') \) is turned off. Given a (connected) tree \( \mathcal{G}(\mathcal{N}, \mathcal{E}) \) and a path \( \mathcal{E}(0, 0') \) between bus \( 0 \) and \( 0' \), denote by \( \mathcal{G}_0(\mathcal{N}_0, \mathcal{E}_0) \) and \( \mathcal{G}_{0'}(\mathcal{N}_{0'}, \mathcal{E}_{0'}) \) the two
are defined as conditions [15]–[18]. For radial networks SOCP relaxation is exact under some mild exact if every solution \( p_0 + p_0' \) equals the aggregate load (real power consumption) in the network and the total real power loss, if the loads are fixed, then minimizing \( p_0 + p_0' \) also minimizes power loss. For simplicity we will also refer to OFR-branch as OFR in this paper.

A naive solution to OFR is to enumerate all the lines in \( E(0,0') \) and compare the objective value for each case. It is inefficient as it requires solving two optimal power flow (OPF) problems [3] and [2] for each line. This can be computationally expensive if the size of \( E(0,0') \) is large. In the following we will develop an algorithm to solve OFR that involves solving at most three OPF problems regardless of the size of \( E(0,0') \).

The OPF problem is itself a nonconvex problem thus even one OPF problem is hard to solve in general. However, the OPF problems involved in the proposed algorithm can be solved through a convex relaxation. Next, we briefly describe SOCP (second-order cone program) relaxation of OPF recently developed in [15]–[18].

D. OPF and convex relaxation

The optimal power flow problem seeks to optimize a certain objective over the feasible set \( X(G) \) specified by the power flow equations [11]–[14] and the operation constraints [3]–[7]:

\[
\text{OPF-}G \colon \min_{x \in X(G)} \Gamma(p_0, p_0')
\]

It is a non-convex problem due to the quadratic equalities [4]. Relaxing [4] to inequalities:

\[
\ell_{ij}v_i \geq P_{ij}^2 + Q_{ij}^2
\]

leads to a second order cone program (SOCP) relaxation. Formally define \( X_c(G) := \{ x \mid x \text{ satisfies } (1) - (3), (5) - (7), (10) \} \). The SOCP relaxation of OPF-\( G \) is:

\[
\text{SOPF-}G \colon \min_{x \in X_c(G)} \Gamma(p_0, p_0')
\]

SOPF-\( G \) is convex and can be solved efficiently. Clearly SOPF-\( G \) provides a lower bound for OPF-\( G \) since \( X \subseteq X_c \). It is called exact if every solution \( x^* \) of SOPF-\( G \) attains equality in (10). For radial networks SOC relaxation is exact under some mild conditions [15]–[18].

Since the network graph is radial after we join two feeders severed by different substations as shown in Fig. 2a we will assume that the SOCP relaxation of OPF is always exact throughout this paper. In that case we can solve SOPF-\( G \) and recover an optimal solution to the original non-convex OPF-\( G \). A similar approach can be applied to the OPF problems defined in (8) and (9).

III. ALGORITHM OF BRANCH EXCHANGE FOR FEEDER RECONFIGURATION

OFR seeks to minimize \( \Gamma(p_0, p_0') \) by opening the switch on a line in \( E(0,0') \). Let \( k_0, k_0' \in N \) denote the buses such that \( (0, k_0), (k_0', 0') \in E(0,0') \). The algorithm for OFR is given in Algorithm 1.

**Algorithm 1: Branch Exchange Algorithm for OFR**

**Input:** objective \( \Gamma(p_0, p_0') \), network constraints \( \{p, q, \bar{q}, \bar{v}, \bar{v}, \bar{v}, \bar{v} \} \).

**Output:** line \( e^* \).

Solve OPF-\( G \); let \( x^* \) be an optimal solution.

1) \( P_{k_0,k_0}^* \leq 0; \) \( e^* \leftarrow (0, k_0) \).
2) \( P_{k_0',k_0'}^* \geq 0; \) \( e^* \leftarrow (k_0', 0') \).
3) \( \exists (k_1, k_2) \in E(0,0') \) such that \( P_{k_1,k_2}^* \geq 0 \) and \( P_{k_2,k_1}^* \geq 0; \) \( e^* \leftarrow (k_1, k_2) \).
4) \( \exists (k_1, k_2), (k_2, k_3) \in E(0,0') \) such that \( P_{k_2,k_1}^* \leq 0 \) and \( P_{k_2,k_3}^* \leq 0 \). Calculate \( p_0^*, p_0'^*, p_0^* \) and \( p_0'^* \).
   - \( \Gamma(p_0^*, p_0'^*) \geq \Gamma(p_0^*, p_0'^*); \) \( e^* \leftarrow (k_2, k_3) \).
   - \( \Gamma(p_0^*, p_0'^*) \leq \Gamma(p_0^*, p_0'^*); \) \( e^* \leftarrow (k_1, k_2) \).

The basic idea of Algorithm 1 is simple and we illustrate it using the line network in Fig. 4. After we solve OPF-\( G \) with \( x^* \):

1) if bus 0 receives positive real power from bus 1 through line (0, 1), open line (0, 1).
2) if bus 0' receives positive real power from bus n through line (n, 0'), open line (n, 0').
3) if there exists a line \( (k, k+1) \) where positive real power is injected from both ends, open line \( (k, k+1) \).
4) if there exists a bus \( k \) that receives positive real power from both sides, open either line \( (k-1, k) \) or \( (k, k+1) \).

We are interested in the performance of Algorithm 1, specifically:

- Is the solution \( x^* \) to OPF-\( G \) unique and satisfies exactly one of the cases 1) – 4)?
- Is the line \( e^* \) returned by Algorithm 1 optimal for OFR?

We next state our assumptions and answer these two questions under those assumptions.

IV. PERFORMANCE OF ALGORITHM 1

For ease of presentation we only prove the results for a line network as shown in Fig. 4. They generalize in a
straightforward manner to radial networks as shown in Fig. 2a.

Our analysis is divided into two parts. First we show that, OPF-$G$ has a unique solution $x^*$ and it satisfies exactly one of the cases 1) - 4) in Algorithm 1. This means that Algorithm 1 terminates correctly. Then we prove that the performance gap between the solution $e^*$ given by Algorithm 1 and an optimum of OFR is zero when the voltage magnitude of every bus is fixed at the same nominal value, and bound the gap by a small value when the voltage magnitudes are fixed but different.

A. Assumptions

For the line network in Fig. 4, let the buses at the two ends be substation buses and buses in between be load buses. Hence the path between substations $0$ and $0'$ is $E(0,0') = E$; we sometimes use $0'$ and $n + 1$ interchangeably for ease of notation. We collect the assumptions we need as follows:

- **A1**: $\bar{p}_k < 0$ for $1 \leq k \leq n$ and $\bar{p}_k > 0$ for $k = 0, 0'$.
- **A2**: $\theta_k = \psi_k, \bar{q}_k = -\bar{q}_k = \infty$ for $k \in \mathcal{N}$.
- **A3**: $|\theta_i - \theta_j| < \arctan(x_{ij}/r_{ij})$ for $(i, j) \in E$.
- **A4**: The feasible set $X(G)$ is compact. A1 is a key assumption and it says that buses 0 and $0'$ are substation buses that inject positive real power while buses 1, ..., $n$ are load buses that absorb real power. A2 says that the voltage magnitude at each bus is fixed at their nominal value. To achieve this we also require that the reactive power injections are unconstrained. This is a reasonable approximation for our purpose since there are Volt/VAR control mechanisms on distribution networks that maintain voltage magnitudes within a tight range around their nominal values as demand and supply fluctuate [15]. Our simulation results on real SCE feeders show that the algorithm also works well without A2. A3 is a technical assumption that bounds the angle difference between adjacent buses. It, together with A2, guarantees that SOCP relaxation of OPF is exact [18]. A4 is an assumption that is satisfied in practice and guarantees that our optimization problems are feasible.

B. Main results

Algorithm 1 needs to solve up to three OPF problems. The result of [18] implies that we can solve these problems through their SOCP relaxation.

**Theorem 1**: Suppose A2 and A3 hold. Then, for any subgraph $G'$ of $G$ (including $G$ itself),

1) SOPF-$G'$ is exact provided the objective function $\Gamma(p)$ is a convex nondecreasing function of $p$.

2) OPF-$G'$ has a unique solution provided the objective function $\Gamma(p)$ is convex in $p$.

The next result says that Algorithm 1 terminates correctly because any optimal solution to OPF-$G$ will satisfy exactly one of the four cases in Algorithm 1 under assumption A1.

**Theorem 2**: Suppose A1 holds. Given an optimal solution $x^*$ of OPF-$G$, exactly one of the following holds:

1. Although voltage phase angles $\theta_i$ are relaxed in the relaxed branch flow model [15][16], they are uniquely determined by $\theta_i - \theta_j = \angle(v_i - z_{ij}^*S_{ij})$ in a radial network [15].

C1 : $P_{0,1}^* \leq 0$.

C2 : $P_{0,0'}^* \geq 0$.

C3 : $\exists k \in \mathcal{N}$ such that $P_{k,k+1}^* \geq 0$ and $P_{k+1,k}^* \geq 0$.

C4 : $\exists k \in \mathcal{N}$ such that $P_{k,k-1}^* \leq 0$ and $P_{k+1,k}^* \leq 0$.

The intuition behind Theorem 2 is that if more than one of C1-C4 are satisfied, there will be at least one load bus in $(1 \ldots n)$ that injects positive real power, which violates A1. Theorems 1 and 2 guarantee that Algorithm 1 is feasible and terminates correctly under A1-A4. Next, we will study the suboptimality gap of Algorithm 1. Some of the structure properties that will be used to prove the results are relegated to Appendix E.

To obtain the suboptimality bound of Algorithm 1, we need to define an OPF problem as sequel.

**OPF-$G$:** $f(p_0) := \min_{x \in X(G)} p_0'$ s.t. $p_0$ is a constant (11)

Based on Theorem 2 there exists a unique solution $x^*$ for any OPF problems with convex objective function under A2 and A3. Hence there is also a unique solution to OPF-$G$'s $x^*$ for any feasible real power injection $p_0$ at substation 0. In other words, $x^*$ is a function of $p_0$ and let $x(p_0;G) := (p, q, P, Q)$ represents the solution to OPF-$G$'s real power injection $p_0$ at substation 0. We skip $v$ and $\ell$ in $x$ since $v_i$ is fixed by A2 and $\ell_{ij}$ is uniquely determined by $P_{ij}$ and $Q_{ij}$ according to [11]. By Maximum theorem, $x(p_0;G)$ is a continuous function of $p_0$.

Let $I_{p_0} := \{p_0 \mid \exists x \in X(G)\}$ represent the projection of $X(G)$ on real line. $I_{p_0}$ is compact since $X(G)$ is compact by A4. $f(p_0)$ is strictly convex and monotone decreasing by Corollary 1. it is right differentiable and denote its right derivative by $f'_+(p_0)$, which is monotone increasing and right differentiable and denote its right derivative by $f''_+(p_0)$. Let

$$\kappa_f := \inf_{p_0 \in I_{p_0}} f''_+(p_0) \geq 0.$$ (12)

$\kappa_f$ represents the minimal value of the curvature on a compact interval if $f(p_0)$ is twice differentiable.

Before formally state the result, we will first explain the intuition of the suboptimality gap using Fig. 5. Since OPF-$G$ can be written equivalently as

$$\min \Gamma(p_0, f(p_0)),$$

solving OPF-$G$ is equivalent to find a point when the level set of $(p_0, f(p_0))$ first hits the curve $(p_0, f(p_0))$ on a two

![Fig. 5: Graph interpretations for Theorem 3 and 4](image-url)
dimensional plane, where the x-axis and y-axis are the real power injections from substation 0 and 0′, as shown in Fig. 5. On the other hand, OFR can be written as
\[
\min_{(i,j) \in \mathcal{E}(0,0')} \Gamma(p^i_0, p^j_0)
\]
and solving OFR is equivalent to find a point when the level set of \(\Gamma(p_0, p_{0'})\) first hits one point in \(\{(p^i_0, p^j_0) \mid (i,j) \in \mathcal{E}(0,0')\}\) on the two dimensional plane.

When the voltage magnitude of each bus is fixed at the same value, all the \((p^i_0, p^j_0)\) located exactly on the curve \((p_0, f(p_0))\) as shown in Fig. 5a. Thus, we can obtain exactly the optimal solution to OFR by checking the points \((p^i_0, p^j_0)\) adjacent to the optimal solution to OFR. When the voltage magnitude of each bus is different, \((p^i_0, p^j_0)\) does not locate on the curve \((p_0, f(p_0))\) as shown in Fig. 5b. Thus, the points \((p^i_0, p^j_0)\) adjacent to the optimal solution to OFR may not be optimal for OFR. However, we show that the suboptimality gap is related to three aspects: 1) the distance of \((p^i_0, p^j_0)\) to the curve \((p_0, f(p_0))\) (depicted by line loss), 2) the distance between each point in \((p^i_0, p^j_0)\) (depicted by the power injection at each bus) and 3) the convexity of \(f(p_0)\) (depicted by the curvature \(\kappa_f\)). Since the line loss is much smaller than the power injection at load buses, the suboptimality gap is usually very small, as discussed after Theorem 4.

Now, we will formally state our results on the suboptimality bound of Algorithm 1. When the voltage magnitude of all the buses are fixed at the same reference value, e.g. 1 p.u., Algorithm 1 finds an optimal solution to OFR.

**Theorem 3:** Suppose A1–A4 hold. If the voltage magnitudes of all buses are fixed at the same value, then the line \(e^*\) returned by Algorithm 1 is optimal for OFR.

When the voltage magnitudes are fixed but different at different buses, Algorithm 1 is not guaranteed to find a global optimum of OFR. However it still gives an excellent suboptimal solution to OFR. By nearly optimal, it means the suboptimality gap of Algorithm 1 is negligible.

Define \(L_k\) for each line \((k, k+1) \in \mathcal{E}(0,0')\) as sequel.
\[
L_k := \frac{\delta v^2_k}{\ell_{k,k+1}} \left(\frac{z_{k,k+1}}{2}z_{k,k+1}\right) + \sqrt{(v_k + v_{k+1})^2 - \delta v^2_k \left(\frac{z_{k,k+1}}{2}z_{k,k+1}\right) + 1}
\]
where \(\delta v_k := v_k - v_{k+1}\). \(L_k\) represents the thermal loss of line \((k, k+1)\) when either \(P_{k,k+1}\) or \(P_{k+1,k}\) is 0. Conceptually it means all the real power sending from bus on one end of the line is converted to thermal loss and the other bus receives 0 real power, namely either \(P_{k,k+1} = \ell_{k,k+1}r_{k,k+1}\) or \(P_{k+1,k} = \ell_{k,k+1}r_{k+1,k}\). Then the expression of \(L_k\) can be obtained by substituting either \(P_{k,k+1} = \ell_{k,k+1}r_{k,k+1}\) or \(P_{k+1,k} = \ell_{k,k+1}r_{k+1,k}\) into (3) and (4). \(L_k\) is negligible compared to the power consumption of a load in a distribution system. Therefore the ratio of these two quantity, defined as \(R_k := \frac{\delta v_k}{L_k}\), is usually quite large.

Let \(R := \min R_k\) and \(\kappa_f\) as defined in (12), which is a constant depending on the network. Let \(\Gamma^*\) be the optimal objective value of OFR and \(\Gamma_A\) be the objective value if we open the line \(e^*\) given by Algorithm 1.

| TABLE I: The aggregate power injection from substation 1 for each configuration |
|-----------------|-----------------|-----------------|-----------------|-----------------|
|                | (1, 2)          | (2, 4)          | (4, 20)         | (20, 23)        |
| Power injection (MW) | 3.8857          | 3.8844          | 3.8719          | 3.8718          |
| Opened line      | (23, 25)        | (25, 20)        | (26, 32)        | (32, 1)         |
| Power injection (MW) | 3.8719          | 3.8721          | 3.8755          | 3.9550          |

Fig. 6: The aggregated power injection \(f(p_1) + p_1\) as a function of \(p_1\). The red dots are the operating points for each configuration.

**Theorem 4:** Suppose A1–A4 hold and for all \(i \in \mathcal{N}\). Then
\[
\Gamma^* \leq \Gamma_A \leq \Gamma^* + \max \left\{ \frac{c_0^2}{c_0'},\frac{c_0'}{c_0}\right\} \frac{2}{R^2\kappa_f},
\]
if \(\Gamma(p_0, p_{0'}) = c_0p_0 + c_0'p_{0'}\) for some positive \(c_0, c_0'\).

**Remark:** \(R\) is large, usually on the order of \(10^3\), in a distribution system when there is no renewable generation. Although it is difficult to estimate the value of \(\kappa_f\) in theory, our simulation shows that \(\kappa_f\) is typically around 0.025 MW\(^{-1}\) for a feeder with loop size of 10, thus the bound is approximately \(80\) for \(c_0 = c_0' = 1\), which is quite small. Moreover simulations of two SCE distribution circuits show that Algorithm 1 always finds the global optima of OFR problem; see section V. Therefore the bound in the theorem, already negligible, is not always tight.

The suboptimality bound depends on the expression of the objective function \(\Gamma(p_0, p_{0'})\) if it is not linear but strictly convex. According to Theorem 4, we know that a more convex \(f(p_0)\) (larger \(\kappa_f\)) gives a smaller suboptimality gap. In general, a more convex objective function also suggests a smaller suboptimality gap, which can also be interpreted using Fig. 5b.

V. SIMULATION

In this section we present an example to illustrate the effectiveness of Algorithm 1. The simulation is implemented using the CVX optimization toolbox [19] in Matlab. We use a 56-bus SCE distribution feeder whose circuit diagram is shown in Fig. 7. The network data, including line impedances and real power demand of loads, are listed in Table II. Since there is no loop in the original feeder we added a tie line between bus 1 and bus 32, which is assumed to be initially open.

In our simulation the voltage magnitude of the substation (bus 1) is fixed at 1 p.u.. We relax the assumption A2 needed for our analysis that their voltage magnitudes at all other buses...
are fixed and allow them to vary within $[0.97, 1.03]$ p.u., as required in the current distribution system. The demand of real power is fixed for each load and shown in Table I. The reactive power at each bus, which is kept within 10% of the real power to maintain a power factor of at least 90%, is a control variable, as in Volt/VAR control of [15].

We use the aggregate power injection as our objective, namely

$$\Gamma(p_0, p_0^\prime) := p_0 + p_0^\prime.$$  

It also represents the power loss in this case since we have fixed real power demand of each load. Our addition of the line between buses 1 and 32 creates a loop 1-2-4-20-23-25-26-32-1 that must be broken by turning off the switch on one line from $\{(1, 2), (2, 4), (4, 20), (20, 23), (23, 25), (25, 26), (26, 32), (32, 1)\}$. In Table II we list the corresponding aggregate power injection for all the possible configurations. The optimal configuration is to open line (20, 23) at an optimal cost of 3.8718 MW.

After we run Algorithm 1 bus 23 receives real power from both sides and our algorithm returns line (20, 23), which is the optimal solution to OFR.

We have explained the suboptimality gap derived in Theorem 2 using Fig. 5b. We claim that the suboptimality gap is small because the distance of the point $p_1$ from the curve $\{p_0, f(p_0)\}$ is small for real system, which suggests a negligible gap. Now we verify the claim using our simulation results. Let $p_1$ and $f(p_1)$ denote the real power flow on branch (1, 2) and (32, 1) respectively. Instead of plotting $f(p_1)$ versus $p_1$ as Fig. 5b we plot the aggregate power injection $f(p_1)+p_1$ at substation 1 as a function of $p_1$ for $p_1 \in [0, 4]$ in Fig. 6 for better illustration. In Fig. 6 the red dots corresponding to the operational points for each configuration in Table I and the blue curve is $f(p_1)+p_1$. All the red dots are much closer to the curve than their adjacent points, which illustrates why the suboptimality gap is small in real system.

Even though the voltage magnitudes in our simulation are not fixed at the nominal values as assumed in our analysis, Algorithm 1 still gives the optimal solution to OFR by solving a convex relaxation of OPF. The underlying reason is that the voltage magnitude does not vary much between adjacent buses in real network, hence the performance of the algorithm is not limited by the assumption of fixed voltage magnitudes.

We have also tested our algorithm in another SCE 47 bus distribution feeder and it again yields the optimal solution to OFR.

VI. CONCLUSION

We have proposed an efficient algorithm to optimize the branch exchange step in feeder reconfiguration, based on SOCP relaxation of OPF. We have derived a bound on the suboptimality gap and argued that it is very small. We have proved that the algorithm computes an optimal solution when all voltage magnitudes are the same. We have demonstrated the effectiveness of our algorithm through simulations of real-world feeders.

We actually run the topology where the substation is virtually broken into two substations as Fig. 2a.

REFERENCES


APPENDIX A

PROOF OF THEOREM I

Proof of Part 1: SOCP relaxation for bus injection model has been shown in [13] to be exact under A2 and A3. Since the bus injection model and the branch flow model are equivalent [20], this implies that SOCP-\( G \) is exact under A2 and A3.

Proof of Part 2: We will prove it by contradiction. By part 1), SOCP-\( G \) is exact. Hence every solution of SOCP-\( G \) is also feasible for OPF-\( G \). Assume there are two optimal solutions \( \hat{x} \) and \( \overline{x} \)
and $\pi$ such that they achieve the same objective value, namely $\Gamma(\overline{p}) = \Gamma(\hat{p})$. In addition, $v_i \hat{e}_{ij} = \bar{P}^2_{ij} + \bar{Q}^2_{ij}$ and $v_i \bar{e}_{ij} = \bar{P}_{ij} + \bar{Q}_{ij}$ for $(i, j) \in \mathcal{E}$ since the relaxation is exact. Let $x(\lambda) := \lambda \bar{x} + (1 - \lambda) \pi$ for $0 \leq \lambda \leq 1$. Then $x(\lambda)$ is feasible for SOPF-$\mathcal{G}$ since $\mathcal{X}_e(\mathcal{G})$ is convex. Since $\Gamma(\overline{p})$ is convex, 

$$\Gamma(p(\lambda)) \leq \Gamma(\overline{p}) = \Gamma(\hat{p}).$$

Note that $\Gamma(\overline{p})$ and $\Gamma(\hat{p})$ are optimal for OPF-$\mathcal{G}$ by assumption, the only possibility is $\Gamma(p(\lambda)) = \Gamma(\overline{p}) = \Gamma(\hat{p})$, which means $x(\lambda)$ is also an optimal solution. However

$$P^2_{ij}(\lambda) + Q^2_{ij}(\lambda) = (\lambda \bar{P}_{ij} + (1 - \lambda) \bar{P}_{ij})^2 + (\lambda \bar{Q}_{ij} + (1 - \lambda) \bar{Q}_{ij})^2 \leq \chi(P^2_{ij} + Q^2_{ij} + (1 - \lambda)(\bar{P}_{ij}^2 + \bar{Q}_{ij}^2)) = v_i \bar{e}_{ij}(\lambda)$$

The equality is attained for any $0 < \lambda < 1$ if and only if $\bar{P}_{ij} = P^2_{ij}$ and $\bar{Q}_{ij} = Q^2_{ij}$. Since the convex relaxation is exact, $P^2_{ij}(\lambda) + Q^2_{ij}(\lambda) = v_i \bar{e}_{ij}(\lambda)$, which indicates $\bar{e}_{ij} = \bar{e}_{ij}$. Finally, we have $s_i = \pi_i$ by (1) and (2). Therefore $\hat{x} = \pi$, which contradicts that there are more than one optimal solutions.

**APPENDIX B**

**PROOF OF THEOREM 2**

For any solution $x^*$ to OPF-$\mathcal{G}$, define

$$\mathcal{I}_p := \{P^*_{i,1}, -P^*_{i,0}, P^*_{i,2}, -P^*_{i,1}, \ldots, P^*_{i,n}, -P^*_{i,n} \}.$$ 

We will show that the elements in $\mathcal{I}_p$ are in a descending order and at most two consecutive elements are equal. This implies that exactly one of C1–C4 holds.

By power balance across each line $(i, i + 1) \in \mathcal{E}(0, 0')$ we have

$$P^*_{i,i+1} = -P^*_{i,i+1} + \ell^*_{i,i+1} \geq -P^*_{i,i+1}$$

By power balance at each bus $i \in \mathcal{N} \setminus \{0, 0' \}$ we have

$$P^*_{i,i+1} = -P^*_{i,i+1} + p^*_{i} < -P^*_{i,i+1}$$

under assumption A1 because $p^*_{i} \leq p_i < 0$. Hence, $\mathcal{I}_p$ is a nonincreasing sequence and we are left to show that there exists at most two equal element in $\mathcal{I}_p$. We will show it by contradiction. Since $P^*_{i,i+1}$ is strictly less than $-P^*_{i,i-1}$ under A1, the equality can only happen between $P^*_{i,i+1}$ and $-P^*_{i,i+1}$. Suppose there exist two lines $(k_1, k_1 + 1), (k_2, k_2 + 1) \in \mathcal{E}(0, 0')$ such that $P^*_{k_1,k_1+1} = -P^*_{k_1,k_1+1}$ and $P^*_{k_2,k_2+1} = -P^*_{k_2,k_2+1}$. It means $\ell_{k_1,k_1+1} = \ell_{k_2,k_2+1} = 0$, which indicates $P^*_{k_1,k_1+1} = P^*_{k_2,k_2+1} = 0$ by (4). Assume
\( k_1 < k_2 \) without loss of generality and by power balance equation (\([1]\)),
\[
P_{k_1+1,k_1} + P_{k_2,k_2+1} = \sum_{i=k_1+1}^{k_2} p_i^* - \sum_{i=k_1+1}^{k_2-1} \ell_{i,i+1}^* r_{i,i+1} < 0,
\]
which contradicts \( P_{k_1+1,k_1} = P_{k_2,k_2+1} = 0 \). Thus, there are at most two equal elements.

**APPENDIX C**

**PROOF OF THEOREM** (3)

By Lemma (8) \( P_{k,k+1}(p_0; \mathcal{G}) \) is an increasing and continuous function of \( p_0 \), hence there exists a unique \( p_0 \) to \( P_{k,k+1}(p_0; \mathcal{G}) = 0 \) and let it to be \( p_0(k) \). Recall that \( p_0^k \) and \( p_0^{k+1} \) defined in (8) and (9), we will first show \( p_0^k = p_0(k) \) and \( p_0^{k+1} = f(p_0(k)) \) for any \((k,k+1) \in \mathcal{E}(0,0')\) provided the voltage magnitude of each bus is the same. By symmetry, it suffices to show \( p_0^k = p_0(k) \). \( P_{k+1,k}(p_0(k); \mathcal{G}) = 0 \) indicates \( Q_{k,k+1}(p_0(k); \mathcal{G}) = 0 \) according to (3). Hence, \( p_0(k) \) is a feasible power injection for subtree \( \mathcal{G}_0 \) and it means \( p_0^k \leq p_0(k) \). Next, we will show that it is the smallest possible power injection for \( \mathcal{G}_0^k \). Suppose we have \( p_0^k < p_0(k) \), then \((p_0^k, f(p_0(k)))\) is a feasible power injection for network \( \mathcal{G} \) with \( p_0^k < p_0(k) \). It contradicts \((p_0(k), f(p_0(k))) \in \mathcal{O}(\mathcal{P})\). Therefore we have \( p_0^k = p_0(k) \), \( p_0^{k+1} = f(p_0(k)) \) and
\[
(p_0^k, p_0^{k+1}) = (p_0(k), f(p_0(k))) \in \mathcal{O}(\mathcal{P}),
\]
which means the minimal power injection for each partition of graph \( \mathcal{G} \) locates exactly on the Pareto front of the feasible power injection region of \( \mathcal{P} \). Therefore OFR is equivalent to the following problem:
\[
\min_{0 \leq x \leq n} \Gamma(p_0^k, p_0^{k+1}) = \min_{0 \leq x \leq n} \Gamma(p_0(k), f(p_0(k))),
\]
whose minimizer is denoted by \( k^* \). Similarly, \( \mathcal{O}(\mathcal{P}) \)-can be rewritten as
\[
\min_{p_0 \in \mathcal{P}} \Gamma(p_0, f(p_0)),
\]
whose unique minimizer is denoted by \( p_0^* \) and let \( x^* \) be the optimal solution. By Lemma (8) \( P_{k,k+1}(p_0; \mathcal{G}) \) is an increasing function of \( p_0 \). Therefore we have
1. \( P_{0,1}(p_0^*, \mathcal{G}) < 0 \Leftrightarrow k^* = 0 \).
2. \( P_{0,n}(p_0^*, \mathcal{G}) < 0 \Leftrightarrow k^* = n. \)
3. \( \exists k \in [0, n] \) such that \( P_{k,k+1}(p_0^*, \mathcal{G}) \geq 0 \Leftrightarrow k^* = k - 1 \) or \( k \).

The above 3 cases correspond to 1), 2) and 4) in Algorithm 1. Note that 3) would never happen when all the bus voltages are fixed at the same magnitude.

**APPENDIX D**

**PROOF OF THEOREM** (4)

Let \( p_0^k \) be the solution to \( P_{k,k+1}(p_0; \mathcal{G}) = 0 \) and \( p_0^{k+1} \) be the solution to \( P_{k+1,k}(p_0; \mathcal{G}) = 0 \). The uniqueness of \( p_0^k \) and \( p_0^{k+1} \) can be shown in a similar manner as the uniqueness of \( p_0(k) \) in the proof of Theorem (3). When the voltage magnitude of each bus is the same, \( p_0^k = p_0(k) \) and they degenerate to \( p_0(k) \).

**Lemma 1:** Suppose A1-A4 hold. For any \((k,k+1) \in \mathcal{E}(0,0')\), \( p_0^k = p_0^f(k) \) and \( p_0^{k+1} = f(p_0^k(k)) \).

**Proof:** It suffices to show \( p_0^f(k) \) is optimal for \( \mathcal{G}_0^k \) due to symmetry. First, we show \( p_0^k(k) \) is feasible for \( \mathcal{G}_0^k \). Given a solution \( x(p_0^f(k); \mathcal{G}) \), let \( \tilde{x} := (\mathcal{P}, Q, \tilde{p}, \tilde{\rho}, \tilde{q}) \), where
\[
\tilde{s}_{i,i+1} = S_{i,i+1}(p_0^f(k); \mathcal{G}) \quad i < k
\]
\[
\tilde{\rho}_k = p_0(k; \mathcal{G})
\]
\[
\tilde{q}_k = q_0(k; \mathcal{G}) - Q_{k,k+1}(p_0^f(k); \mathcal{G})
\]
Thus, we have \( \tilde{x} \in \mathcal{X}(\mathcal{G}_0^k) \), which means \( p_0^f(k) \) is feasible for \( \mathcal{G}_0^k \). Next, we will show \( p_0^k(k) \) is the minimal power injection for \( \mathcal{G}_0^k \). Suppose \( \tilde{p}_0 < p_0^f(k) \) is feasible for \( \mathcal{G}_0^k \), then we can construct a feasible solution \( \tilde{x} \in \mathcal{X}(\mathcal{G}) \) and the real power injection at node 0 and 0’ are \( \tilde{p}_0 \) and \( f(p_0^f(k)) \), respectively. Therefore it contradicts that \((p_0^k(k), f(p_0(k))) \in \mathcal{O}(\mathcal{P})\). The construction process is as follows:
\[
\tilde{s}_{i,i+1} = \begin{cases} S_{i,i+1}(\tilde{p}_0; \mathcal{G}) & i < k \\ S_{i,i+1}(p_0^f(k); \mathcal{G}) & i \geq k \end{cases}
\]
\[
\tilde{\rho}_i = \begin{cases} p_0(k; \mathcal{G}) + Q_{k,k+1}(p_0^f(k); \mathcal{G}) & i = k \\ s_i(p_0^f(k); \mathcal{G}) & i > k \end{cases}
\]
It can be verified that \( \tilde{x} \in \mathcal{X}(\mathcal{G}) \). Therefore, we show that \( p_0^k(k) \) is the minimal power injection for \( \mathcal{G}_0^k \).

**Lemma 2:** Suppose A1-A4 hold. Then we have
\[
\frac{p_0^f(k) - p_0^k(k)}{p_0^k(k) + 1 - p_0^k(k)} \leq \frac{1}{R_k}.
\]

**Proof:** By Lemma (8) \( P_{k,k+1}(p_0, \mathcal{G}) \) is a concave increasing function with respect to \( p_0 \). Hence \(-P_{k,k+1}(p_0, \mathcal{G}) = \phi(P_{k,k+1}(p_0, \mathcal{G}))\) is also a concave increasing function of \( p_0 \). Recall that \(-P_{k,k+1}(p_0^f(k), \mathcal{G}) = -L_k - P_{k,k+1}(p_0^f(k), \mathcal{G}) = 0 \) and \(-P_{k,k+1}(p_0^k(k), \mathcal{G}) = -P_{k,k+1} \geq -P_{k,k+1} \), we have
\[
0 - (-L_k) \geq \frac{-p_{k+1} - 0}{p_0^k(k) - p_0^f(k)} \frac{p_0^k(k) - 1 - p_0^f(k)}{p_0^k(k) - p_0^f(k)} \leq \frac{1}{R_k}.
\]
by definition of a concave function. Rearrange the above inequality, we obtain
\[
\frac{p_0^k(k) - p_0^f(k)}{p_0^k(k) + 1 - p_0^f(k)} \leq \frac{0 - (-L_k)}{-p_{k+1} - 0} := \frac{1}{R_k}.
\]

**Lemma 3:** Let \( g(x) \) be a strictly convex decreasing function supported on \([a, b] \) and \( \kappa_g := \inf_{x \in (a, b]} g^* + (p_0) \). Define \( G(x) := c_1 g(x) + c_2 x \) \( (c_1, c_2 > 0) \), which is also strictly convex with a unique minimizer \( x^* \) on \([a, b] \). Let \( a \leq y_1 \leq \cdots \leq y_{2n} \leq b \) be a partition on \([a, b] \) such that
\[
\frac{y_{2i} - y_{2i-1}}{y_{2i+1} - y_{2i}} \leq \frac{1}{R} (1 \leq i \leq n - 1)
\]
for some \( R > 0 \). Then there exists a \( 0 \leq k \leq 2n \) such that \( y_k \leq x^* \leq y_{k+1} \), where \( y_0 = a \) and \( y_{2n+1} = b \). Let \( G_i :=
\]
Define
\[
G_A := \begin{cases} 
G_1 & \text{if } k = 0 \\
G_n & \text{if } k = 2n \\
G_{(k-1)/2} & \text{if } k \text{ is odd} \\
\min\{G_{k/2}, G_{k/2+1}\} & \text{if } k \neq 0, 2n \text{ and is even}
\end{cases}
\]
Then
\[
G^* \leq G_A \leq G^* + \max\left\{\frac{c_1^2}{c_2 - c_1}, \frac{2}{R^2k_g}\right\}
\]
Proof: Without loss of generality, assume \(c_1 \leq c_2\) and let \(\lambda := \frac{c_2}{c_1}\). The unique minimizer \(x^*\) of \(G(x) := c_1g(x) + c_2x\) is
\[
x^* = \arg\sup_{x \in [a,b]} \{x \mid G'_+ (x) \leq 0\} = \arg\sup_{x \in [a,b]} \{x \mid g'_+ (x) \leq -\lambda\}.
\]
In addition, let
\[
x_l := \arg\sup_{x \in [a,b]} \left\{x \mid g'_+ (x) \leq -\lambda(1 + \frac{1}{R})\right\}
x_r := \arg\inf_{x \in [a,b]} \left\{x \mid g'_+ (x) \geq -\lambda(1 - \frac{1}{R})\right\}
\]
Then we have \(x_l \leq x^* \leq x_r\) because \(g(x)\) is strictly convex. Let
\[
t_1 := \max\{i \mid y_{2i-1} \leq x^*\} \quad t_2 := \min\{i \mid y_{2i} \geq 2x_l - x^*\}
\]
\[
t_3 := \min\{i \mid y_{2i} \geq x^*\} \quad t_4 := \max\{i \mid y_{2i-1} \leq 2x_r - x^*\}
\]
Next, we will prove the result for different \(k\) as sequel.
Case I: \(k = 2n\). In this case, we have \(d_n = G_n, t_1 = n\). We need to further divide it into two categories.
(1.a) \([y_{2i-1} \leq x_l \text{ or } y_{2i-1} \in [x_l, x^*] \text{ and } t_1 = t_2]\).
For any \(i < t_1\),
\[
\frac{G_i - G_{i+1}}{c_0} = \frac{\lambda g(y_{2i}) + y_{2i-1} - g(y_{2i+2})}{c_0} \geq \frac{\lambda g(y_{2i}) + y_{2i-1} - g(y_{2i+1})}{c_0} \geq \lambda \frac{y_{2i-1} - y_{2i}}{R} G(y_{2i}) - G(y_{2i+1}) \geq \frac{\lambda}{R} (y_{2i-1} - y_{2i}) + G(y_{2i}) - G(y_{2i+1}) \geq 0
\]
The first inequality follows from \(g(x)\) is an increasing function and the second inequality follows from the assumption \([13]\).
For the last inequality, if \(y_{2i-1} \leq x_l\), we have \(G'_+(y_{2i+1}) < -\lambda/R\) for all \(i < n\) and the inequality holds according to mean value theorem. If \(y_{2i-1} \in [x_l, x^*]\) and \(t_1 = t_2 = n\), the inequality holds for \(i < n - 1\) due to similar reason above.
When \(i = n - 1\),
\[
G(y_{2i}) - G(y_{2i+1}) \geq G'_+(x_l)(y_{2i-1} - y_{2i+1}) \geq -\lambda \frac{r}{R} (y_{2i-1} - y_{2i+1})
\]
because \(y_{2n-2} \leq 2x_l - x^*\) by definition of \(t_2\) and \(G(x)\) is convex. \(\{G_i\}\) means the sequence \(\{G_i\}\) is of descending order and \(G^* = G_n = G_A\), thus \(G^* - G_A = 0\).
(1.b) \([y_{2i-1} \in [x_l, x^*] \text{ and } t_2 < t_1]\).
In this case, \(y_{2i+1} - y_{2i} \leq 2(x^* - x_l)\) for \(t_2 < i < t_1\). Denote
\[
\delta_i := y_{2i} - y_{2i-1} \leq \frac{y_{2i+1} - y_{2i}}{R} \leq \frac{2(x^* - x_l)}{R}
\]
for \(t_2 < i < t_1\). Note that the curvature of \(g(x)\) is bounded below by \(\kappa_g\), then \(x^* - x_l \leq \lambda/(R\kappa_g)\). Substitute it into \([15]\), we have for \(t_2 < i < t_1\)
\[
\delta_i \leq \frac{2\lambda}{R^2\kappa_g}.\]
Then for \(t_2 < i < t_1\),
\[
G_i - G_{t_1} = c_1g(y_{2i}) + c_2y_{2i-1} - G_{t_1} \geq -c_2\delta_i + G(y_{2i}) - G(y_{2i-1}) \geq -\frac{2c_2^2}{c_1R^2\kappa_g}
\]
Clearly the first inequality holds. The second inequality follows from \([10]\) and \(G(x)\) is monotone decreasing for \(x \leq x^*\).
For \(i \leq t_2\), \(G_i < G_{t_1}\), can be shown in a similar manner as \(1.a). \) Thus, we have \(G_i - G_n \geq -\frac{2c_2^2}{c_1R^2\kappa_g}\) for any \(i \leq n\), which indicates \(G_A - G^* \leq -\frac{2c_2^2}{c_1R^2\kappa_g}\).
Case II: \(k = 0\). In this case, \(G_A = G_1\) and the bound can be established in a similar manner as Case I.
Case III: \(k\) is odd. In this case, \(G_A = G_{(k-1)/2}\), \(t_1 = t_3 = (k-1)/2\). Similar approach can be applied as Case I and Case II to show
\[
G_i \geq \begin{cases} 
\frac{G(y_{2i-1}) - \frac{2c_2^2}{c_1R^2\kappa_g}}{c_0} & \text{if } i \leq t_1 \\
\frac{G(y_{2i}) - \frac{2c_2^2}{c_1R^2\kappa_g}}{c_0} & \text{if } i \geq t_3
\end{cases}
\]
And \(G_A = G_{(k-1)/2} \leq \max\{G(y_{2i-1}), G(y_{2i})\} \leq G_i + \frac{2c_2^2}{c_1R^2\kappa_g}\) for \(1 \leq i \leq n\).
Case IV: \(k \neq 0\) and \(k\) is even. In this case, \(G_A = \min\{G_{k/2}, G_{k/2+1}\}\) and \(t_1 = k/2, t_3 = k/2 + 1\). Similar approach can be applied as Case I and Case II to show
\[
G_i \geq \begin{cases} 
\frac{G_{t_1} - \frac{2c_2^2}{c_1R^2\kappa_g}}{c_0} & \text{if } i \leq t_1 \\
\frac{G_{t_3} - \frac{2c_2^2}{c_1R^2\kappa_g}}{c_0} & \text{if } i \geq t_3
\end{cases}
\]
and we arrive at our conclusion. 

Consider the sequence \(p_i^0(0) \leq p_i^0(0) \leq \ldots \leq p_i^0(n) \leq p_i^0(n)\) as the partition on \(I_{p_0}\) and \(f(p_0)\) as the function \(g(x)\) in Lemma \(3\) we can prove Theorem \(4\).

APPENDIX E

STRUCTURAL PROPERTIES OF OPF

Given two real vectors \(x, y \in \mathbb{R}^n, x \leq y\) means \(x_i \leq y_i\) for \(1 \leq i \leq n\) and \(x < y\) means \(x_i < y_i\) for at least one component. The Pareto front of a compact set \(A \subseteq \mathbb{R}^n\) is defined as
\[
\mathcal{O}(A) := \{x \in A \mid \exists \tilde{x} \in A \setminus \{x\} \text{ such that } \tilde{x} \leq x\}
\]
Let \(\mathbb{P} := \{(p_0, p^0) \mid \exists x \in \mathcal{X}(G)\}\) represent the projection of \(\mathcal{X}(G)\) on \(\mathbb{R}^2\) and \(\mathbb{P}_c := \{(p_0, p^0) \mid \exists x \in \mathcal{X}_c(G)\}\) be the
projection of $X_c(G)$ on $\mathbb{R}^2$. Based on Theorem 1, the convex relaxation is exact under A2 and A3, it means $O(P) = O(P_c)$ since the objective $\Gamma(p_0, p_v)$ is a convex and nondecreasing function. Next, we begin by studying some properties of $f(p_0)$ defined in (14).

**Lemma 4:** $(p_0, f(p_0)) \in O(P)$ provided the optimization problem OPF-$G$ is feasible.

**Proof:** Suppose $(p_0, \hat{p}_v) \in O(P)$. Let $\Gamma^*(p_0, p_v)$ be the objective function such that $(p_0, \hat{p}_v)$ solves OPF-$G$. Note that OPF-$G$ can be written equivalently as

$$\min_{p_0} \Gamma^*(p_0, p_v) \quad \text{s.t.} \quad p_0 = f(p_0)$$

Therefore, at optimality, $f(p_0) = \hat{p}_v$ and $(p_0, f(p_0)) \in O(P)$. ■

By property of Pareto Front [21], for each point $(p_0, p_v) \in O(P_c) = O(P)$, there exists a convex nondecreasing function $\Gamma^*: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $(p_0, p_v)$ is an optima for OPF-$G$. Therefore OPF-$G$s and OPF-$G$ are equivalent in the sense that if we fix $p_0$ and solve OPF-$G$ with $x^*$, there exists an objective function $\Gamma^*(p_0, p_v)$ such that $x^*$ solves the corresponding OPF-$G$ by Lemma 4.

**Lemma 5:** Let $A$ be a compact and convex set in $\mathbb{R}^2$. Define $g(x) := y$ for any $(x, y) \in O(A)$. Then $y = g(x)$ is a convex decreasing function of $x$ for $(x, y) \in O(A)$.

**Proof:** We first show $g(x)$ is a decreasing function and then show $g(x)$ is also convex.

Let $(x_1, g(x_1))$ and $(x_2, g(x_2))$ be two points in $O(A)$. Without loss of generality, assume $x_1 > x_2$. If $g(x_1) \geq g(x_2)$, it violates the fact that $(x_1, g(x_1)) \in O(A)$ and hence $g(x_1) < g(x_2)$, which means that $g(x)$ is a decreasing function.

Next, we will show $g(*)$ is convex. Recall that $A$ is a compact set, we have $(x_1, g(x_1)), (x_2, g(x_2)) \in O(A) \subseteq A$. $A$ is also a convex set, thus $(2x_1 + x_2, g((x_1 + x_2)/2)) \in A$. By definition of Pareto front, $g(x_1 + x_2/2) = \inf_{(x_1 + x_2)/2, (x_1 + x_2)/2} \{y \leq g(x_1) + g(x_2)/2\}$, which shows $g(x)$ is a convex function. ■

Note that $X_c(G)$ is convex and compact by A4, hence its projection on a two dimensional space $P_c$ is also compact and convex. By Lemma 4, $(p_0, f(p_0))$ characterizes the Pareto front $O(P) = O(P_c)$, then we have the following corollary.

**Corollary 1:** $f(p_0)$ is a strictly convex decreasing function of $p_0$ under assumption A2-A4.

For a line $(i, j)$ between two buses $i$ and $j$ with fixed voltage magnitude, $(P_{ij}, Q_{ij}, \ell_{ij})$ are governed by (3)-(4) and $Q_{ij}, \ell_{ij}$ can be uniquely solved given a $P_{ij}$ if A3 holds. Let $\phi(P_{ij}) := -P_{ij} = P_{ij} - \ell_{ij} r_{ij}$ and we have the following result.

**Lemma 6:** Suppose A2 and A3 hold, $\phi(P_{ij})$ is a concave increasing function of $P_{ij}$ for $(i, j) \in E$.

**Proof:** By (4), we have $\ell_{ij} = (P_{ij}^2 + Q_{ij}^2)/v_i$ and substitute it in $\phi(P_{ij})$, we have

$$\phi(P_{ij}) = P_{ij} - \frac{r_{ij}}{v_i} \left(P_{ij}^2 + Q_{ij}^2\right).$$

The relation between $P_{ij}$ and $Q_{ij}$ is governed by (5). Let $\theta_{ij} := \theta_i - \theta_j$ and then $P_{ij}$ and $Q_{ij}$ can be written as

$$P_{ij} = \frac{v_{ij} r_{ij}}{r_{ij}^2 + x_{ij}^2} + \sqrt{\frac{v_{ij}^2 r_{ij}^2}{r_{ij}^2 + x_{ij}^2} + \sin(\theta_{ij} - \beta_{ij})}$$

$$Q_{ij} = \frac{v_{ij} x_{ij}}{r_{ij}^2 + x_{ij}^2} - \sqrt{\frac{v_{ij}^2 r_{ij}^2}{r_{ij}^2 + x_{ij}^2} + \cos(\theta_{ij} - \beta_{ij})},$$

where $\beta_{ij} := \arctan r_{ij}/x_{ij}$. Substitute them into $\phi(P_{ij})$, we obtain

$$\phi(P_{ij}) = -\frac{v_{ij} r_{ij}}{r_{ij}^2 + x_{ij}^2} + \sqrt{\frac{v_{ij}^2 r_{ij}^2}{r_{ij}^2 + x_{ij}^2} + \sin(\theta_{ij} + \beta_{ij})}.$$  

Take derivative of $\phi(P_{ij})$ with respect to $P_{ij}$, we have

$$\frac{d\phi(P_{ij})}{dP_{ij}} = \frac{\cos(\theta_{ij} + \beta_{ij})}{\cos(\theta_{ij} - \beta_{ij})},$$

which is always positive by assumption A3 that $|\theta_{ij}| < \arctan x_{ij}/r_{ij}$. Furthermore,

$$\frac{d^2\phi(P_{ij})}{dP_{ij}^2} = -\sqrt{\frac{r_{ij}^2 + x_{ij}^2}{v_{ij} r_{ij}}} \sin(2\beta_{ij}),$$

which is always negative by assumption A3 that $|\theta_{ij}| < \arctan x_{ij}/r_{ij}$. Thus, $\phi(P_{ij})$ is a concave increasing function of $P_{ij}$. ■

**Lemma 6** means if the one end of the line increases its real power injection on the line, the other end should receive more real power under assumption A2 and A3.

**Lemma 7:** Suppose A2-A4 hold. Given a solution $x(p_0; G)$ to OPF-$G$s, $P_{k,k+1}(p_0; G)$ is a nondecreasing function of $p_0$ for all $(k, k + 1) \in E(0, 0')$.

**Proof:** The following argument holds without assuming $\phi = \infty$ and $\phi' = -\infty$.

Suppose $P_{k,k+1}(p_0; G)$ is not a nondecreasing function of $p_0$ at $p_0^*$ for a line $(k, k + 1) \in E(0, 0')$, then either C1 or C2 below will hold for arbitrary small $\epsilon > 0$.

C1: $\exists p_0 \in (p_0^* - \epsilon, p_0^* + \epsilon)$ such that $P_{k,k+1}(p_0^*; G) < P_{k,k+1}(p_0^*; G)$.

C2: $\exists p_0 \in (p_0^* - \epsilon, p_0^* + \epsilon)$ such that $P_{k,k+1}(p_0^*; G) > P_{k,k+1}(p_0^*; G)$.

We will show by contradiction that $(p_0^*, f(p_0^*)) \not\in O(P)$ in this case, which violates Lemma 4. Assume without loss of generality that $P_{i,i+1}(p_0^*)$ is a nondecreasing function of $p_0$ for $0 \leq i < k$.

**Case 1:** $q_k(p_0^*; G) > q_k$. Suppose C1 holds, then there exists a monotone decreasing sequence $p_0^{(m)} \downarrow p_0^*$ such that $\{p_0^{(m)}; G\}$, $m \in \mathbb{N}$ is a monotone increasing sequence that converges to $P_{k,k+1}(p_0^*; G)$ because $x(p_0; G)$ is continuous over $p_0$. By power balance equation (1) at bus $k$, for any $m$, we have

$$p_k(p_0^{(m)}; G) = p_{k,k+1}(p_0^{(m)}; G) - p_{k,k+1}(p_0^{(m)}; G) < p_{k,k+1}(p_0^{(m+1)}; G) - p_{k,k+1}(p_0^{(m+1)}; G) = p_k(p_0^{(m+1)}; G).$$

Thus $\{p_k(p_0^{(m)}; G), m \in \mathbb{N}\}$ is a monotone increasing sequence that converges to $p_k(p_0^*; G)$. We now construct a point $\hat{x} = \ldots$.
\((\tilde{P}, \tilde{Q}, \hat{p}, \hat{q})\) as follows. First, pick up \((\tilde{P}_{k,k+1}, \tilde{Q}_{k,k+1}, \hat{p}_k, \hat{q}_k)\) such that \(\hat{p}_k \in \{p_k(p_0^{(m)}; \mathcal{G}), m \in \mathbb{N}\}, \hat{q}_k \in (\tilde{q}_k, q_k(p_0^*; \mathcal{G}))\)

and they satisfy the following equations:

\[\begin{align*}
\hat{p}_{k+1} &= P_{k,k+1}(p_0^*; \mathcal{G}) - p_k(p_0^*; \mathcal{G}) + \tilde{p}_k \\
\hat{q}_{k+1} &= \frac{P_{k,k+1}(p_0^*; \mathcal{G}) - q_k(p_0^*; \mathcal{G}) + \tilde{q}_k}{v_k} \\
v_{k+1} &= v_k - 2(r_{k,k+1} \tilde{P}_{k,k+1} + x_{k,k+1} \tilde{Q}_{k,k+1}) v_k + \tilde{P}_{k,k+1}^2 + \tilde{Q}_{k,k+1}^2 |z_{k,k+1}|^2
\end{align*}\]

The existence of \((\tilde{P}_{k,k+1}, \tilde{Q}_{k,k+1}, \hat{p}_k, \hat{q}_k)\) is guaranteed by the following two facts:

- \(p_k(p_0^{(m)}; \mathcal{G})\) is a monotone increasing sequence that converges to \(p_k(p_0^*; \mathcal{G})\).
- \(q_k\) is a continuous decreasing function of \(\tilde{p}_k\) if they satisfy (17).

Since \(\tilde{P}_{k,k+1} \in [P_{k,k+1}(p_0^{(1)}; \mathcal{G}), P_{k,k+1}(p_0^*; \mathcal{G})]\) and \(x(p_0; \mathcal{G})\)

are continuous over \(p_0\), then there exists a \(p_0^* \in [p_0^{(1)}, p_0^*]\) such that \(S_{i,k+1}(p_0^*; \mathcal{G}) = S_{i,k+1}\).

Next, we will construct the feasible physical variable for \(i \neq k\). For \(0 \leq i < k\), let \(\bar{s}_i = s_i(p_0^*; \mathcal{G})\) and \(\bar{s}_{i,i+1} = S_{i,i+1}(p_0^*; \mathcal{G})\). For \(k < i \leq n\), let \(\bar{s}_i = s_i(p_0^*; \mathcal{G})\) and \(\bar{s}_{i,i+1} = S_{i,i+1}(p_0^*; \mathcal{G})\). Clearly that \(\bar{x} \in X(\mathcal{G})\) with \((p_0^*, f(p_0^*))\) as the real power injection at substation 0 and \(0'\). However, \(f(p_0^*) < f(p_0^*)\), which contradicts \((p_0^*, f(p_0^*)) \in \mathcal{O}(\mathcal{P})\).

Case II: \(q_k(p_0^*; \mathcal{G}) < \bar{q}_k\). Similar approach can be used to show C2 does not hold by contradiction.

So far, we have shown that \(P_{k,k+1}(p_0; \mathcal{G})\) is non-decreasing either on its left or right neighborhood. Thus \(P_{k,k+1}(p_0; \mathcal{G})\) is non-decreasing of \(p_0\) if \(\tilde{q}_k < \bar{q}_k\) because \(P(p_0; \mathcal{G})\) is a continuous function of \(p_0\). The case where \(\tilde{q}_k = \bar{q}_k\) can be covered by taking limitation of the case of \(\tilde{q}_k < \bar{q}_k\).

**Lemma 8:** Suppose A2-A4 hold. Given a solution \(x(p_0; \mathcal{G})\) to OPF-\(\mathcal{G}_s\), \(P_{k,k+1}(p_0; \mathcal{G})\) is a concave increasing function of \(p_0\) for all \((k, k + 1) \in \mathcal{E}(0, 0')\).

**Proof:** It is shown that \(P_{k,k+1}(p_0; \mathcal{G})\) is a nondecreasing function of \(p_0\) in Lemma[7] We now show it is also a concave function of \(p_0\). Let \(\mathcal{G}_1 = (\mathcal{N}_1, \mathcal{E}_1)\), where \(\mathcal{N}_1 = \{i \mid 0 \leq i \leq k + 1\}\) and \(\mathcal{E}_1 = \{(i, i + 1) \mid 1 \leq i \leq k\}\). All the physical constraints are the same as \(\mathcal{G}\) except the bus injection power \(s_{k+1}\) at node \(k + 1\), which is relaxed to be a free variable.

Mathematically, it means

\[\begin{align*}
\bar{f}_{i,i+1}(\mathcal{G}_1) &= \bar{f}_{i,i+1}(\mathcal{G}) \quad i \leq k \\
\bar{f}_i(\mathcal{G}_1) &= \bar{f}_i(\mathcal{G}) \quad i \leq k \\
\bar{f}_{k+1}(\mathcal{G}_1) &= \infty \\
\bar{g}_i(\mathcal{G}_1) &= \bar{g}_i(\mathcal{G}) \quad i \leq k \\
\bar{g}_{k+1}(\mathcal{G}_1) &= -\infty
\end{align*}\]

Consider the following OPF problem:

**OPF-\(\mathcal{G}_s1\):** \[\min_{x \in X(\mathcal{G}_1)} P_{k,k+1} \quad \text{s.t. } p_0 \text{ is a fixed}\]

Let \(p_{k,k+1}(p_0^*; \mathcal{G}_1)\) be the optimal value for OPF-\(\mathcal{G}_s1\) and \(P_{k,k+1}(p_0^*; \mathcal{G}_1)\) be the real branch power flow across line \((k, k + 1)\), respectively.

Next, we will show \(P_{k,k+1}(p_0^*; \mathcal{G}) = P_{k,k+1}(p_0^*; \mathcal{G}_1)\).

Clearly that \(P_{k,k+1}(p_0^*; \mathcal{G}) \leq P_{k,k+1}(p_0^*; \mathcal{G}_1)\). Otherwise, by