Time-dependent mean-field approximations for many-body observables

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The excitation of a many-body system by a time-dependent perturbation is considered within the framework of functional integration. The stationary phase approximation to a functional-integral representation of the final expectation values of many-body observables in the interaction picture leads to a new time-dependent mean-field theory. The resulting equations of motion depend upon the observable itself and consequently are nonlocal in time. The method is illustrated by an analytically soluble application to the forced harmonic oscillator.

[NUCLEAR REACTIONS Functional integral representation of expectation values of many-body observables. Stationary phase approximation. Time-dependent mean fields.]

I. INTRODUCTION

Time-dependent mean-field approximations have recently been proposed in the framework of the functional integrals for the description of nuclear many-body scattering.[1-3] In Ref. 1, a functional integral based on the Hubbard-Stratonovich transformation[4] was used to represent many-body S-matrix elements for a system under a time-dependent one-body perturbation. In the stationary phase approximation, this leads to very satisfactory results for the S matrix in a simple spin model[1] and in more realistic atomic physics situations.[3] It is therefore a promising starting point for the description of the full many-body scattering. Indeed, any measurement involving several final and initial channels can be expressed formally in terms of exclusive measurements which involve specific outgoing and incoming channels. As was emphasized in Ref. 2, such a rough representation of inclusive measurements would imply, at the mean-field level, a very complicated average over a large number of exclusive mean fields, each involving a well-defined exit channel. To overcome this difficulty, the authors of Ref. 1 proposed to represent the expectation values of few-body inclusive observables as functional integrals of the corresponding few-body density correlation functions.[2] As a result, the stationary phase approximation for the expectation value of one- and two-body operators[2] led to an averaging of the density and the density-density correlation functions over all numerous exit channels through the time dependent Hartree-Fock (TDHF) approximation.

The purpose of the present paper is to develop a formulation of the full many-body scattering within a universal functional integral representation. Apart from the fact that this unifies the description of the various aspects of many-body scattering and clarifies the structure of the resulting mean field, our method also has several physical advantages. First, the application of the stationary phase approximation is unambiguous since it applies to the effective expectation values of both exclusive and inclusive observables. In the latter case, the time-dependent mean-field configuration depends self-consistently upon the observable itself. It means equivalently that the averaging over a large number of exit channels is achieved coherently since each contributing inclusive transition amplitude is weighted by the matrix elements of the observable within the mean field itself. As a result of this additional self-consistency, we will see that our mean-field equations are nonlocal in time, in contrast to the TDHF equations. A similar dependence of the mean field upon the observable has also been obtained in Ref. 5 without resorting to functional integrals methods.

To present our method, we have organized this paper as follows. In Sec. II, we use the formalism of Ref. 7 to construct a very general functional integral representation of the many-body evolution operator. The nuclear scattering is simulated by a time-dependent one-body potential. In Sec. III, we use these functional integrals to calculate the expectation values of many-body observables which are constants of the unperturbed motion. In Sec. IV, we derive the time-dependent mean-field equations from the stationary phase approximation. Finally, we devote Sec. V to an illustration of our method using the forced harmonic oscillator, which has the advantage of being analytically soluble.

II. FUNCTIONAL INTEGRAL REPRESENTATIONS OF THE TIME-EVOLUTION OPERATOR

In the absence of any external interaction, the nucleons within a given nucleus are commonly taken to interact through a static effective two-body interaction, $\hat{V}$, so that their intrinsic motion is described by the effective Hamiltonian

$$\hat{H}_0=\sum_{\alpha\beta}K_{\alpha\beta}a^\dagger_\alpha a_\beta+\tfrac{1}{2}\sum_{\alpha\beta\gamma\delta}V_{\alpha\beta\delta\gamma}a^\dagger_\alpha a^\dagger_\beta a_\gamma a_\delta\ , \tag{1}$$

where we use the nonantisymmetrized form of the matrix elements of the two-body interaction

$$V_{\alpha\beta\gamma\delta}=(\alpha\beta\mid V\mid\gamma\delta)$$

in the single particle basis. Of course, the kinetic energy operator $\hat{K}$ represents the free motion of the nucleons. We also assume that in the presence of an interacting target or projectile the resulting modifications of the intrinsic motion can be roughly described by an external *time-dependent* one-body potential

$$\hat{W}(t)=\sum_{\alpha\beta} W_{\alpha\beta}(t)a^\dagger_\alpha a_\beta \;, \tag{2a}$$

with the boundary conditions

$$\lim_{|t|\to\infty} W_{\alpha\beta}(t)=0 \;. \tag{2b}$$

The problem of finding the intrinsic nuclear excitations of the various interacting systems is now reduced to solving for the many-body evolution operator

$$U(t_1,t_2)=Te^{-i\int_{t_2}^{t_1}\hat{H}(t)dt} \;, \tag{3}$$

with the time-dependent Hamiltonian

$$\hat{H}(t)=\hat{H}_0+\hat{W}(t) \;. \tag{4}$$

We now employ a very general functional integral representation for $U(t_1,t_2)$ developed in Ref. 7. We simply sketch the steps which are the most important for the present work and we advise the interested reader to refer to Ref. 7 for further details.

We introduce two complex bidimensional fields σ and φ as well as two arbitrary matrices u and v, and rewrite expression (3) as the following:

$$U(t_1,t_2)=\lim_{\epsilon\to 0}\int D[\sigma,\sigma^*]D[\varphi,\varphi^*]e^{i(\epsilon/2)\sum_{\alpha\beta,k}(|\sigma|^2+|\varphi|^2)_{\alpha\beta,k}}\,U_{\sigma\varphi}(t_1,t_2) \;, \tag{5}$$

where $U_{\sigma\varphi}(t_1,t_2)$ is the effective evolution operator

$$U_{\sigma\varphi}(t_1,t_2)=T\prod_{k=1}^{N}\left[1-i\epsilon\hat{K}-i\epsilon\hat{W}(t_k)-i\frac{\epsilon}{2}(\sigma^*_k v\hat{\rho}+\sigma_k v^*\hat{\rho}^\dagger)-i\epsilon(\varphi_k u^*\hat{\eta}^\dagger+\varphi^*_k u\hat{\eta})-\frac{\epsilon^2}{4}\sigma^*_k V\hat{\Omega}\sigma_k\right] \;. \tag{6}$$

The purpose of introducing additional degrees of freedom with respect to Ref. 7 through a *complex* σ field is to provide the Hermiticity of $U_{\sigma\varphi}$ for any arbitrary value of the σ field and the matrix v. As we will see later on, such a generalization is essential for the present work in order to ensure the Hermiticity of the mean-field itself. (This generalization was made in collaboration with Kerman.) The quantity ϵ divides the time interval (t_1,t_2) into N slices which define a time lattice for the time-dependent fields $\sigma(t_k)$ and $\varphi(t_k)$ as well as for $\hat{W}(t_k)$. The normalization constant in Eq. (5) has been absorbed in the two measures $D[\sigma,\sigma^*]$ and $D[\varphi,\varphi^*]$, which are otherwise simple Riemann differential elements in the complex plane. The one-body operators which involve matrix products in Eq. (6) are defined as

$$\sigma_k v\hat{\rho}=\sum_{\alpha\beta\gamma\delta}\sigma_{\alpha\beta,k}v_{\alpha\beta\gamma\delta}a^\dagger_\gamma a_\delta \;, \tag{7}$$

and $\hat{\eta}^\dagger,\hat{\eta}$ represent the creation or the annihilation of two particles

$$\hat{\eta}^\dagger_{\alpha\beta}=a^\dagger_\alpha a^\dagger_\beta \;, \tag{8a}$$

$$\hat{\eta}_{\alpha\beta}=a_\beta a_\alpha \;. \tag{8b}$$

That term in the integrand (6) which is responsible for the two-body interaction $\hat{V}$ reads similarly

$$\sigma^*_k V\hat{\Omega}\sigma_k=\sum_{\substack{\text{all}\\ \text{indices}}}\sigma^*_{\beta\gamma,k}V_{\beta\alpha\gamma\delta}a^\dagger_\rho a^\dagger_\alpha a_\delta a_\mu a_{\rho\mu,k} \tag{9}$$

and appears to be a second order in the time step. Since the mean-field equations for functional integrals such as (5) are exclusively governed by the terms of first order in time, and since the two matrices u and v can be chosen arbitrarily, it is clear that the stationary phase approximation for the many-body propagator (3) leads to a very general time-dependent mean-field propagator.[7]

In Ref. 6, the two matrices u and v are shown to be relevant only through the combinations $v^\dagger v$ and $u^\dagger u$, which have the physical meaning of trial two-body interactions. In particular, the *"Hartree representation,"* where

$$(v^\dagger v)_{\alpha\beta\gamma\delta}=V_{\beta\gamma\alpha\delta}$$

and $u^\dagger u=0$, allows us to take the following continuous limit[6] of the lattice functional integral (5):

$$U(t_1,t_2)=\int D[\sigma(t),\sigma^*(t)]e^{(i/2)\int_{t_2}^{t_1}\sigma^*(t)V\sigma(t)dt}:U_\sigma(t_1,t_2): \;, \tag{10}$$

where the normal ordering operator (::) takes into account the self-energy of the two-body interaction $\hat{V}$ which is due to the term of order ϵ^2 in the expansion of the *effective* evolution operator,

$$U_\sigma(t_1,t_2)=T\exp\left\{-i\int_{t_2}^{t_1}\hat{h}[\sigma(t)]dt\right\} . \tag{11}$$

Since the matrix elements of $\hat{V}$ can always be chosen to be real, $\hat{h}[\sigma(t)]$ is the σ-dependent one-body Hamiltonian

$$\hat{h}[\sigma(t)]=\hat{K}+\hat{W}(t)+\tfrac{1}{2}[\sigma^*(t)V\hat{\rho}+\sigma(t)V\hat{\rho}^\dagger] \;. \tag{12}$$

In expressions (10) and (12) the single particle indices of the two matrixial products are implicit, i.e.,

$$\sigma^*(t)V\sigma(t)=\sum_{\alpha\beta\gamma\delta}\sigma^*_{\alpha\beta}(t)V_{\beta\gamma\alpha\delta}\sigma_{\gamma\delta}(t) \;, \tag{13a}$$

$$\sigma^*(t)V\hat{\rho}=\sum_{\alpha\beta\gamma\delta}\sigma^*_{\alpha\beta}(t)V_{\beta\gamma\alpha\delta}a^\dagger_\gamma a_\delta\ . \tag{13b}$$

From Eq. (12), it is clear that making the σ field complex provides the extra degrees of freedom needed to render $\hat{h}[\sigma(t)]$ Hermitian for any arbitrary $\sigma(t)$. On the other hand, a single real bidimensional field $\tau(\vec{X},t)$ is sufficient enough to make the evolution operator $U_\tau(t_1,t_2)$ Hermitian when the functional integral is expressed in the $\vec{X}$ representation. Indeed, when the effective two-body interaction $\hat{V}$ is local and when $V(\vec{X},\vec{X}')$ has no derivatives and also commutes with the field operators $\psi^\dagger(\vec{X})$ and $\psi(\vec{X})$, $U(t_1,t_2)$ can be written as:

$$U(t_1,t_2)=\int D[\tau(\vec{X},t)]e^{(i/2)\int_{t_2}^{t_1}\int\int\tau(\vec{X},t)V(\vec{X},\vec{X}')\tau(\vec{X}',t)d\vec{X}\,d\vec{X}'dt}:U_\tau^{(\vec{X})}(t_1,t_2):\ . \tag{14}$$

The effective operator $U_\tau^{(\vec{X})}(t_1,t_2)$ is now Hermitian for any value of the field $\tau(\vec{X},t)$:

$$U_\tau^{\vec{X}}(t_1,t_2)=T\exp\left\{-i\int_{t_2}^{t_1}\left[\frac{1}{2m}\int d\vec{X}(\vec{\nabla}\Psi^\dagger(\vec{X}))(\vec{\nabla}\Psi(\vec{X}))+\int d\vec{X}\,W(\vec{X},t)\Psi^\dagger(\vec{X})\Psi(\vec{X})\right.\right.$$
$$\left.\left.+\int\int d\vec{X}d\vec{X}'\tau(\vec{X},t)\vec{V}(\vec{X},\vec{X}')\Psi^\dagger(\vec{X}')\Psi(\vec{X}')\right]dt\right\}. \tag{15}$$

We will use the expressions above to construct a functional integral representation for the expectation values of many-body constants of the unperturbed motion. For reasons of simplicity, we will use only the continuous limit of the *Hartree representation,* as embodied in either (10) or (14), but will show how to generalize the results to the case of an arbitrary representation.

III. FUNCTIONAL INTEGRAL REPRESENTATIONS OF MANY-BODY EXPECTATION VALUES

If the nucleus is prepared in the state $|i\rangle$ at time t_2, and if we measure at time t_1 the expectation value of any many-body observable $\hat{O}$ which is a constant of the unperturbed motion, the result in the interaction picture is

$$O(t_1)=\langle i\mid U^\dagger(t_1,t_2)U_0(t_1,t_2)\hat{O}U_0^\dagger(t_1,t_2)U(t_1,t_2)\mid i\rangle\ , \tag{16}$$

where $U(t_1,t_2)$ is the full evolution operator introduced in (3) and $U_0(t_1,t_2)$ effects the unperturbed time evolution,

$$U_0(t_1,t_2)=e^{-i\hat{H}_0(t_1-t_2)}\ . \tag{17a}$$

Although this operator appears to be irrelevant in Eq. (16) for $O(t_1)$ since

$$U_0(t_1,t_2)\hat{O}U_0^\dagger(t_1,t_2)=\hat{O}\ , \tag{17b}$$

we will see in Secs. III and IV that the interaction picture is essential to the functional integral representation of $O(t_1)$ since it removes the spurious dispersion due to the time evolution of the mean-field itself.

To obtain such a functional representation for the expectation value (16) of $\hat{O}$ in the interaction picture, we simply introduce four complex bidimensional fields σ_1, σ_1', σ_2, and σ_2' and write, with the help of (10):

$$O(t_1)=\int D[\sigma_1,\sigma_1^*]D[\sigma_2,\sigma_2^*]D[\sigma_1',\sigma_1'^*]D[\sigma_2',\sigma_2'^*]e^{(i/2)\int_{t_2}^{t_1}(\sigma_1^*V\sigma_1-\sigma_2^*V\sigma_2+\sigma_2'^*V\sigma_2'-\sigma_1'^*V\sigma_1')(t)dt}$$
$$\times\langle i\mid:U^\dagger_{\sigma_1'}(t_1,t_2)U^0_{\sigma_2'}(t_1,t_2)\hat{O}U^{0\dagger}_{\sigma_2}(t_1,t_2)U_{\sigma_1}(t_1,t_2):\mid i\rangle\ , \tag{18}$$

where the evolution operator U_{σ_1} has been already introduced in (11) and (12), and $U^0_{\sigma_2'}$ represents the time evolution

$$U^0_{\sigma_2'}(t_1,t_2)=T\exp\left\{-i\int_{t_2}^{t_1}\hat{h}_0[\sigma_2'(t)]dt\right\} \tag{19a}$$

for the "unperturbed" time-dependent one-body Hamiltonian:

$$\hat{h}_0[\sigma_2'(t)]=\hat{K}+\tfrac{1}{2}[(\sigma_2'^*(t)V\hat{\rho}+\sigma_2'(t)V\hat{\rho}^\dagger)]\ . \tag{19b}$$

The operator $U^{0\dagger}_{\sigma_2}$ represents the backward propagation in time through the unperturbed one-body Hamiltonian

$$\hat{h}_0^\dagger[\sigma_2(t)]=\hat{K}+\tfrac{1}{2}[\sigma_2^*(t)V\hat{\rho}+\sigma_2(t)V\hat{\rho}^\dagger]\ , \tag{20}$$

since the matrix elements of $\hat{V}$ can always be chosen to be real. Similarly, $U^\dagger_{\sigma_1'}$ propagates backward in time through the one-body Hamiltonian:

$$\hat{h}^\dagger[\sigma_1'(t)]=\hat{K}+\hat{W}(t)+\tfrac{1}{2}[\sigma_1'^*(t)V\hat{\rho}+\sigma_1'(t)V\hat{\rho}^\dagger]\ . \tag{21}$$

It should be noted that the introduction of the four different fields σ_i's and σ_i''s on the rhs of (18) generally breaks locally the time reversal symmetry of the many-body propagation represented in (16). However, as we will see in the following section, this symmetry is entirely restored in the stationary phase approximation to the functional integral (18).

IV. STATIONARY PHASE APPROXIMATION: TIME-DEPENDENT MEAN-FIELD APPROXIMATION

In order to obtain the stationary phase approximation to $O(t_1)$, one usually separates the imaginary and the real part of the action, so that

$$O(t_1)=\int D[\sigma_1,\sigma_1^*]D[\sigma_2,\sigma_2^*]D[\sigma_1',\sigma_1'^*]D[\sigma_2',\sigma_2'^*]e^{i\varphi_{\rm eff}}e^{-(1/2){\rm Im}\int_{t_2}^{t_1}(\sigma_1^*V\sigma_1-\sigma_2^*V\sigma_2+\sigma_2'^*V\sigma_2'-\sigma_1'^*V\sigma_1')(t)dt}$$
$$\times|\langle i|:U_{\sigma_1'}^\dagger U_{\sigma_2'}^0\hat{O}U_{\sigma_2}^{0\dagger}U_{\sigma_1}:|i\rangle| , \tag{22}$$

where the argument $\varphi_{\rm eff}$ reads

$$\varphi_{\rm eff}=\tfrac{1}{2}{\rm Re}\int_{t_2}^{t_1}(\sigma_1^*V\sigma_1-\sigma_2^*V\sigma_2+\sigma_2'^*V\sigma_2'-\sigma_1'^*V\sigma_1')(t)dt+{\rm Im}\log\langle i|:U_{\sigma_1'}^\dagger U_{\sigma_2'}^0\hat{O}U_{\sigma_2}^{0\dagger}U_{\sigma_1}:|i\rangle . \tag{23}$$

Due to the rapid oscillations of the integrand (22), the classical values of the fields σ_i and σ_i' which contribute most to $O(t_1)$ are solutions of the coupled equations

$$\frac{\partial\varphi_{\rm eff}}{\partial\sigma_i}=\frac{\partial\varphi_{\rm eff}}{\partial\sigma_i^*}=\frac{\partial\varphi_{\rm eff}}{\partial\sigma_i'}=\frac{\partial\varphi_{\rm eff}}{\partial\sigma_i'^*}=0;\quad i=1,2 . \tag{24}$$

To simplify the notation, we henceforth write U instead of $U(t_1,t_2)$ when there is no ambiguity about the time interval of the propagation as in (22) and (23).

We now introduce the particle-hole operator in the interaction picture

$$\hat{\rho}_{\sigma,\alpha\beta}(t)=U_\sigma(t_2,t)a_\alpha^\dagger a_\beta U_\sigma(t,t_2) , \tag{25a}$$

as well as its Hermitian conjugate

$$\hat{\rho}^\dagger_{\sigma,\alpha\beta}(t)=U_\sigma^\dagger(t,t_2)a_\beta^\dagger a_\alpha U_\sigma^\dagger(t_2,t) , \tag{25b}$$

in order to define a generalized density operator in the interaction picture

$$\hat{r}(t)=\frac{1}{2}\begin{Bmatrix}\hat{\rho}^\dagger(t)+\hat{\rho}(t)\\ i[\hat{\rho}^\dagger(t)-\hat{\rho}(t)]\end{Bmatrix} . \tag{26a}$$

Similarly we introduce a real superfield, $\Sigma(t)$, which takes into account the degrees of freedom of the complex field $\sigma(t)$:

$$\Sigma(t)=\frac{1}{2}\begin{bmatrix}\sigma^*(t)+\sigma(t)\\ i[\sigma^*(t)-\sigma(t)]\end{bmatrix} . \tag{26b}$$

With these definitions, the solutions of Eq. (24) can be represented by the following expectation values of the Hermitian operator $\hat{r}(t)$:

$$\Sigma_1(t)={\rm Re}\frac{\langle i\,|\,U_{\sigma_1'}^\dagger U_{\sigma_2'}^0\hat{O}U_{\sigma_2}^{0\dagger}U_{\sigma_1}\hat{r}_{\sigma_1}(t)\,|\,i\rangle}{\langle i\,|\,U_{\sigma_1'}^\dagger U_{\sigma_2'}^0\hat{O}U_{\sigma_2}^{0\dagger}U_{\sigma_1}\,|\,i\rangle} ; \tag{27a}$$

$$\Sigma_2(t)={\rm Re}\frac{\langle i\,|\,U_{\sigma_1'}^\dagger U_{\sigma_2'}^0\hat{O}U_{\sigma_2}^{0\dagger}\hat{r}_{\sigma_2}(t)U_{\sigma_1}\,|\,i\rangle}{\langle i\,|\,U_{\sigma_1'}^\dagger U_{\sigma_2'}^0\hat{O}U_{\sigma_2}^{0\dagger}U_{\sigma_1}\,|\,i\rangle} ; \tag{27b}$$

$$\Sigma_1'(t)={\rm Re}\frac{\langle i\,|\,\hat{r}_{\sigma_1'}(t)U_{\sigma_1'}^\dagger U_{\sigma_2'}^0\hat{O}U_{\sigma_2}^{0\dagger}U_{\sigma_1}\,|\,i\rangle}{\langle i\,|\,U_{\sigma_1'}^\dagger U_{\sigma_2'}^0\hat{O}U_{\sigma_2}^{0\dagger}U_{\sigma_1}\,|\,i\rangle} ; \tag{27c}$$

$$\Sigma_2'(t)={\rm Re}\frac{\langle i\,|\,U_{\sigma_1'}^\dagger U_{\sigma_2'}^0\hat{r}_{\sigma_2'}(t)\hat{O}U_{\sigma_2}^{0\dagger}U_{\sigma_1}\,|\,i\rangle}{\langle i\,|\,U_{\sigma_1'}^\dagger U_{\sigma_2'}^0\hat{O}U_{\sigma_2}^{0\dagger}U_{\sigma_1}\,|\,i\rangle} ; \tag{27d}$$

where the Wick ordering has been taken out of the expectation values (27) since there is no longer any integration over the fields σ_i and σ_i' and the terms like $\epsilon^2a^\dagger aa^\dagger a$ in the expansion of U or U_0 have become irrelevant in the limit where $\epsilon\to 0$.

Since all of the Σ fields are real, it is clear that the set of Eqs. (27) has the particular solutions

$$\sigma_1(t)=\sigma_1'(t) ; \tag{28a}$$

$$\sigma_2(t)=\sigma_2'(t) , \tag{28b}$$

as well as their complex conjugates. This restores the broken time reversal symmetry discussed previously and leads to the two coupled equations,

$$\Sigma_1(t)={\rm Re}\frac{\langle i\,|\,U_{\sigma_1}^\dagger U_{\sigma_2}^0\hat{O}U_{\sigma_2}^{0\dagger}U_{\sigma_1}\hat{r}_{\sigma_1}(t)\,|\,i\rangle}{\langle i\,|\,U_{\sigma_1}^\dagger U_{\sigma_2}^0\hat{O}U_{\sigma_2}^{0\dagger}U_{\sigma_1}\,|\,i\rangle} , \tag{29a}$$

$$\Sigma_2(t)={\rm Re}\frac{\langle i\,|\,U_{\sigma_1}^\dagger U_{\sigma_2}^0\hat{O}U_{\sigma_2}^{0\dagger}\hat{r}_{\sigma_2}(t)U_{\sigma_1}\,|\,i\rangle}{\langle i\,|\,U_{\sigma_1}^\dagger U_{\sigma_2}^0\hat{O}U_{\sigma_2}^{0\dagger}U_{\sigma_1}\,|\,i\rangle} , \tag{29b}$$

where the explicit dependence of Σ_i upon σ_i has been defined in (26b). As before, $U_{\sigma_1}(t_1,t_2)$ propagates between t_2 and t_1 with the Hamiltonian $\hat{h}_{\sigma_1}(t)$, which has matrix elements

$$h_{\sigma_1,\alpha\beta}(t)=K_{\alpha\beta}+W_{\alpha\beta}(t)+\tfrac{1}{2}\sum_{\gamma\delta}[\sigma_{1,\gamma\delta}^*(t)V_{\delta\alpha\gamma\beta}+\sigma_{1,\gamma\delta}(t)V_{\delta\beta\gamma\alpha}] . \tag{30a}$$

The effect of the unperturbed motion is cancelled by the backward propagation between t_1 and t_2 with the Hamiltonian $\hat{h}^0_{\sigma_2}(t)$, which has matrix elements

$$h^0_{\sigma_2,\alpha\beta}(t)=K_{\alpha\beta}+\tfrac{1}{2}\sum_{\gamma\delta}[\sigma^*_{2,\gamma\delta}(t)V_{\delta\alpha\gamma\beta}+\sigma_{2,\gamma\delta}(t)V_{\delta\beta\gamma\alpha}] \; . \tag{30b}$$

From the equations of motion (29), we see that the mean-field configurations $\sigma_1(t)$ and $\sigma_2(t)$ depend upon the operator $\hat{O}$, whose expectation is the very thing being sought. This is the major issue of this new time-dependent mean-field theory. Since most of the symmetries of the unperturbed Hamiltonian are expected to be broken by the mean field itself,

$$[\hat{O},U^{0\dagger}_{\sigma_2}U_{\sigma_1}]\neq 0 \; , \tag{31}$$

the dependence upon the observable $\hat{O}$ is in general responsible for a time nonlocality in Eqs. (29). Indeed, in order to determine the fields $\sigma_1(t)$ and $\sigma_2(t)$ at any given time t, we need to know their values between the boundaries t_2 and t_1. We recall that a similar time nonlocality has already been obtained in Ref. 1 and has led to a very satisfactory mean-field description of S-matrix elements of a schematic nuclear model.[1] Through the present functional integral representation, we now extend these nonlocal equations of motion to the mean-field description of inclusive expectation values of few-body observables. This does not mean that the mean field has the same structure, independent of the physical quantity which is being measured. Rather, it is obtained in a universal way for each particular observable.

The solutions $\sigma_1(t)$ and $\sigma_2(t)$ to Eqs. (29) are highly nonlinear in most cases. However, when $\hat{O}$ slightly violates the corresponding symmetry of the mean field, the fields $\sigma_1(t)$ and $\sigma_2(t)$ represent an *approximate* averaging of the one-body density matrix $\hat{r}(t)$, (26a), over all the possible exit channels, *weighted* by the corresponding matrix elements of $\hat{O}$. On the other hand, it is well known[2] that in the TDHF approximation the one-body density correlation function is approximated by an equally weighted average over the numerous exit channels. As we will see in more detail at the end of this section, it is therefore not surprising that the TDHF equations are *formally* recovered from the present theory only for the trivial case where $\hat{O}=\hat{1}$. However, since the TDHF approximation has been found[2] to be successful for the description of the expectation values of few-body observables, it is expected to be in good agreement with the present approximations in such cases.

From Sec. II and Ref. 7, it is clear that the structure of the equations of motion does not change when we vary the representation defined by the trial two-body interactions $v^\dagger v$ and $u^\dagger u$ introduced in Sec. II. Recall that the presence of the nonantisymmetrized matrix elements $V_{\delta\alpha\gamma\beta}$ in Eqs. (30) and (31) is characteristic of the *Hartree representation* where $(v^\dagger v)_{\alpha\beta\gamma\delta}=V_{\beta\gamma\alpha\delta}$ and $u^\dagger u=0$. Since the HF solution is the normal phase of the superconducting HFB solution, it is natural to treat simultaneously the special cases of the *"Hartree-Fock"* and *"Hartree-Fock-Bogolyubov representations."* Indeed, for the *Hartree-Fock-Bogolyubov representation* where $(v^\dagger v)_{\alpha\beta\gamma\delta}=V^A_{\beta\gamma\alpha\delta}$ and $u^\dagger u=V^A/8$, the equations of motion for the mean field are easily derived by introducing the generalized Hermitian density operator:

$$\hat{R}_{\alpha\beta}=\frac{1}{2}\begin{bmatrix} 2\hat{r} \\ (\hat{\eta}^\dagger+\hat{\eta}) \\ i(\hat{\eta}^\dagger-\eta) \end{bmatrix}_{\alpha\beta}=\frac{1}{2}\begin{bmatrix} a^\dagger_\beta a_\alpha+a^\dagger_\alpha a_\beta \\ i(a^\dagger_\beta a_\alpha-a^\dagger_\alpha a_\beta) \\ a^\dagger_\alpha a^\dagger_\beta+a_\beta a_\alpha \\ i(a^\dagger_\alpha a^\dagger_\beta-a_\beta a_\alpha) \end{bmatrix} , \tag{32}$$

as well as two superfields $\Phi_1(t)$ and $\Phi_2(t)$:

$$\Phi(t)=\frac{1}{2}\begin{bmatrix} \sigma^*+\sigma \\ i(\sigma^*-\sigma) \\ \varphi^*+\varphi \\ i(\varphi^*-\varphi) \end{bmatrix}(t) \; . \tag{33}$$

These two real fields Φ_1 and Φ_2 are dynamically coupled through the equations of motion analogous to (29):

$$\Phi_1(t)=\mathrm{Re}\frac{\langle i\mid U^\dagger_{\Phi_1}U^0_{\Phi_2}\hat{O}U^{0\dagger}_{\Phi_2}U_{\Phi_1}\hat{R}_{\Phi_1}(t)\mid i\rangle}{\langle i\mid U^\dagger_{\Phi_1}U^0_{\Phi_2}\hat{O}U^{0\dagger}_{\Phi_2}U_{\Phi_1}\mid i\rangle} \; ; \tag{34a}$$

$$\Phi_2(t)=\mathrm{Re}\frac{\langle i\mid U^\dagger_{\Phi_1}U^0_{\Phi_2}\hat{O}U^{0\dagger}_{\Phi_2}\hat{R}_{\Phi_2}(t)U_{\Phi_1}\mid i\rangle}{\langle i\mid U^\dagger_{\Phi_1}U^0_{\Phi_2}\hat{O}U^{0\dagger}_{\Phi_2}U_{\Phi_1}\mid i\rangle} \; . \tag{34b}$$

The corresponding one-body Hamiltonians for the evolution operators U_{Φ_1} and U_{Φ_2} now include pairing correlations and violate particle number conservation through the pair creation and annihilation operators $\hat{\eta}^\dagger=a^\dagger a^\dagger$ and $\hat{\eta}=aa$:

$$\hat{h}_{\Phi_1}(t)=\hat{K}+\hat{W}(t)-\mu_F(t)\hat{A}+\tfrac{1}{2}[\sigma^*_1(t)V^A\hat{\rho}+\sigma_1(t)V^A\hat{\rho}^\dagger]+\varphi_1(t)\frac{V^A}{4}\hat{\eta}^\dagger+\varphi^*_1(t)\frac{V^A}{4}\hat{\eta} \tag{35a}$$

and

$$\hat{h}^0_{\Phi_2}(t)=\hat{K}-\mu_F(t)\hat{A}+\tfrac{1}{2}[\sigma^*_2(t)V^A\hat{\rho}+\sigma_2(t)V^A\hat{\rho}^\dagger]+\varphi_2(t)\frac{V^A}{4}\hat{\eta}^\dagger+\varphi^*_2(t)\frac{V^A}{4}\hat{\eta} \; . \tag{35b}$$

As mentioned earlier, the equations of motion for the *Hartree-Fock representation* follow naturally as the nonsuperconducting solutions of Eqs. (34) and (35), where $\varphi(t)=0$. As before, the Hamiltonian $\hat{h}^0_{\Phi_2}$ cancels the contribution of the unperturbed motion to (34); we emphasize that this is done self-consistently through the coupling of (34a) and (34b). The quantity $\mu_F(t)$ is the time-dependent chemical potential, which ensures the correct number of particles at each time t:

$$\langle \hat{A} \rangle(t) \approx \langle \hat{A} \rangle^0(t) = A \ , \tag{36}$$

where we approximate the exact many-body value on the lhs of (36) by its mean-field expectation value

$$\langle \hat{A} \rangle^0(t) = \langle i \mid U^\dagger_{\Phi_1}(t,t_2) U^0_{\Phi_2}(t,t_2) \hat{A} U^{0\dagger}_{\Phi_2}(t,t_2) U_{\Phi_1}(t,t_2) \mid i \rangle \ . \tag{37}$$

Since $\Phi_1(t)$ and $\Phi_2(t)$ are themselves functions of $\mu_F(t)$, the inclusion of pairing correlations within the mean field introduces an additional self-consistency through Eq. (37). The reader who is interested in a further treatment of mean-field dynamics in the presence of pairing correlations using functional integrals is referred to Ref. 7.

So far, we have kept totally arbitrary the representations defined from the trial two-body interactions $v^\dagger v$ and $u^\dagger u$, and have not chosen between TDH, TDHF, TDHFB, or any other which would be the optimal *representation* for this time-dependent mean field. In Ref. 7, by varying the trial two-body interaction $v^\dagger v$ while $u^\dagger u=0$, it was shown that, among the large variety of possible static mean fields which do not take into account pairing correlations, HF is the optimal static mean-field approximation to the exact nuclear grand potential. Similarly, when we allow the mean field to include pairing correlations and vary the two trial interactions $v^\dagger v$ and $u^\dagger u$, the HFB approximation is optimal for the mean-field grand potential. Since these HF and HFB mean fields are optimal for the description of time-independent phenomena, it is natural to consider similarly TDHF and TDHFB as special configurations for the description of time-dependent processes.

As a result, we end up at the stationary phase approximation for $\langle \hat{O} \rangle(t_1)$:

$$O(t_1) \approx \langle i \mid U^\dagger_{\Phi_1}(t_1,t_2) U^0_{\Phi_2}(t_1,t_2) \hat{O} U^{0\dagger}_{\Phi_2}(t_1,t_2) U_{\Phi_1}(t_1,t_2) \mid i \rangle \ , \tag{38}$$

where $\sigma_1(t)$ and $\sigma_2(t)$ are solutions of Eqs. (29)—(31) within the TDHF representation, namely when the two-body interaction $\hat{V}$ enters the equations of motion (29) through its antisymmetrized matrix elements $V^A_{\alpha\beta\gamma\delta}$. If we decide to include the pairing in the mean field, the stationary phase approximation then leads to

$$O(t_1) \approx \langle i \mid U^\dagger_{\Phi_1}(t_1,t_2) U^0_{\Phi_2}(t_1,t_2) \hat{O} U^{0\dagger}_{\Phi_2}(t_1,t_2) U_{\Phi_1}(t_1,t_2) \mid i \rangle \ , \tag{39}$$

where $\Phi_1(t)$ and $\Phi_2(t)$ are solutions of Eqs. (34)—(36) within the TDHFB representation. This last approximation might be particularly relevant when the nucleus exhibits strongly correlated nucleonic pairs that are susceptible of being excited during the nuclear scattering.

We briefly sketch the result when we choose coordinate space ($\vec{X}$) to construct our functional integral. As expected, the equations of motion can then be expressed in terms of the density operator $\hat{\rho}(\vec{X}) = \Psi^\dagger(\vec{X})\Psi(\vec{X})$:

$$\sigma_1(\vec{X},t) = \mathrm{Re}\frac{\langle i \mid U^\dagger_{\sigma_1} U^0_{\sigma_2} \hat{O} U^{0\dagger}_{\sigma_2} U_{\sigma_1} \hat{\rho}(\vec{X},t) \mid i \rangle}{\langle i \mid U^\dagger_{\sigma_1} U^0_{\sigma_2} \hat{O} U^{0\dagger}_{\sigma_2} U_{\sigma_1} \mid i \rangle} \ , \tag{40a}$$

$$\sigma_2(\vec{X},t) = \mathrm{Re}\frac{\langle i \mid U^\dagger_{\sigma_1} U^0_{\sigma_2} \hat{O} U^{0\dagger}_{\sigma_2} \hat{\rho}(\vec{X},t) U_{\sigma_1} \mid i \rangle}{\langle i \mid U^\dagger_{\sigma_1} U^0_{\sigma_2} \hat{O} U^{0\dagger}_{\sigma_2} U_{\sigma_1} \mid i \rangle} \ , \tag{40b}$$

with the corresponding Hamiltonians expressed in coordinate space. Since these equations of motion have the same character as the more abstract ones above, we will not repeat our previous discussion. Rather, we simply mention the two continuity equations satisfied by the average densities:

$$\frac{\partial \langle \hat{\rho}(\vec{X}) \rangle_1(t)}{\partial t} + \mathrm{div} \langle \vec{j}(\vec{X}) \rangle_1(t) = 0 \ ; \tag{41a}$$

$$\frac{\partial \langle \hat{\rho}(\vec{X}) \rangle_2(t)}{\partial t} + \mathrm{div} \langle \vec{j}(\vec{X}) \rangle_2(t) = 0 \ , \tag{41b}$$

where the expectation values $\langle \ \rangle_1$ and $\langle \ \rangle_2$ are defined in analogy with Eqs. (40) and $\vec{j}(\vec{X})$ is the current operator:

$$\vec{j}(\vec{X}) = \frac{1}{2im} \{ \Psi^\dagger(\vec{X}) \vec{\nabla} \Psi(\vec{X}) - [\vec{\nabla} \Psi^\dagger(\vec{X})] \Psi(\vec{X}) \} \ . \tag{41c}$$

It is interesting to note the apparent time locality and decoupling of these continuity equations. These are indeed only apparent since the initial conditions for (41) embody both the time nonlocality and the original dynamical coupling of the equations of motion (40). Indeed, in order to know the initial values of the various fields, we need to know them at any later time.

Finally, we consider the trivial case where $\hat{O}$ is simply the unit operator, $\hat{1}$. Since the normalization of the state of the system is conserved by unitarity during the entire evolution, we have the identity

$$\langle \hat{1} \rangle(t) = \langle i \mid U^\dagger(t,t_2) U(t,t_2) \mid i \rangle = \langle i \mid i \rangle \ . \tag{42a}$$

For this trivial example, the two equations of motion (40a) and (40b) reduce simply to the decoupled set,

$$\sigma_1(\vec{X},t) = \langle i \mid U^\dagger_{\sigma_1}(t,t_2) \Psi^\dagger(\vec{X}) \Psi(\vec{X}) U_{\sigma_1}(t,t_2) \mid i \rangle \ , \tag{42b}$$

$$\sigma_2(\vec{X},t)=\langle i \mid U^{\dagger}_{\sigma_2}(t,t_2)\Psi^{\dagger}(\vec{X})\Psi(\vec{X})U_{\sigma_2}(t,t_2)\mid i\rangle \;, \tag{42c}$$

which are nothing else but the TDHF (or correspondingly, TDHFB) equations of motion in the interaction picture. Of course, the stationary phase approximation to $\langle \hat{1}\rangle(t)$ leads to the exact value since it represents the conserved norm of the TDHF wave function in the interaction picture:

$$\begin{aligned}\langle \hat{1}\rangle^0(t)&=\langle i \mid U^{\dagger}_{\sigma_1}(t,t_2)U^0_{\sigma_2}(t,t_2)\\&\quad\times U^{0\dagger}_{\sigma_2}(t,t_2)U_{\sigma_1}(t,t_2)\mid i\rangle\\&=\langle i\mid i\rangle=\langle \hat{1}\rangle(t)\;.\end{aligned} \tag{42d}$$

We can also represent the resolution of unity $\hat{1}$ as the complete sum of projectors $\sum_{m=0}^{\infty}\mid m\rangle\langle m\mid$, so that the inclusive expectation value (42a) can also be regarded as a linear superposition of *equally weighted* exclusive expectation values. For this extreme case, the averaging over the numerous exit channels leads to an averaging of the one-body density matrix in the TDHF approximation.[2]

V. ILLUSTRATIVE EXAMPLE: THE FORCED HARMONIC OSCILLATOR

In this section, we illustrate our method with the example of the forced harmonic oscillator. Since it is analytically soluble, it appropriately demonstrates how self-consistency operates in this time-dependent mean-field theory. In addition, this example also justifies the choice of the interaction picture, which has been introduced in a somewhat *ad hoc* manner in (16).

The unperturbed Hamiltonian is chosen to be that of a single harmonic oscillator

$$\hat{H}_0=\frac{\hat{p}^2}{2}+\frac{\hat{q}^2}{2}\;, \tag{43}$$

where $\hat{q}$ and $\hat{p}$ are canonically conjugate variables satisfying the usual commutation relation $[\hat{q},\hat{p}]=i$. The external time-dependent potential is defined as

$$\hat{W}(t)=f(t)\hat{q}\;, \tag{44}$$

so that the time-dependent Hamiltonian which describes the entire evolution of the system reads

$$\hat{H}(t)=\frac{\hat{p}^2}{2}+\frac{\hat{q}^2}{2}+f(t)\hat{q}\;. \tag{45}$$

In contrast to the previous sections, where the initial time t_2 and the final time t_1 were arbitrary, we choose in this section $t_2=0$, but still keep the final time $t_1=T$ arbitrary. This choice has the advantage of simplifying the search for the mean-field configuration and obviously does not affect the physics itself.

The system is initially prepared in an eigenstate $\mid n\rangle$ of $\hat{H}_0$ and the expectation value at time T of any operator $\hat{O}$ which is a constant of the unperturbed motion is given by the expression

$$O(T)=\langle n \mid U^{\dagger}(T,0)U_0(T,0)\hat{O}U_0^{\dagger}(T,0)U(T,0)\mid n\rangle\;, \tag{46}$$

where

$$U_0(T,0)=e^{-i\hat{H}_0T} \tag{47a}$$

and

$$U(T,0)=T\exp\left[-i\int_0^T\hat{H}(t)dt\right]\;. \tag{47b}$$

There is an analogy between the operator $\hat{q}^2$ and the effective static two-body interaction $\hat{V}$, Eq. (1), as well as one between the time-dependent operator $f(t)\hat{q}$ and the time-dependent external one-body potential, $\hat{W}(t)$, introduced in (2a). Therefore, without any additional justification, we employ the Gaussian transformation used in (18) and linearize the operator $\hat{q}^2$ which appears in (4.6) to obtain

$$\begin{aligned}O(T)=&\int D[\sigma_1]D[\sigma_2]D[\sigma_1']D[\sigma_2']e^{(i/2)\int_0^T[\sigma_1^2(t)-\sigma_2^2(t)+\sigma_2'^2(t)-\sigma_1'^2(t)]dt}\\&\times\langle n\mid U^{\dagger}_{\sigma_1'}(T,0)U^0_{\sigma_2'}(T,0)\hat{O}U^{0\dagger}_{\sigma_2}(T,0)U_{\sigma_1}(T,0)\mid n\rangle\;,\end{aligned} \tag{48}$$

where these evolution operators are defined as in Sec. IV [see Eqs. (19)—(21)] with the obvious modifications

$$\hat{h}_0[\sigma(t)]=\frac{\hat{p}^2}{2}+\sigma(t)\hat{q} \tag{49}$$

for the unperturbed "effective one-body" motion and

$$\hat{h}[\sigma(t)]=\frac{\hat{p}^2}{2}+\sigma(t)\hat{q}+f(t)\hat{q} \tag{50}$$

for the overall "effective one-body" motion.

When applying the stationary phase approximation to Eq. (48) we can, as above, look for time-reversal invariant solutions which satisfy

$$\sigma_1(t)=\sigma_1'(t) \tag{51a}$$

and

$$\sigma_2(t)=\sigma_2'(t)\;, \tag{51b}$$

so that the mean-field configuration satisfies the equations $(T\geq t\geq 0)$

$$\sigma_1(t)=\mathrm{Re}\frac{\langle n\mid U^{\dagger}_{\sigma_1}U^0_{\sigma_2}\hat{O}U^{0\dagger}_{\sigma_2}U_{\sigma_1}\hat{q}(t)\mid n\rangle}{\langle n\mid U^{\dagger}_{\sigma_1}U^0_{\sigma_2}\hat{O}U^{0\dagger}_{\sigma_2}U_{\sigma_1}\mid n\rangle} \tag{52a}$$

and

$$\sigma_2(t)=\mathrm{Re}\frac{\langle n\mid U^{\dagger}_{\sigma_1}U^{0}_{\sigma_2}\hat{O}U^{0\dagger}_{\sigma_2}\hat{q}(t)U_{\sigma_1}\mid n\rangle}{\langle n\mid U^{\dagger}_{\sigma_1}U^{0}_{\sigma_2}\hat{O}U^{0\dagger}_{\sigma_2}U_{\sigma_1}\mid n\rangle}\,, \tag{52b}$$

where all the evolution operators run from 0 to T.

Due to the simple structure of the harmonic oscillator, $\sigma_1(t)$ and $\sigma_2(t)$ obey the two equations of motion $(T\geq t\geq 0)$

$$\ddot{\sigma}_1(t)=-\sigma_1(t)-f(t)\,, \tag{53a}$$

$$\ddot{\sigma}_2(t)=-\sigma_2(t)\,. \tag{53b}$$

These can be directly integrated to give

$$\sigma_1(t)=\sigma_1(0)\cos t+\dot{\sigma}_1(0)\sin t - \int_0^t f(\tau)\sin(t-\tau)d\tau\,, \tag{54a}$$

$$\sigma_2(t)=\sigma_2(0)\cos t+\dot{\sigma}_2(0)\sin t\,, \tag{54b}$$

where $\sigma_i(0)$ and $\dot{\sigma}_i(0)$ remain to be defined self-consistently from (52a) and (52b). To do so, we reexpress all the evolution operators which appear in (52) in the interaction picture defined by $\hat{p}^2$,

$$\hat{U}_\sigma(t_1,t_2)=\exp(i\hat{p}^2t_1/2)U_\sigma(t_1,t_2)\exp(-i\hat{p}^2t_2/2)\,. \tag{55}$$

This transformation leaves both $\sigma_i(0)$ and $\dot{\sigma}_i(0)$ invariant and greatly simplifies their evaluation since, for any function $\varphi(t)$, we have the parametrization (see Refs. 1 and 8)

$$T\exp\left\{-i\int_0^T\left[\frac{\hat{p}^2}{2}+\varphi(t)\hat{q}\right]dt\right\}=e^{i\mu(T)}\vec{A}[\alpha(T)]\,, \tag{56a}$$

with

$$A[\alpha(t)]=e^{\alpha(T)a^{\dagger}-\alpha^*(T)a^{\dagger}}\,. \tag{56b}$$

The value $\mu(T)$ is real in order to ensure the unitarity of (56a),

$$\mu(T)=\int_0^T\mathrm{Im}(\dot{\alpha}\alpha^*)(t)dt\,, \tag{57}$$

and $\alpha(t)$ is a complex function of time defined between 0 and T,

$$\alpha(t)=-\frac{i}{\sqrt{2}}\int_0^t\varphi(\tau)(1+i\tau)d\tau\,. \tag{58}$$

It is then a matter of simple algebra to evaluate the stationary phase approximation to (48),

$$O(T)\approx O^0(T)=\langle n\mid A[-\beta(T)]\hat{O}A[\beta(T)]\mid n\rangle\,, \tag{59a}$$

in terms of the complex parameter $\beta(T)$ which can be expanded in the following form:

$$\begin{aligned}\beta(T)=-\frac{i}{\sqrt{2}}\int_0^T(1+it)[f(t)+\sigma_1(t)-\sigma_2(t)]dt&=-\frac{i}{\sqrt{2}}[\sigma_1(0)-\sigma_2(0)]\int_0^T(1+it)\cos t\,dt\\&\quad-\frac{i}{\sqrt{2}}[\dot{\sigma}_1(0)-\dot{\sigma}_2(0)]\int_0^T(1+it)\sin t\,dt\\&\quad-\frac{i}{\sqrt{2}}\int_0^T(1+it)\left[f(t)-\int_0^t f(\tau)\sin(t-\tau)d\tau\right]dt\,.\end{aligned} \tag{59b}$$

The initial values $\sigma_i(0)$ and $\dot{\sigma}_i(0)$ are simple linear functions of $\beta(T)$,

$$\sigma_1(0)-\sigma_2(0)=\sqrt{2}\,\mathrm{Re}\beta(T)\,, \tag{60a}$$

$$\dot{\sigma}_1(0)-\dot{\sigma}_2(0)=-\sqrt{2}\,\mathrm{Im}\beta(T)\,. \tag{60b}$$

The choice of this forced harmonic oscillator is now clear since its analytical simplicity reduces the self-consistency of the time-dependent mean-field configuration to a linear two-dimensional system. Indeed, this leads after integration by parts to the compact form,

$$\beta(T)=-\frac{i}{\sqrt{2}}\int_0^T f(t)e^{it}dt\,. \tag{61}$$

For some particular choices of $\hat{O}$, we have taken the unperturbed Hamiltonian $\hat{H}_0$, as well as the projector $\mid m\rangle\langle m\mid$ for any eigenstate $\mid m\rangle$ of $\hat{H}_0$. These particular values of $\hat{O}$ obviously satisfy the required commutation relation

$$[\hat{O},\hat{H}_0]=0\,. \tag{62}$$

By substituting the value (61) of $\beta(T)$ into Eq. (59a), the total unperturbed energy at time T,

$$\langle\hat{H}_0\rangle(T)=\langle n\mid U^{\dagger}(T,0)\hat{H}_0U(T,0)\mid n\rangle\,, \tag{63a}$$

is approximated at the mean-field level by the energy

$$\langle\hat{H}_0\rangle^0(T)=n+\tfrac{1}{2}+\mid\beta(T)\mid^2\,. \tag{63b}$$

The time of measurement T has been kept arbitrary for the sake of comparison with the exact value, although in most of the physical applications, the limit $T\to\infty$, for which $\mid f(T)\mid\to 0$, is the only one relevant.

The expectation value of the projector $\mid m\rangle\langle m\mid$ is the probability

$$P_{mn}(T)=\mid\langle m\mid U(T,0)\mid n\rangle\mid^2 \tag{64a}$$

that the system, initially in the unperturbed state $\mid n\rangle$, is excited (or deexcited) to the unperturbed state $\mid m\rangle$, by the external potential acting during the time T. It is similarly approximated with the stationary phase level by the quantity

$$P^0_{mn}(T)=\frac{n_i!}{n_s!}e^{-x}x^{n_s-n_i}(L^{n_s-n_i}_{n_s}[x])^2 , \quad (64b)$$

where L^k_n is a Laguerre polynomial and $x=|\beta(T)|^2$. The integers n_i and n_s are related to the initial and final quantum numbers n and m by the relations:

$$n_i=\inf(n,m) , \quad (65a)$$

$$n_s=\sup(n,m) : \quad (65b)$$

We would like to emphasize that although the probability $P^0_{mn}(T)$ obtained in Eq. (64b) is identical to the one derived in Ref. 1, the fields $\sigma_1(t)$ and $\sigma_2(t)$ are entirely different since they are self-consistently defined by the operator $|m\rangle\langle m|$ through the set of Eqs. (52).

In order to compute the exact values (63a) and (64a), we use the interaction picture defined by $\hat{H}_0$ for the evolution operator,

$$\hat{U}(t_1,t_2)=e^{i\hat{H}_0t_1}T\exp\left[-i\int_{t_2}^{t_1}\hat{H}(t)dt\right]e^{-i\hat{H}_0t_2} . \quad (66)$$

We then find the remarkable result that the mean-field expectation value of any observable which satisfies (62) coincides with the exact corresponding expectation value. In particular, for any arbitrary $f(t)$, we have the identities

$$\langle\hat{H}_0\rangle(T)=\langle\hat{H}_0\rangle^0(T) , \quad (67a)$$

$$P_{mn}(T)=P^0_{mn}(T) . \quad (67b)$$

We close this section with a justification of the use of the interaction picture by pointing out the deficiencies encountered in the direct Schrödinger picture. To do so, we rewrite the functional integral (48) in the Schrödinger picture by only introducing two real bidimensional fields σ_1 and σ_1'

$$O(t_1)=\int D[\sigma_1]D[\sigma_1']e^{(i/2)\int_0^T[\sigma_1^2(t)-\sigma_1'^2(t)]dt}\times\langle n|U^\dagger_{\sigma_1'}(T,0)\hat{O}U_{\sigma_1}(T,0)|n\rangle . \quad (68)$$

The derivation of the equations of motion for the resulting mean field proceeds formally as before and we will not repeat it. The associated self-consistent equations are, however, in the general case, more difficult to solve analytically. This is why at the end of this section we choose to test the time-dependent mean-field theory following from (68) through its asymptotic static limit by taking $f(t)=0$. For a further simplification, we assume the system to be initially in the ground state $|0\rangle$ of the harmonic oscillator.

Since there is no external potential, the system will always remain in its ground state $|0\rangle$ so that at any time T,

$$P^0_{00}(T)=|\langle 0|e^{-i\hat{p}^2T/2}|0\rangle|^2=\frac{1}{[(1+T^2/4)]^{1/2}} . \quad (69)$$

A quick look at Eqs. (68) and (69) shows that the failure of the mean field, maximal at $T\to\infty$ where $P^0_{00}(\infty)\to 0$, results from the dispersion of the "mean-field" wave function. This dispersion comes from the free (kinetic) propagation, which should clearly be removed as in (48) in order to give a sensible mean-field description.

A similar deficiency occurs when we look for a mean-field approximation to the constant energy of the system

$$\langle\hat{H}_0\rangle(T)=\langle 0|e^{i\hat{H}_0T}\hat{H}_0e^{-i\hat{H}_0T}|0\rangle=\tfrac{1}{2} . \quad (70)$$

The free propagation then delocalizes the mean-field wave function from the center of the harmonic oscillator where it is initially peaked. Equivalently, this propagation excites the system to higher quantum numbers and, in any event, leads to the approximation

$$\langle\hat{H}_0\rangle^0(T)=\tfrac{1}{2}\left[1+\frac{T^2}{2}\right] , \quad (71)$$

which diverges as $T\to\infty$. This dramatically illustrates how, in the general case, the interaction picture removes the dispersion due to the free propagation in the mean field itself.

VI. CONCLUSION

We have proposed a time dependent mean-field theory for the expectation values of many-body observables which are constants of the unperturbed motion. Although we have treated the case of a static two-body interaction in a one-body external potential, Sec. I is easily generalized to the case of any static effective interaction which includes many-body forces in the presence of any time-dependent many-body external potential.

We have shown that the mean-field configuration depends upon the measured observable itself and that the stationarity equations are highly nonlocal in time, in contrast to the TDHF equations. Namely, the determination of the various fields at any time requires their determination at any arbitrary time between the initial and final times. We have also proposed a way of including the pairing correlations within the time dependent mean-field approximation.

As an application, we have tested our method on the analytically soluble model of the forced harmonic oscillator. The mean-field expectation values of the constants of the unperturbed motion are identical with the exact values. We have also demonstrated that the functional integral should be formulated in the interaction picture in order to eliminate the spurious mean-field dispersion.

Finally, we emphasize once more that the present scheme allows the treatment of any many-body observable which is a constant of the unperturbed motion. As a major consequence, we are able to describe, within the same functional integral formulation, exclusive measurements which involve many-body observables (such as any eigenstate projector), as well as inclusive measurements involving few-body observables (such as the Hamiltonian itself). It is evident that our description of exclusive measurements is not complete since only moduli of the S-matrix elements emerge from our theory. This is in contrast to Ref. 1, where the S-matrix elements were directly approximated through a path integral formulation which has a structure similar to that used in this work.

The present time-dependent mean-field theory is a good candidate to unify and describe the various aspects of the many-body scattering problem. In particular, it is of interest in future work to correlate the degree of time nonlocality in the determination of the mean-field configurations $\sigma_1(t)$ and $\sigma_2(t)$ with the nature of the observable itself and to establish a closer comparison with the TDHF approximation. The next step in this direction would be to make quantitative evaluations on more realistic models of scattering.

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