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Approximate capacity region of the two-pair bidirectional Gaussian relay network

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***Abstract*—We study the capacity of the Gaussian two-pair full-duplex directional (or two-way) relay network with a single-relay supporting the communication of the pairs. This network is a generalization of the well known bidirectional relay channel, where we have only one pair of users. We propose a novel transmission technique which is based on a specific superposition of lattice codes and random Gaussian codes at the source nodes. The relay attempts to decode the Gaussian codewords and the superposition of the lattice codewords of each pair. Then it forwards this information to all users. We analyze the achievable rate of this scheme and show that for all channel gains it achieves to within 2 bits/sec/Hz per user of the cut-set upper bound on the capacity region of the two-pair bidirectional relay network.**

I. Introduction

Cooperative communication and relaying is one of the main research topics in multi-user information theory. A basic model to study this problem is the 3-node relay channel which was first introduced in 1971 by van der Meulen [1] and the most general strategies for this network were developed by Cover and El Gamal [2].

While much of the focus so far is on the one-way-relay channel, bidirectional communication has also attracted attention. Bidirectional or two-way communication between two nodes was first studied by Shannon himself in [3]. Recently, there has been focus on two-way communication where an additional node acting as a relay is supporting the exchange of information between the two nodes (or one pair). Some achievable rate regions for this one-pair two-way relay channel using different strategies at the relay, such as decode-and-forward, compress-and-forward and amplify-and-forward, have been analyzed in [4]. Network coding type techniques have been proposed by [5] (and others) in order to improve the transmission rate. Similarly, in [6] the one-pair half-duplex two-way relay channel where the channel gains are all equal to one is investigated. It was shown that a combination of a decode-and-forward strategy using lattice codes and a joint decoding strategy is asymptotically optimal. Furthermore, in [7], the capacity region of the full-duplex two-way relay channel was approximated to within 3 bits/sec/Hz per user for the general case, where channel gains are all different.

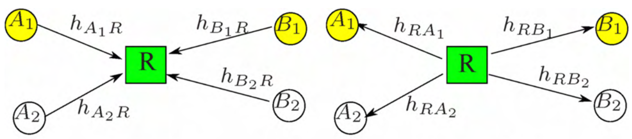
For multi-pair two-way relaying, the optimal power allocation and bit error rate analysis was investigated in [8] assuming that common spreading signatures were used by the pairs in order to distinguish themselves from the other pairs. However, so far no attempt has been done to characterize the capacity region of this network, and the optimal strategy is unknown.

In [9] we made progress on this problem by using a simpler deterministic channel model introduced in [10], which simplifies the wireless network interaction model by eliminating the noise and allows us to focus on the interaction between signals. This approach was successfully applied to the relay network in [10], and resulted in insight in terms of transmission techniques which further led to an approximate characterization of the noisy wireless relay network problem [11]. It has also been recently applied to the bidirectional relay channel problem [7], which again resulted in approximating the capacity region of the noisy (Gaussian) bidirectional relay channel. Inspired by these results, in [9] we characterized the capacity region of the multi-pair bidirectional relay network and showed that it is achieved by an *equation-forwarding* scheme, in which different pairs are orthogonalized on the signal level space and the relay just re-orders the received equations created from the superposition of the transmitted signals on the wireless medium and forwards them.

In this paper we use these insights to find a near optimal transmission technique for the Gaussian case. More specifically, we propose a specific superposition of lattice codes and random Gaussian codes at the source nodes. The relay attempts to decode the Gaussian codewords and the superposition of the lattice codewords of each pair. The relay then forwards this information to the intended destinations. We analyze the achievable rate region of this scheme and show that for all channel gains it achieves to within 2 bits/sec/Hz per user of the cut-set upper bound on the capacity region of the two-pair bidirectional relay network.

II. System model

As shown in Figure 1, we consider two single-antenna transceiver pairs, (A_1, B_1) and (A_2, B_2), communicating to each other by exploiting a relay R. The relay is operating in the full-duplex mode, i.e. it can listen and transmit at the same time. We use a complex AWGN channel model for all channels in this network. Hence, the received signals at the nodes are given by $y_R = h_{A_1R}x_{A_1} + h_{B_1R}x_{B_1} + h_{A_2R}x_{A_2} + h_{B_2R}x_{B_2} + z_R$, $y_{A_i} = h_{RA_i}x_R + z_{A_i}$, and $y_{B_i} = h_{RB_i}x_R + z_{B_i}$, with $i = 1, 2$, where x_{A_1}, x_{B_1}, x_{A_2}, x_{B_2}, and x_R are the signals transmitted from nodes A_1, B_1, A_2, B_2, and R, respectively. The transmit power constraint



(a) Uplink (b) Downlink

Fig. 1. Two-Pair bidirectional full-duplex relay network

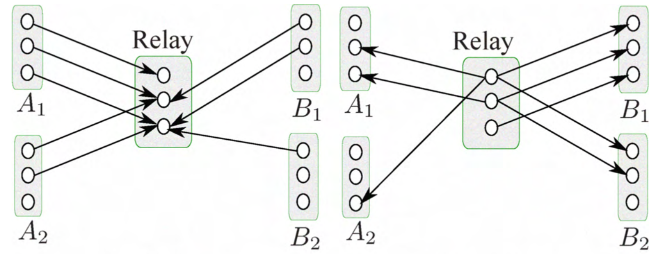
is $\mathbb{E}\left[|x_{A_i}|^2\right] = \mathbb{E}\left[|x_{B_i}|^2\right] = \mathbb{E}\left[|x_R|^2\right] \leq 1$ and the noises z_{A_1}, z_{B_1}, z_{A_2}, z_{B_2}, and z_R are all distributed as $\mathcal{CN}(0,1)$. Note that the uplink channels gains (h_{A_iR} and h_{B_iR}) are not necessarily equal to the down-link channel gains (h_{RA_i} and h_{RB_i}), i.e. channel reciprocity is not assumed. For each pair (A_i,B_i), R_{A_i} is the rate at which A_i transmit data to B_i and R_{B_i} is the transmission rate of B_i to A_i.

III. Insights from the Deterministic Multi-Pair Two Way Relay Network

In our previous work in [9] we analyzed the deterministic M-pair bidirectional relay network shown in Fig. 2 (for two pairs) based on the deterministic channel model introduced in [10]. In this figure each little circle represents a signal level and what is sent on it is a bit. The transmit and received signal levels are sorted from MSB to LSB from top to bottom. The channel gain between two nodes i and j indicates how many of the first MSB transmitted signal levels of node i are received at destination node j. For the deterministic multi-pair bidirectional relay network, we have been able in [9] to exactly identify the capacity region. More specifically, we showed that the capacity is achieved by a simple *divide and conquer* scheme. The result basically says that it is optimal to divide the signal level space and allocate these orthogonal subspaces to the different pairs. Furthermore, it suggests that the stronger user of each pair (the user with stronger uplink channel, say A_i) splits its message into two parts; The second part has the same rate as the weak user (R_{B_i}) and is transmitted at the same power level of the signal from the weak user. The first part- the remaining ($R_{A_i} - R_{B_i}$) bits- are transmitted at some higher signal levels. The same strategy is used for all other pairs at non-overlapping signal levels.

From the relay view point, four chunks of bits are received at different signal levels. Those are the bits that are created from the superposition of the signals of both users of each pair (referred to as equations in the following) or from the exclusive signals of the strong transmitter of each pair. The relay forwards these signals at non-overlapping signal levels to the end users so that the superposed signals (i.e. equations) are received by both users whereas the exclusive bits (from the strong transmitters) are received by the corresponding end users only. This way each user can easily decode its message having the received equations, received bits and what it has originally transmitted. For more details, the interested reader is referred to [9].

Going from the deterministic model to the more realistic Gaussian channel model one will face three immediate challenges. The first one is the effect of the additive noise which is the primitive of the Gaussian channels. The second



(a) Uplink (b) Downlink

Fig. 2. Deterministic model for multi-pair bidirectional relaying

issue is the power leakage from the signals of lower levels (e.g. superposition of chunks of signals) to those transmitted at higher levels; should one try to break the messages into superpositions of low power and high power signal as in the deterministic case. The third complication is to decode the equations (i.e. superposition of signals) at the relay.

We propose the following solutions to overcome these difficulties. The noise issue can be simply resolved by using an appropriate block symbol coding scheme. The leakage problem is inevitable, since in the wireless Gaussian channel the interference will always exist. However, a compensation in the capacity region allows for a leakage tolerance. In other words, rather than showing the cut-set bound is tight, we show that the cut-set upper bound is achievable to within a constant. Finally, using an appropriate lattice code, the third challenge is resolvable too. In a lattice structure, the superposition of every two codewords is also a lattice codeword and can be therefore decoded at the relay. These will be addressed in the coming sections.

IV. Two-Pair Two Way Gaussian Relay Network

In this section we analyze the capacity region of the two-pair bidirectional Gaussian relay network defined in Section II. We begin by describing the cut-set upper bound [12], denoted by $\bar{\mathcal{C}}$, on the capacity region of this network:

$$
\begin{aligned}
\bar{\mathcal{C}} = \Big\{ & (R_{A_1}, R_{B_1}, R_{A_2}, R_{B_2}) \in \mathbb{R}_+^4 : \\
& R_{A_i} \leq \min\left(C\left(|h_{A_iR}|^2\right), C\left(|h_{RB_i}|^2\right)\right) && (1)\\
& R_{B_i} \leq \min\left(C\left(|h_{B_iR}|^2\right), C\left(|h_{RA_i}|^2\right)\right) && (2)\\
R_{A_1} + R_{A_2} & \leq \min\Big(C\left(|h_{A_1R}|^2 + |h_{A_2R}|^2\right), \\
& \qquad C\left(\max\left(|h_{RB_1}|^2, |h_{RB_2}|^2\right)\right)\Big) && (3)\\
R_{B_1} + R_{B_2} & \leq \min\Big(C\left(|h_{B_1R}|^2 + |h_{B_2R}|^2\right), \\
& \qquad C\left(\max\left(|h_{RA_1}|^2, |h_{RA_2}|^2\right)\right)\Big) && (4)\\
R_{A_1} + R_{B_2} & \leq \min\Big(C\left(|h_{A_1R}| + |h_{B_2R}|^2\right), \\
& \qquad C\left(\max\left(|h_{RB_1}|^2, |h_{RA_2}|^2\right)\right)\Big) && (5)\\
R_{B_1} + R_{A_2} & \leq \min\Big(C\left(|h_{B_1R}|^2 + |h_{A_2R}|^2\right), \\
& \qquad C\left(\max\left(|h_{RA_1}|^2, |h_{RB_2}|^2\right)\right)\Big)\Big\}, && (6)
\end{aligned}
$$

where $C(x) = \log(1+x)$. Next, we define the up-link and down-link cut-set regions. The up-link cut-set region, $\mathcal{C}_u$, is the set of rates satisfying equations (1)-(6) when the down-link channel gains are assumed infinity. This means that the only restricting factors in determining the capacity regions are assumed to be the up-link channel gains. Likewise, the down-link cut-set region, $\mathcal{C}_d$, is the set of rates satisfying (1)-(6) in which the up-link channel gains are set to infinity. Note that $\bar{\mathcal{C}} = \mathcal{C}_d \cap \mathcal{C}_u$.

We say that a 4-tuple $(R_{A_1}, R_{B_1}, R_{A_2}, R_{B_2})$ is achievable if simultaneously A_i can communicate to B_i at rate R_{A_i} and B_i can communicate to A_i at rate R_{B_i} with arbitrary small error probability. The union of all achievable rate tuples is defined as the capacity region. We are now ready to state our main result.

Theorem 1: The capacity region of the two pair full-duplex bidirectional relay network is within 2 bits/sec/Hz per user of its cut-set upper bound described in (1)-(6). Or, more precisely, if $(R_{A_1}, R_{B_1}, R_{A_2}, R_{B_2}) \in \overline{\mathcal{C}}$ and $R_{A_i}, R_{B_i} \geq 2$ for $i = 1, 2$, then the rate tuple $(R_{A_1} - 2, R_{B_1} - 2, R_{A_2} - 2, R_{B_2} - 2)$ is achievable.

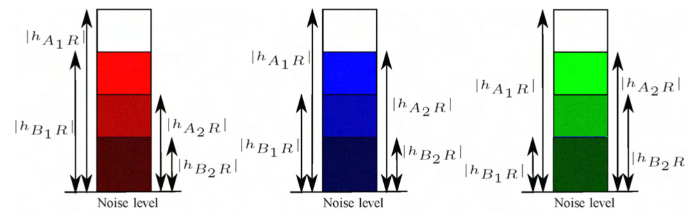
The rest of this section is devoted to proving this Theorem. First, we state the following lemma which helps us by limiting the number of rate configurations that we have to consider.

Lemma 1: Let $\mathbf{R} = (R_{A_1}, R_{B_1}, R_{A_2}, R_{B_2})$ be a rate tuple in the cut-set region $\overline{\mathcal{C}}$. Assume $R_{A_i} \geq R_{B_i}$, $i = 1, 2$. Then it is always possible to sufficiently reduce the transmit powers at the uplink and add extra noise to the received signals at the downlink, such that new effective channel gains satisfy $|\tilde{h}_{A_iR}| \geq |\tilde{h}_{B_iR}|$ and $|\tilde{h}_{RB_i}| \geq |\tilde{h}_{RA_i}|$ for $i = 1, 2$, and $\mathbf{R}$ is still in the shrunk cut-set region.

Proof: See Appendix A. ∎

This lemma basically reduces the number of relevant channel gain orderings that we have to consider in order to prove Theorem 1. Assume that the rate tuple that we want to show it is achievable (within 2 bits per user) satisfies $R_{A_i} \geq R_{B_i}$ for $i = 1, 2$. By Lemma 1, we can without loss of generality (wlog) assume that $|h_{A_iR}| \geq |h_{B_iR}|$ for $i = 1, 2$. We can also wlog assume that $|h_{A_1R}| \geq |h_{A_2R}|$ (otherwise we can re-label pair 1 and pair 2). Therefore, we only need to consider three different channel gain orderings for the uplink. Those three cases are shown in Fig. 3(a), 3(b) and 3(c). Similarly, we only need to consider three cases for the downlink.

To prove Theorem 1, first we describe the encoding strategy at the transmission nodes. As mentioned earlier, the idea is that strong transmitters of each pair split their signals into a Gaussian codeword and a lattice codeword, while the weak user only transmits a lattice codeword. While stating this encoding strategy we leave the power allocation parameters unspecified. In other words, the power level at which the user breaks up its message into the superposition of Gaussian and a lattice codeword remains as parameters. In the next step we mention the decoding at the relay where the superposition of lattice points and the Gaussian codewords are decoded. Afterwards, the relay maps each of the four decoded codewords into a random Gaussian codeword, and broadcasts their weighted superposition to all users. The last step is the decoding at the nodes, where every receiver first decodes the undesired codewords that have larger weights than the desired codewords. Thus, those codewords are decoded and successively canceled from the received signal one by one. Afterwards, both the weak and the strong receivers of each pair decode the Gaussian codeword corresponding to the lattice codeword belonging to that pair. In addition to that, the strong receivers decode one more codeword. This codeword corresponds to the Gaussian codeword, which was received by the relay from their transmitting strong counterpart. Eventually as a result of this scheme the rates that the users will successfully transmit will be a function of the power parameters that we set at the beginning. We will finally show that by choosing these parameters appropriately any rate tuple within 2 bits per user of the cut set is achievable.



(a) Case I (b) Case II (c) Case III

Fig. 3. Three relevant configurations for the uplink and their corresponding received signal at the relay. At the lowest level, all signals are superposed, while at the next level (medium shade), all but one signals are superposed. At the top level (white) only one signal remains.

A. Encoding at the nodes

Wlog assume that $R_{A_iR} \geq R_{B_iR}$. By Lemma 1 this means that we can assume $|h_{A_iR}| \geq |h_{B_iR}|$ and $|h_{RB_i}| \geq |h_{RA_i}|$. Then, the transmit signals at the nodes are given by

$$\mathbf{x}_{A_i} = \sqrt{\alpha_{A_i}^{(1)}}\mathbf{x}_{A_i}^{(1)} + \sqrt{\alpha_{A_i}^{(2)}}\mathbf{x}_{A_i}^{(2)} \ , \ \mathbf{x}_{B_i} = \sqrt{\alpha_{B_i}^{(2)}}\mathbf{x}_{B_i}^{(2)} \quad i = 1, 2$$

$$\mathbf{x}_R = \sum_{j=1}^{4} \sqrt{\alpha_R^{(j)}}\mathbf{x}_R^{(j)} \qquad \text{with } \sum_{j=1}^{4} \alpha_R^{(j)} = 1, \tag{7}$$

where $\mathbf{x}_{A_i}^{(1)}$ and $\mathbf{x}_R^{(j)}$ are codewords chosen from a random Gaussian codebook of size $2^{nR_{A_i}^{(1)}}$, $i = 1, 2$, and $2^{nR_R^{(j)}}$, for $j = 1, \ldots, 4$, respectively. $\mathbf{x}_{A_i}^{(2)}$ and $\mathbf{x}_{B_i}^{(2)}$, $i = 1, 2$, are lattice coded [6] using lattice Λ_c of dimension n, where Λ_c is a subgroup of $\mathbb{R}^n$, giving a codebook of size $2^{nR_{A_i}^{(2)}}$ and $2^{nR_{B_i}^{(2)}}$ with $i = 1, 2$, respectively. We assume that the second moment per dimension of the fundamental Voronoi region [6] of Λ_c is $1/2$. At nodes A_i we have two messages $\mathbf{m}_{A_i}^{(1)}$ and $\mathbf{m}_{A_i}^{(2)}$ of size $2^{nR_{A_i}^{(1)}}$ and $2^{nR_{A_i}^{(2)}}$ that are mapped to $x_{A_i}^{(1)}$ and $x_{A_i}^{(2)}$, respectively. In other words, the strong transmitter of each pair transmits a superposition of a lattice code and a random Gaussian code, while the weaker user only transmits a lattice code. Thus, the transmit signals of nodes B_1 and B_2 are given by $x_{B_1} = \sqrt{\alpha_{B_1}^{(2)}}x_{B_1}^{(2)} = \sqrt{\alpha_{B_1}^{(2)}}\,(t_2 - d_2) \mod \Lambda_c$, $x_{B_2} = \sqrt{\alpha_{B_2}^{(2)}}x_{B_2}^{(2)} = \sqrt{\alpha_{B_2}^{(2)}}\,(f_2 - e_2) \mod \Lambda_c$, with lattice points t_2 and f_2 and dithers d_2 and e_2 [6]. For the nodes A_1 and A_2, we have a superposition code (cf. (7)) with $x_{A_1}^{(2)} = (t_1 - d_1) \mod \Lambda_c$, $x_{A_2}^{(2)} = (f_1 - e_1) \mod \Lambda_c$ with lattice points t_1 and f_1 and dithers d_1 and e_1. Note that $t = (t_1 + t_2) \mod \Lambda_c$, $f = (f_1 + f_2) \mod \Lambda_c$ where t and f are also lattice points due to the group structure of the lattice [6]. It is important to realize that $\mathbb{E}[t] = \mathbb{E}[t_i]$, for $i = 1, 2$, and similarly for $\mathbb{E}[f]$.

The power parameters (i.e. α_{A_i} and α_{B_i}) are assigned such that the lattice codes of each pair arrive at the same power level, so that the relay can decode the sum codeword correctly. Thus we set,

$$\alpha_{A_i}^{(2)} = \frac{|h_{B_iR}|^2}{|h_{A_iR}|^2}\alpha_{B_i}^{(2)}. \tag{8}$$

Furthermore, we should have $\alpha_{A_i}^{(1)} + \alpha_{A_i}^{(2)} \leq 1$ and $\alpha_{B_i}^{(2)} \leq 1$.

B. Uplink: Decoding at the relay

Recall that as discussed in Section III and illustrated in Figure 3 we have to analyze three cases only. Here, the analysis for the first case (cf. 3(a)) is given. However, the other cases are very similar and therefore omitted. In the first case we have $|h_{A_1R}| \geq |h_{B_1R}| \geq |h_{A_2R}| \geq |h_{B_2R}|$.

The decoding order at the relay is as follows. First the relay decodes the Gaussian $x_{A_1}^{(1)}$, then the lattice point t from A_1 and B_1, followed by $x_{A_2}^{(1)}$ and finally the lattice point f from A_2 and B_2. We can show that for any choice of $\alpha_{A_i}^{(j)}$ and $\alpha_{B_i}^{(2)}$, this can be done successfully as long as,

$$R_{A_1}^{(1)} \leq C\left(\frac{|h_{A_1R}|^2\alpha_{A_1}^{(1)}}{2\alpha_{B_1}^{(2)}|h_{B_1R}|^2 + \alpha_{A_2}^{(1)}|h_{A_2R}|^2 + 2\alpha_{B_2}^{(2)}|h_{B_2R}|^2 + 1}\right) \tag{9}$$

$$R_{A_1}^{(2)}, R_{B_1} \leq \log\left(\frac{1}{2} + \frac{|h_{B_1R}|^2\alpha_{B_1}^{(2)}}{\alpha_{A_2}^{(1)}|h_{A_2R}|^2 + \alpha_{B_2}^{(2)}|h_{B_2R}|^2 + 1}\right)^+ \tag{10}$$

$$R_{A_2}^{(2)}, R_{B_2} \leq \left(\log\left(\alpha_{B_2}^{(2)}|h_{B_2R}|^2\right)\right)^+, \quad R_{A_2}^{(1)} \leq C\left(\frac{|h_{A_2R}|^2\alpha_{A_2}^{(1)}}{|h_{B_2R}|^2\alpha_{B_2}^{(2)} + 1}\right). \tag{11}$$

The structure of the above expressions results from the decoding strategy described above and the exploitation of lattice properties. Details of the derivations are omitted due tu lack of space and will be given in [13]. Now we state the following lemma whose proof is given in Appendix B.

Lemma 2: Suppose that the nodes are using the transmit strategy described in Section IV-A. Then for any 4-tuple $(r_{A_1}, r_{B_1}, r_{A_2}, r_{B_2})$ satisfying

$$r_{A_1} \leq C\left(|h_{A_1R}|^2\right) - 2\ , r_{B_1} \leq C\left(|h_{B_1R}|^2\right) - 1 \tag{12}$$
$$r_{A_2} \leq C\left(|h_{A_2R}|^2\right) - 2\ , r_{B_2} \leq C\left(|h_{B_2R}|^2\right) - 1 \tag{13}$$
$$r_{A_1} + r_{A_2} \leq C\left(|h_{A_1R}|^2 + |h_{A_2R}|^2\right) - 4 \tag{14}$$
$$r_{A_1} + r_{B_2} \leq C\left(|h_{A_1R}|^2 + |h_{B_2R}|^2\right) - 3 \tag{15}$$
$$r_{B_1} + r_{B_2} \leq C\left(|h_{B_1R}|^2 + |h_{B_2R}|^2\right) - 2 \tag{16}$$
$$r_{B_1} + r_{A_2} \leq C\left(|h_{B_1R}|^2 + |h_{A_2R}|^2\right) - 3, \tag{17}$$

there exists a choice of power assignments ($\alpha_{A_i}^{(j)}$ and $\alpha_{B_i}^{(2)}$) such that the relay can use the decoding strategy described earlier to decode the Gaussian $x_{A_i}^{(1)}$ of rate $R_{A_i}^{(1)} = r_{A_i} - r_{B_i}$, the lattice point t of rate $R_{A_1}^{(2)} = R_{B_1} = r_{B_1}$, and the lattice point f of rate $R_{A_2}^{(2)} = R_{B_2} = r_{B_2}$, with arbitrary small error probability.

C. Encoding at the relay

The relay maps the decoded $x_{A_1}^{(1)}$, t, $x_{A_2}^{(1)}$, and f to a Gaussian codeword $x_R^{(1)}$ of size $2^{nR_{A_1}^{(1)}}$, $x_R^{(2)}$ of size $2^{nR_{B_1}}$, $x_R^{(3)}$ of size $2^{nR_{A_2}^{(1)}}$, and $x_R^{(4)}$ of size $2^{nR_{B_2}}$, respectively.

D. Downlink: Decoding at the nodes

As in the uplink, we have to consider three cases only, from which we provide the detailed analysis for $|h_{RB_1}| \geq |h_{RA_1}| \geq |h_{RB_2}| \geq |h_{RA_2}|$. The other cases follow similar lines of arguments.

The relay uses a superposition of four messages. One message is decoded by all users. Another message is decoded by both users of the first pair and the strong receiver of the second pair. Yet another message is decoded by only the strong receiver of the first pair, and finally the remaining message is decoded by both users of the first pair. We can show that for any choice of $\alpha_{A_i}^{(j)}$ and $\alpha_{B_i}^{(2)}$, this can be done successfully as long as,

$$R_{A_1}^{(2)}, R_{B_1} \leq \min\left(C\left(\frac{|h_{RB_1}|^2\alpha_R^{(2)}}{1 + |h_{RB_1}|^2\alpha_R^{(1)}}\right), C\left(|h_{RA_1}|^2\alpha_R^{(2)}\right)\right),$$

$$R_{A_2}^{(2)}, R_{B_2} \leq \min\left(C\left(\frac{|h_{RB_2}|^2\alpha_R^{(4)}}{1 + |h_{RB_2}|^2\sum_{j=1}^{3}\alpha_R^{(j)}}\right), C\left(\frac{|h_{RA_2}|^2\alpha_R^{(4)}}{1 + |h_{RA_2}|^2\left(\alpha_R^{(1)} + \alpha_R^{(2)}\right)}\right)\right),$$

$$R_{A_1}^{(1)} \leq C\left(|h_{RB_1}|^2\alpha_R^{(1)}\right), \quad R_{A_2}^{(1)} \leq C\left(\frac{|h_{RB_2}|^2\alpha_R^{(3)}}{1 + |h_{RB_2}|^2\left(\alpha_R^{(1)} + \alpha_R^{(2)}\right)}\right).$$

Details of the derivation are given in [13]. Now we state the following lemma whose proof is very similar to the proof of Lemma 2 and hence omitted due to lack of space.

Lemma 3: Suppose that the relay is using the transmit strategy described above. Then for any 4-tuple $(r_{A_1}, r_{B_1}, r_{A_2}, r_{B_2})$ satisfying

$$r_{A_1} \leq C\left(|h_{RB_1}|^2\right) - 2\ , r_{B_1} \leq C\left(|h_{RA_1}|^2\right) - 2 \tag{18}$$
$$r_{A_2} \leq C\left(|h_{RB_2}|^2\right) - 2\ , r_{B_2} \leq C\left(|h_{RA_2}|^2\right) - 2 \tag{19}$$
$$r_{A_1} + r_{A_2} \leq C\left(\max\left(|h_{RB_1}|^2, |h_{RB_2}|^2\right)\right) - 3 \tag{20}$$
$$r_{A_1} + r_{B_2} \leq C\left(\max\left(|h_{RB_1}|^2, |h_{RA_2}|^2\right)\right) - 3 \tag{21}$$
$$r_{B_1} + r_{B_2} \leq C\left(\max\left(|h_{RA_1}|^2, |h_{RA_2}|^2\right)\right) - 3 \tag{22}$$
$$r_{B_1} + r_{A_2} \leq C\left(\max\left(|h_{RA_1}, |h_{RB_2}|^2\right)\right) - 3 \tag{23}$$

there exists a choice of power assignments ($\alpha_R^{(j)}$'s) such that B_1 can decode the Gaussian codewords $x_R^{(1)}$ of rate $R_{A_1}^{(1)} = r_{A_1} - r_{B_1}$, A_1 and B_1 can both decode the Gaussian codeword

$x_R^{(2)}$ of rate $R_{A_1}^{(2)} = R_{B_1} = r_{B_1}$, B_2 can decode the Gaussian codeword $x_R^{(3)}$ of rate $R_R^{(3)} = r_{A_2} - r_{B_2}$, and A_2 and B_2 can both decode the Gaussian codeword $x_R^{(4)}$ of rate $R_{A_2}^{(2)} = R_{B_2} = r_{B_2}$, with arbitrary small error probability.

Now note that if

$$(R_{A_1}, R_{B_1}, R_{A_2}, R_{B_2}) \in \overline{\mathcal{C}}$$

and $R_{A_i}, R_{B_i} \geq 2$ for $i = 1, 2$, then the rate tuple

$$(r_{A_1}, r_{B_1}, r_{A_2}, r_{B_2}) = (R_{A_1} - 2, R_{B_1} - 2, R_{A_2} - 2, R_{B_2} - 2)$$

satisfies the conditions of both Lemma 2 and 3. Therefore by the proposed strategy the rate tuple $(R_{A_1} - 2, R_{B_1} - 2, R_{A_2} - 2, R_{B_2} - 2)$ is achievable, and this completes the proof of Theorem 1.

V. Conclusion

Based on insights from a recently proposed deterministic channel model, we proposed a transmission strategy for the Gaussian two-pair two-way full-duplex relay network and found an approximate characterization of the capacity region. In fact, we proposed a specific superposition coding scheme that achieves to within 2 bits per user of the cut-set upper bound on the capacity of the two-pair two-way relay network. Possible directions for future work are the extension to the half-duplex mode as well as the generalization to $M > 2$ pairs.

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Appendix A
Proof of Lemma 1

Since the proof for both pairs are similar, we only bring the proof for pair $i = 1$. We claim that if $|h_{B_1R}| > |h_{A_1R}|$ and $\mathbf{R} \in \mathcal{C}_u$, then $\mathbf{R} \in \mathcal{C}'_u$, where $\mathcal{C}'_u$ is the up-link cut-set region of the network resulted by weakening $|h_{B_1R}|$ and setting it equal to $|h_{A_1R}|$. We call the new (undermined) uplink channel gains $(h'_{A_1R}, h'_{B_1R}, h'_{A_2R}, h'_{B_2R})$. The claim is justified by check marking equations (1) to (6) for new capacities (with infinite down-link channel gains). The only non-obvious inequalities are the ones in which h'_{B_1R} appears. By symmetry we only have to verify that (2) and (6) hold. Start with the original equations for $(h_{A_1R}, h_{B_1R}, h_{A_2R}, h_{B_2R})$ and note that the LHS of equations (2) and (6) are less than or equal to the LHS of (1) and (3) respectively and thus less than their RHS. Now replace h_{A_1R} with h'_{B_1R} and h_{A_2R} with h'_{A_2R} to get the desired inequalities. A similar argument on the down-link cut-set region shows that we can make the down-link channel gains of each pair consistent (in ordering) with the transmission rate and this completes the proof.

Appendix B
Proof of Lemma 2

Consider a 4-tuple $(r_{A_1}, r_{B_1}, r_{A_2}, r_{B_2})$ satisfying (12)-(17). Starting with (11), we equate

$$\left(\log\left(\alpha_{B_2}^{(2)}|h_{B_2R}|^2\right)\right)^+ = r_{B_2} \Rightarrow \alpha_{B_2}^{(2)} = \frac{2^{r_{B_2}}}{|h_{B_2R}|^2}. \tag{24}$$

Now from (13) we know that

$$\alpha_{B_2}^{(2)} \leq \frac{1+|h_{B_2R}|^2}{2|h_{B_2R}|^2} \overset{|h_{B_2R}|\geq 1}{\leq} 1, \tag{25}$$

which shows that this is a valid choice of $\alpha_{B_2}^{(2)}$. Next we equate $r_{A_2} - r_{B_2} =$ RHS of (11), by setting

$$\alpha_{A_2}^{(1)} = \frac{\left(2^{r_{A_2}-r_{B_2}} - 1\right)\left(2^{r_{B_2}}+1\right)}{|h_{A_2R}|^2}. \tag{26}$$

Using (8) and (24) and adding this to (26) we get

$$\alpha_{A_2}^{(1)} + \alpha_{A_2}^{(2)} = \frac{2^{r_{A_2}} + 2^{r_{A_2}-r_{B_2}} - 1}{|h_{A_2R}|^2} \leq \frac{2^{r_{A_2}+1}-1}{|h_{A_2R}|^2} \overset{(13)}{\leq} 1, \tag{27}$$

verifying that this is a valid choice of $\alpha_{A_2}^{(1)}$, $\alpha_{A_2}^{(2)}$. Then we equate $r_{B_1} =$ RHS of (10), by setting

$$\alpha_{B_1}^{(2)} = \frac{\left(2^{r_{B_1}} - \frac{1}{2}\right)2^{r_{A_2}-r_{B_2}}\left(2^{r_{B_2}}+1\right)}{|h_{B_1R}|^2} \leq \frac{2^{r_{B_1}+r_{A_2}+1}-1}{|h_{B_1R}|^2} \overset{(17)}{\leq} 1, \tag{28}$$

verifying that this is a valid choice of $\alpha_{B_1}^{(2)}$. Finally we equate $r_{A_1} =$ RHS of (9), by setting

$$\alpha_{A_1}^{(2)} = \frac{\left(2^{r_{A_1}-r_{B_1}} - 1\right)\left(2^{r_{A_2}+r_{B_1}+1}\left(1+2^{-r_{B_2}}\right)+2^{r_{B_2}}\right)}{|h_{A_1R}|^2}. \tag{29}$$

Using (8) and (28) and adding this to (29) we get

$$\alpha_{A_1}^{(1)} + \alpha_{A_1}^{(2)} = \frac{\left(2^{r_{B_1}} - \frac{1}{2}\right)2^{r_{A_2}-r_{B_2}}\left(2^{r_{B_2}}+1\right)}{|h_{A_1R}|^2} + \left(2^{r_{A_1}-r_{B_1}} - 1\right)$$
$$\frac{\left(2^{r_{A_2}+r_{B_1}+1}\left(1+2^{-r_{B_2}}\right)+2^{r_{B_2}}\right)}{|h_{A_1R}|^2} \leq \frac{5\cdot 2^{r_{A_1}+r_{A_2}}-1}{|h_{A_1R}|^2} \overset{(14)}{\leq} 1.$$

which shows that this is a valid choice of $\alpha_{A_1}^{(1)}$, $\alpha_{A_1}^{(2)}$.