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Upper Bounds for Mixed H^2/H^∞ Control[1]

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Abstract

We consider the mixed H^2/H^∞ control problem of choosing a controller to minimize the H^2 norm of a given closed-loop map, subject to the H^∞ norm of another closed-loop map being less than a prescribed value γ. Let d_2 and γ_2 denote the H^2 and H^∞ norms for the pure H^2-optimal solution (without any H^∞ constraint), and let d_c and $\gamma_c < \gamma$ denote the H^2 and H^∞ norms for *any* solution that yields an H^∞ norm strictly less than γ (such as, say, the central solution). Then if d_m denotes the optimal H^2 norm that can be achieved in the mixed H^2/H^∞ control problem, we show that

$$\frac{d_m^2 - d_2^2}{d_c^2 - d_2^2} \leq \left(\frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_c}\right)^2 < \left(\frac{\gamma_2^2 - \gamma^2}{\gamma_2^2 - \gamma_c^2}\right)^2 < 1.$$

1 Introduction

The problem of mixed H^2/H^∞ control has received considerable attention in the last few years [1, 2, 3, 4, 5, 6, 7]. The main objective in this area is to design controllers that are robust and that, at the same time, have optimal average performance. Unfortunately, unlike the H^2 and H^∞ control problems, the mixed H^2/H^∞ control problem has proven to be surprisingly difficult, and in [8, 9] it has been shown that, except for some trivial cases, the solution is nonrational so that finite-dimensional state-space solutions do not exist. Therefore to date, most authors have either resorted to finding suboptimal solutions with finite-dimensional state-space structure [3, 5, 10], or to considering auxiliary problems where the objective H^2 cost is replaced by a suitable upper bound [1, 2, 4, 6].

In this paper, we develop a simple upper bound formula for the achievable H^2 cost in the mixed H^2/H^∞ control problem. This bound is of importance for two reasons: first, it allows for a performance comparison with the solutions obtained by finite-dimensional numerical optimization [3, 5, 10] and/or via auxiliary costs [1, 2, 4, 6], and, second, it allows one to obtain a quick "feel" for how much improvement in H^2 performance is possible within the set of all controllers satisfying a certain H^∞ bound.

We should remark that we are not aware of similar bounds in the literature. The only ones that we are aware of are those of [11, 12] that give an upper bound on the H^2 norm of the *central* controller, and are not really concerned with the best achievable H^2 norm for other controllers.

To describe the upper bounds, consider the following mixed H^2/H^∞ problem:

$$\begin{cases} \min_{Q(\cdot)\in H^\infty} & \left\|T_Q^{(2)}(z)\right\|_2 \\ \text{subject to} & \left\|T_Q^{(\infty)}(z)\right\|_\infty \leq \gamma \end{cases} . \quad (1)$$

Here $T_Q^{(2)}(z)$ represents the transfer matrix whose H^2 cost we are attempting to minimize, and $T_Q^{(\infty)}(z)$ represents the transfer matrix whose robustness we are required to ensure. These two transfer matrices are functions of a common transfer matrix $Q(z) \in H^\infty$ which can be thought of as the controller. The mixed H^2/H^∞ problem has two major instances: the four-block problem, which arises in measurement-feedback control, where[1]

$$T_Q^{(i)}(z) = \begin{bmatrix} P_{11}^{(i)} + P_{12}^{(i)}Q(z)P_{21}^{(i)}(z) & P_{12}^{(i)}Q(z) \\ Q(z)P_{21}^{(i)}(z) & Q(z) \end{bmatrix}, \quad (2)$$

with $P_{jk}^{(i)} \in H^\infty$, $i = 2, \infty$, $j,k = 1,2$, and the two-block problem, which arises in estimation and in full-information control, where

$$T_Q^{(i)}(z) = \begin{bmatrix} P_1^{(i)}(z) + P_2^{(i)}(z)Q(z) & Q(z) \end{bmatrix}, \quad (3)$$

with $P_j^{(i)}(z) \in H^\infty$, $i = 2, \infty$, $j = 1, 2$. In what follows, for brevity, we shall only consider the four-block

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[1] The four-block problem mentioned here follows [13], Chapter 11, and is slightly different from the one commonly referred to in the literature (such as in [14]). However, our results also hold for the latter four-block problem, as we expalin in Sec. 4, Remark (vi).

case. Our results also hold for the two-block case, since similar arguments can be used.

We shall make two assumptions on γ. The first is that $\gamma > \gamma_\infty$, where

$$\gamma_\infty \triangleq \inf_{Q(\cdot)\in H^\infty} \left\| T_Q^{(\infty)}(z) \right\|_\infty . \tag{4}$$

This ensures that the set of controllers for which $\left\|T_Q^{(\infty)}(z)\right\|_\infty \le \gamma$ is non-empty, so that problem (1) does indeed have a solution.

The second is that $\gamma < \gamma_2 \triangleq \left\|T_{Q_2}^{(\infty)}(z)\right\|_\infty$, where $Q_2(z) \in H^\infty$ is the solution to the unconstrained H^2 problem, *i.e.*,

$$Q_2(z) \triangleq \arg \min_{Q(\cdot)\in H^\infty} \left\| T_{Q_2}^{(\infty)}(z) \right\|_2 . \tag{5}$$

This ensures that the constraint in problem (1) is indeed active, so that the solution is not trivially equal to $Q_2(z)$.

With these assumptions, we have the following result.

Theorem 1 (Simple Upper Bound) *Consider the mixed H^2/H^∞ problem (1) and assume that $\gamma_2 > \gamma > \gamma_\infty$. Let $Q_c(z) \in H^\infty$ be a transfer matrix that satisfies $\left\|T_{Q_c}^{(\infty)}(z)\right\|_\infty < \gamma$, and define*

$$d_c \triangleq \left\|T_{Q_c}^{(\infty)}(z)\right\|_2 \quad \text{and} \quad \gamma_c \triangleq \left\|T_{Q_c}^{(\infty)}(z)\right\|_\infty < \gamma.$$

Then, if we denote the optimal H^2 cost in problem (1) by d_m, we have

$$\frac{d_m^2 - d_2^2}{d_c^2 - d_2^2} \le \left(\frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_c}\right)^2 < \left(\frac{\gamma_2^2 - \gamma^2}{\gamma_2^2 - \gamma_c^2}\right)^2 < 1. \tag{6}$$

Note that in the above theorem, the upper bound in (6) is valid for "any" transfer matrix $Q_c(z) \in H^\infty$ satisfying $\left\|T_{Q_c}^{(\infty)}(z)\right\|_\infty < \gamma$. One example is the so-called *central* controller corresponding to γ, for which it is wellknown that $\gamma_c < \gamma$. In this case, (6) can be rearranged as

$$d_m^2 \le d_2^2 + \left(\frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_c}\right)^2 (d_c^2 - d_2^2). \tag{7}$$

The upper bound on the RHS is readily computable since d_2, γ_2, d_c, and γ_c are easy to compute and γ is given.

Note, moreover, that the ratio $\frac{d_m^2-d_2^2}{d_c^2-d_2^2}$ represents the relative improvement in the H^2 norm of the mixed H^2/H^∞ optimal solution, compared to the H^2 norm of *any* controller satisfying $\left\|T_{Q_c}^{(\infty)}(z)\right\|_\infty < \gamma$. The ratio $\frac{\gamma_2^2-\gamma^2}{\gamma_2^2-\gamma_c^2}$, on the other hand, represents how much room for "maneuvering" the H^∞ norm is available to the mixed H^2/H^∞ optimal solution, compared to *any* solution satisfying $\left\|T_{Q_c}^{(\infty)}(z)\right\|_\infty < \gamma$. Clearly, since

$$d_2 < d_m < d_c \quad \text{and} \quad \gamma_2 > \gamma > \gamma_c \tag{8}$$

both ratios are strictly less than one. However, the bound (6) shows that

$$\frac{d_m^2 - d_2^2}{d_c^2 - d_2^2} < \left(\frac{\gamma_2^2 - \gamma^2}{\gamma_2^2 - \gamma_c^2}\right)^2 . \tag{9}$$

In other words, the improvement in the H^2 norm performance is quadratically better than the room available for maneuver in the H^∞ norm. This suggests that the mixed H^2/H^∞ problem is a reasonable one, and that, in the trade-off between H^2 and H^∞ performances, it is possible to obtain significant improvement in the H^2 norm without incurring too large a loss in the H^∞ performance.

In fact, we show the stronger result that

$$\frac{d_m^2 - d_2^2}{d_c^2 - d_2^2} \le \left(\frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_c}\right)^2 . \tag{10}$$

The fact that this is a tighter upper bound follows from

$$\left(\frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_c}\right)^2 < \left(\frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_c}\right)^2 \underbrace{\left(\frac{\gamma_2 + \gamma}{\gamma_2 + \gamma_c}\right)^2}_{>1} = \left(\frac{\gamma_2^2 - \gamma^2}{\gamma_2^2 - \gamma_c^2}\right)^2 .$$

Therefore in what follows we shall focus on proving (10). However, before doing so, it will be illustrating to obtain an alternative upper bound.

2 Alternative Upper Bound

The alternative upper bound that we are about to present, uses the triangular inequality for norms. To this end, let $Q_2(z)$ denote the H^2-optimal solution, $Q_c(z)$ denote a solution satisfying $\left\|T_{Q_c}^{(\infty)}(z)\right\|_\infty < \gamma$, and consider the convex combination

$$Q_\alpha(z) = (1-\alpha)Q_2(z) + \alpha Q_c(z), \quad 0 \le \alpha \le 1. \tag{11}$$

Since $T_Q^{(i)}$, $i = 2, \infty$ is affine in $Q(\cdot)$, for this choice of solution, we have

$$T_{Q_\alpha}^{(i)} = (1-\alpha)T_{Q_2}^{(i)} + \alpha T_{Q_c}^{(i)}, \quad i = 2, \infty \tag{12}$$

so that

$$\begin{aligned} \left\|T_{Q_\alpha}^{(2)}\right\|_2 &\le (1-\alpha)\left\|T_{Q_2}^{(2)}\right\|_2 + \alpha\left\|T_{Q_c}^{(2)}\right\|_2 \\ &= (1-\alpha)d_2 + \alpha d_c, \end{aligned}$$

and

$$\left\|T^{(\infty)}_{Q_\alpha}\right\|_\infty \le (1-\alpha)\left\|T^{(\infty)}_{Q_2}\right\|_\infty + \alpha\left\|T^{(\infty)}_{Q_c}\right\|_\infty = (1-\alpha)\gamma_2 + \alpha\gamma_c.$$

It thus follows that

$$d_m \le \left\{ \begin{array}{ll} \min_{0\le\alpha\le 1} & (1-\alpha)d_2 + \alpha d_c \\ \text{subject to} & (1-\alpha)\gamma_2 + \alpha\gamma_c \le \gamma \end{array} \right., \quad (13)$$

since, compared to problem (1), we are minimizing an upper bound on the H^2 cost over a smaller constraint set.

Now, since $d_2 < d_c$, the objective cost on the RHS of (13), $d_2 + \alpha(d_c - d_2)$, is an increasing function of α, and so to minimize it we need to find the smallest value of $\alpha \in [0,1]$ that satisfies $(1-\alpha)\gamma_2 + \alpha\gamma_c \le \gamma$. The corresponding value is clearly $\alpha_{opt} = \frac{\gamma_2-\gamma}{\gamma_2-\gamma_c}$, for which we obtain

$$\begin{aligned} d_m &\le (1 - \frac{\gamma_2-\gamma}{\gamma_2-\gamma_c})d_2 + \frac{\gamma_2-\gamma}{\gamma_2-\gamma_c}d_c \\ &= d_2 + \frac{\gamma_2-\gamma}{\gamma_2-\gamma_c}(d_c - d_2). \end{aligned}$$

We thus have the following result.

Lemma 1 (Alternative Upper Bound) *Consider the setting of Theorem 1. Then we have*

$$\frac{d_m - d_2}{d_c - d_2} \le \frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_c}. \quad (14)$$

Although the bounds obtained in Theorem 1 and Lemma 1 bear certain resemblances, the bounds of Theorem 1 are tighter. Indeed:

Lemma 2 (Comparison of Upper Bounds) *Consider the setting of Theorem 1 and Lemma 1. Then we have*

$$d_2^2 + \left(\frac{\gamma_2-\gamma}{\gamma_2-\gamma_c}\right)^2 (d_c^2 - d_2^2) \le \left[d_2 + \frac{\gamma_2-\gamma}{\gamma_2-\gamma_c}(d_c - d_2)\right]^2, \quad (15)$$

so that the bound of Theorem 1 is tighter than the bound of Lemma 1.

Proof: Subtracing the bound of Theorem 1 from the (squared) bound of Lemma 2, and after some algebraic simplification, we obtain

$$\left[d_2 + \tfrac{\gamma_2-\gamma}{\gamma_2-\gamma_c}(d_c - d_2)\right]^2 - \left[d_2^2 + \left(\tfrac{\gamma_2-\gamma}{\gamma_2-\gamma_c}\right)^2(d_c^2 - d_2^2)\right] =$$

$$2d_2(d_c - d_2)\tfrac{\gamma_2-\gamma}{\gamma_2-\gamma_c}\left(1 - \tfrac{\gamma_2-\gamma}{\gamma_2-\gamma_c}\right) \ge 0,$$

since all the factors on the RHS are non-negative.

■

The reason why the upper bound of Lemma 1 is not as tight as the one of Theorem 1 is that we have not used the facts that the norms in the objective and constraint of problem (1) are H^2 and H^∞ norms. Indeed the arguments leading to Lemma 1 apply for *any* norms, since we only made use of the triangular inequality. When we make explicit use of the fact that the objective cost is an H^2 norm, then we obtain the result of Theorem 1.

3 Proof of Theorem 1

The proof of Theorem 1 uses the following basic intermediate result.

Lemma 3 (H^2 Cost) *Let $Q_2(z)$ denote the H^2-optimal solution of $\mathrm{argmin}_{Q(\cdot)\in H^\infty}\left\|T^{(2)}_Q(z)\right\|_2$, $Q_c(z)$ denote a solution satisfying $\left\|T^{(\infty)}_{Q_c}(z)\right\|_\infty < \gamma$, and consider the convex combination*

$$Q_\alpha(z) = (1-\alpha)Q_2(z) + \alpha Q_c(z), \quad 0 \le \alpha \le 1.$$

Then we have

$$\begin{aligned} \left\|T^{(2)}_{Q_\alpha}\right\|_2^2 &= (1-\alpha^2)\left\|T^{(2)}_{Q_2}\right\|_2^2 + \alpha^2\left\|T^{(2)}_{Q_c}\right\|_2^2 \\ &= (1-\alpha^2)d_2^2 + \alpha^2 d_c^2. \end{aligned} \quad (16)$$

Proof: Defining the canonical spectral factorizations:

$$\begin{aligned} M_1^*(z^{-*})M_1(z) &= I + P_{12}^{(2)*}(z^{-*})P_{12}^{(2)}(z) \\ N_1(z)N_1^*(z^{-*}) &= I + P_{12}^{(2)}(z)P_{12}^{(2)*}(z^{-*}) \end{aligned}$$

with $M_1(z)$, $M_1^{-1}(z)$, $N_1(z)$, $N_1^{-1}(z) \in H^\infty$, and

$$\begin{aligned} M_2^*(z^{-*})M_2(z) &= I + P_{21}^{(2)*}(z^{-*})P_{21}^{(2)}(z) \\ N_2(z)N_2^*(z^{-*}) &= I + P_{21}^{(2)}(z)P_{21}^{(2)*}(z^{-*}) \end{aligned}$$

with $M_2(z)$, $M_2^{-1}(z)$, $N_2(z)$, $N_2^{-1}(z) \in H^\infty$, it is straightforward to see that the transfer matrices

$$\Theta_1(z) \triangleq \begin{bmatrix} M_1^{-*}(z^{-*})P_{12}^{(2)*}(z^{-*}) & M_1^{-*}(z^{-*}) \\ -N_1^{-1}(z) & N_1^{-1}(z)P_{12}^{(2)}(z) \end{bmatrix}$$

$$\Theta_2(z) \triangleq \begin{bmatrix} P_{21}^{(2)*}(z^{-*})N_2^{-*}(z^{-*}) & -M_2^{-1}(z) \\ N_2^{-*}(z^{-*}) & P_{21}^{(2)}(z)M_2^{-1}(z) \end{bmatrix}$$

are unitary, since $\Theta_1(z)\Theta^*(z^{-*}) = I$ and $\Theta_2^*(z^{-*})\Theta_2(z) = I$. With these definitions, some simple algebra shows that

$$\Theta_1 T_Q^{(2)} \Theta_2 =$$

$$\begin{bmatrix} \underbrace{M_1^{-*}P_{12}^{(2)*}P_{11}^{(2)}P_{21}^{(2)*}N_2^{-*}}_{\triangleq S} + M_1QN_2 & -M_1^{-*}P_{12}^{(2)*}P_{11}^{(2)}M_2^{-1} \\ -N_1^{-1}P_{11}^{(2)}P_{21}^{(2)*}N_2^{-*} & N_1^{-1}P_{11}^{(2)}M_2^{-1} \end{bmatrix}, \tag{17}$$

where, for notational simplicity, we have suppressed the dependencies on z. Now since $\Theta_1(z)$ and $\Theta_2(z)$ are unitary, we have

$$\left\|T_Q^{(2)}\right\|_2 = \left\|\Theta_1 T_Q^{(2)} \Theta_2\right\|_2.$$

Moreover, since only the $(1,1)$ block entry of (17) depends on $Q(z)$, we have

$$\left\|T_Q^{(2)}\right\|_2^2 = \|S + M_1QN_2\|_2^2 + \text{ terms independent of } Q.$$

Now if we define $\{A(z)\}_+$ as the causal part of $A(z)$ (*i.e.*, the part analytic in $|z| \geq 1$), and define $\{A(z)\}_- \triangleq A(z) - \{A(z)\}_+$, it is straightforward to see that $\|A(z)\|_2^2 = \|\{A(z)\}_+\|_2^2 + \|\{A(z)\}_-\|_2^2$. Thus, since $\{M_1(z)Q(z)N_2(z)\}_- = 0$ (all transfer matrices in the argument are analytic in $|z| \geq 1$), we may write

$$\begin{aligned}\left\|T_Q^{(2)}\right\|_2^2 &= \|\{S\}_+ + M_1QN_2\|_2^2 + \|\{S\}_-\|_2^2 \\ &\quad + \text{terms independent of } Q.\end{aligned}$$

Moreover, since only the first term on the RHS of the above expression depends on $Q(z)$ it is straightforward to see that the H^2-optimal solution is given by

$$Q_2(z) = -M_1^{-1}(z)\,\{S(z)\}_+\,N_2^{-1}(z), \tag{18}$$

and that

$$\left\|T_Q^{(2)}\right\|_2^2 = \|\{S\}_+ + M_1QN_2\|_2^2 + d_2^2. \tag{19}$$

Specializing this last expression for $Q(z) = Q_\alpha(z)$ yields

$$\begin{aligned}\left\|T_{Q_\alpha}^{(2)}\right\|_2^2 &= \|\{S\}_+ + (1-\alpha)M_1Q_2N_2 + \alpha M_1Q_cN_2\|_2^2 + d_2^2 \\ &= \|\alpha\{S\}_+ + \alpha M_1Q_cN_2\|_2^2 + d_2^2 \\ &= \alpha^2\left(\|\{S\}_+ + M_1QN_2\|_2^2 + d_2^2\right) + (1-\alpha^2)d_2^2 \\ &= \alpha^2 d_c^2 + (1-\alpha^2)d_2^2,\end{aligned}$$

where in the second step we used (18) which states that $\{S\}_+ + M_1Q_2N_2 = 0$. But the last equality is our desired result.

■

Using the result of Lemma 3, it follows that

$$d_m^2 \leq \left\{ \begin{array}{ll} \min_{0\leq\alpha\leq 1} & (1-\alpha^2)d_2^2 + \alpha^2 d_c^2 \\ \text{subject to} & (1-\alpha)\gamma_2 + \alpha\gamma_c \leq \gamma \end{array} \right., \tag{20}$$

since, as in (13), we are minimizing the H^2 cost over a smaller constraint set.

Since $d_2 < d_c$, the objective cost on the RHS of (20), $d_2^2 + \alpha^2(d_c^2 - d_2^2)$, is an increasing function of α. Therefore to minimize it we need to find the smallest value of $\alpha \in [0,1]$ that satisfies $(1-\alpha)\gamma_2 + \alpha\gamma_c \leq \gamma$, which is readily seen to be $\alpha_{opt} = \frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_c}$. For this value of α we obtain

$$\begin{aligned} d_m^2 &\leq \left(1 - \left(\frac{\gamma_2-\gamma}{\gamma_2-\gamma_c}\right)^2\right) d_2^2 + \left(\frac{\gamma_2-\gamma}{\gamma_2-\gamma_c}\right)^2 d_c^2 \\ &= d_2^2 + \left(\frac{\gamma_2-\gamma}{\gamma_2-\gamma_c}\right)^2 (d_c^2 - d_2^2), \end{aligned}$$

or, equivalently,

$$\frac{d_m^2 - d_2^2}{d_c^2 - d_2^2} \leq \left(\frac{\gamma_2-\gamma}{\gamma_2-\gamma_c}\right)^2,$$

which is what we set out to prove.

4 Conclusion and Further Remarks

In this paper we developed a simple upper bound for the best achievable H^2 cost in mixed H^2/H^∞ control problems. This result is of importance since it allows one to check the performance of the mixed H^2/H^∞ solutions currently available in the literature that use either finite-dimensional numerical optimization techniques or auxiliary costs, rather than the objective H^2 cost. Moreover, it allows one to obtain a quick "feel" for how much improvement in H^2 performance is possible within the set of all controllers that satisfy a certain H^∞ bound. In this regard, the bound shows that the possible improvement in the H^2 norm performance is quadratically better than the room available for maneuver in the H^∞ norm. This suggests that the tradeoff between the H^2 and H^∞ performances is a favorable one: it is possible to obtain significant improvement in the H^2 norm without incurring too large a loss in the H^∞ performance.

Remarks:

(i) Our proof of (6) also constructed a solution that achieves the upper bound, namely,

$$Q_\alpha(z) = \left(1 - \frac{\gamma_2-\gamma}{\gamma_2-\gamma_c}\right) Q_2(z) + \frac{\gamma_2-\gamma}{\gamma_2-\gamma_c} Q_c(z), \tag{21}$$

where $Q_2(z)$ is the H^2-optimal solution and $Q_c(z)$ is any solution yielding γ_c and d_c.

(ii) The upper bound (6) can be used with *any* solution satisfying $\|T_Q^{(\infty)}(z)\|_\infty < \gamma$, and its corresponding values of γ_c and d_c. One natural choice is the central solution corresponding to the value of γ, for which $Q_c(z)$ and the values $\gamma_c < \gamma$ and $d_c > d_2$ are readily computable. However, how tight the upper bound is depends on the specific $Q_c(\cdot)$ chosen. Although it is easy to argue that there exists a $Q_c(\cdot)$ that achieves the optimal value d_m (to see this simply choose

$$Q_c(z) = \frac{\gamma_2 - \gamma_c}{\gamma_2 - \gamma} Q_m(z) - \left(\frac{\gamma_2 - \gamma_c}{\gamma_2 - \gamma} - 1\right) Q_2(z),$$

where $Q_m(\cdot)$ is the optimal mixed H^2/H^∞ solution) the problem is how to come up with such a choice ($Q_m(\cdot)$ is obviously not known a priori for the above choice to be viable).

Therefore it would be useful and interesting to study (via simulation or otherwise) how tight the upper bound of (6) is for the central controller, or for controllers randomly chosen from the set $\|T_{Q_c}(z)\|_\infty < \gamma$.

(iii) Another possible choice is $Q_\infty(z)$, the H^∞-optimal solution $Q_\infty(z) = \mathrm{argmin}_{Q(\cdot)\in H^\infty} \|T_Q^{(\infty)}(z)\|_\infty$, with corresponding H^2 and H^∞ norms d_∞ and γ_∞. For this choice, (6) specializes to

$$d_m^2 \le d_2^2 + \frac{d_\infty^2 - d_2^2}{(\gamma_2 - \gamma_\infty)^2}(\gamma_2 - \gamma)^2, \qquad (22)$$

which shows a very explicit quadratic relation between d_m and γ.

(iv) We should also mention that the bound (6) can be used for any mixed-norm problem, as long as the objective cost is the H^2 norm. In other words, (6) is valid for any problem of the type

$$\left\{ \begin{array}{ll} \min_{Q(\cdot)\in H^\infty} & \left\|T_Q^{(2)}(z)\right\|_2 \\ \text{subject to} & \|T_Q(z)\| \le \gamma \end{array} \right. , \qquad (23)$$

where $\|T_Q(z)\|$ represents *any* norm on the transfer matrix $T_Q(z)$. The reason is that in the proof of Theorem 1, the only property of $\|\cdot\|_\infty$ that we used was the triangular inequality, which is valid for any norm.

(v) Moreover, using the approach of this paper, it is also possible to study a "reverse" mixed H^2/H^∞ problem of the form

$$\left\{ \begin{array}{ll} \min_{Q(\cdot)\in H^\infty} & \left\|T_Q^{(\infty)}(z)\right\|_\infty \\ \text{subject to} & \left\|T_Q^{(2)}(z)\right\|_2 \le d \end{array} \right. . \qquad (24)$$

Denoting the best achievable H^∞ norm in problem (24) by γ_m, a similar argument to the one presented in the proof of Theorem 1, or a rearrangement of (22), leads to the following bound

$$\gamma_m \le \gamma_\infty + \left(1 - \sqrt{\frac{d^2 - d_2^2}{d_\infty^2 - d_2^2}}\right)(\gamma_2 - \gamma_m). \qquad (25)$$

Of course, by appropriately modifying γ_∞ and d_∞, this bound can also be used for the more general problem:

$$\left\{ \begin{array}{ll} \min_{Q(\cdot)\in H^\infty} & \|T_Q(z)\| \\ \text{subject to} & \left\|T_Q^{(2)}(z)\right\|_2 \le d \end{array} \right. . \qquad (26)$$

(vi) Finally, we should mention that the four-block problem studied here is of the form given in [13], and is slightly different from the one often referred to in the literature (as in [14]), which takes the form

$$T_Q^{(i)}(z) = \begin{bmatrix} R_{11}^{(i)} + Q(z) & R_{12}^{(i)}(z) \\ R_{21}^{(i)}(z) & R_{22}^{(i)}(z) \end{bmatrix}, \qquad (27)$$

with $R_{jk}^{(i)} \in H^\infty$, $i = 2, \infty$, $j,k = 1,2$. Nonetheless, our upper bound (6) also holds for four-block problems of the type (27). The basic reasoning is as follows. Our proof of (6) relied on two facts: first, that $T_Q(\cdot)$ is affine in $Q(\cdot)$, so that the triangular inequality could be applied to $\|T_Q(z)\|_\infty$, and, second, that the result of Lemma 3 holds. Now $T_Q(\cdot)$ in (27) is clearly affine in $Q(\cdot)$. Secondly, the result of Lemma 3 also holds for four-block transfer matrices of the type (27). To see why, note that the first step in the proof of Lemma 3 was to transform the $T_Q(\cdot)$ of (2) to an equivalent one of the form (27) — see Eq. (17).

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