Polyphase Networks, Block Digital Filtering, LPTV Systems, and Alias-Free QMF Banks: A Unified Approach Based on Pseudocirculants

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Abstract—**In this paper, the relation between block digital filtering and quadrature mirror filter (QMF) banks is explored. Necessary and sufficient conditions for alias cancellation in QMF banks are expressed in terms of an associated matrix, derived from the polyphase components of the analysis and synthesis filters. These conditions, called the pseudocirculant conditions, enable us to directly unite QMF banks with the framework of block digital filtering. Absence of amplitude distortion in an alias-free QMF bank translates into the "losslessness" property of the pseudocirculant matrix involved.**

I. Introduction

In the area of multirate signal processing, one of the topics that has received very wide interest is quadrature mirror filtering [1]–[13]. Fig. 1 shows an M-band maximally decimated quadrature mirror filter (QMF) bank. Its importance, applications, and operational principles can be found in many references [3], [7], [8]–[11]. The signal $\hat{x}(n)$, called the reconstructed signal, is related to $x(n)$ by a fairly complicated expression [7], [10], [38], [39], given by

$$\hat{X}(z) = \frac{1}{M} \sum_{n=0}^{M-1} X(zW^{-n}) \sum_{k=0}^{M-1} H_k(zW^{-n}) F_k(z). \quad (1a)$$

Here, $H_k(z)$ and $F_k(z)$ are the kth analysis and synthesis filters, respectively, for $0 \leq k \leq M-1$ and $W = e^{-j2\pi/M}$. This expression (1a) represents an aliased and distorted version of $X(z)$; the terms with $n \neq 0$ represent aliasing. If these are cancelled [10], the result is

$$\hat{X}(z)/X(z) = T(z) = \sum_{k=0}^{M-1} H_k(z) F_k(z)/M. \quad (1b)$$

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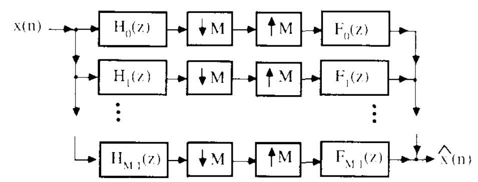

Fig. 1. An M-band maximally decimated parallel QMF bank.

If $T(z)$ is allpass, the reconstructed signal is free from amplitude distortion. If $T(z)$ is a pure delay, then we have perfect reconstruction, which is a topic that has received attention in recent years [6], [14]–[16], [38], [39].

The topic of block digital filtering has also received considerable attention in the past [17]–[20]. The relation between multirate filters and block digital filtering has been studied earlier [21]. The notion of "polyphase transforms" has also been used recently [23] as a step toward such unification (see also [24]). The main purpose of this paper is to explore this relationship further. In Section II we derive a new set of necessary and sufficient conditions for the QMF bank of Fig. 1 to be free from aliasing. Several sets of sufficient conditions derived in the past can be obtained from this set of conditions. These conditions, based on the polyphase framework [3], [22], are called the pseudocirculant conditions, and are much less stringent than the corresponding conditions for perfect reconstruction [14]. Moreover, these conditions directly place in evidence the relation between block digital filtering and multirate QMF banks, as elaborated in Section IV. Conditions (and procedures) for inverting linear periodically time varying systems are also evident from these results, as elaborated in Section IV-C.

Even though the relation between pseudocirculants and QMF banks has probably not been observed earlier, this result itself is not surprising. There are at least two contexts in the literature where pseudocirculants have been encountered either implicitly [19] or explicitly [20], [23]. In the block-filtering context, an $L \times L$ matrix has been mentioned in [19, p. 206] (where L is the block length). This matrix is pseudocirculant if and only if the underlying scalar linear system is time invariant. In fact, the "block-shift invariance" concept introduced in [20] is closely related to the pseudocirculant property, as evidenced from the example of equation (18) in [20]. Second, in an altogether different setting, based on polyphase transforms, the relation between time-invariance and pseudocirculants has been observed in [23] and [24] (even though the name "para-circulants" is used in [23]).

In this paper, however, we take a different approach with the QMF bank of Fig. 1 as the starting point, and show that the pseudocirculant conditions are a direct consequence of enforcing alias-free property. Given an alias-free QMF bank, a set of necessary and sufficient conditions for the distortion function $T(z)$ to be allpass are developed in Section III. Once again, this set of conditions can be stated in terms of the losslessness property, or "LBR" property [25], of an associated "block digital filter transfer matrix." Procedures for meeting certain sufficient conditions of this type have been reported in a number of papers in the past [10], [14]–[16].

Since the framework of this paper is based on a class of matrices called pseudocirculants, some important properties of these matrices are listed in Appendix A.

Notations Used in the Sequel: Superscript T stands for matrix (or vector) transposition, whereas superscript dagger ($\dagger$) stands for transposition followed by complex conjugation. Boldface italic letters indicate matrices and vectors. $\boldsymbol{W}_M$ denotes the $M \times M$ DFT matrix, i.e., $\boldsymbol{W}_M = [W_M^{kl}]$ with $W_M = e^{-j2\pi/M}$; the subscript M is often deleted for simplicity. Superscript asterisk ($*$) stands for complex conjugation, while subscript asterisk denotes conjugation of coefficients of the function or matrix. The tilde accent on a function $\boldsymbol{F}(z)$ is defined such that, on the unit circle, $\tilde{\boldsymbol{F}}(z) = \boldsymbol{F}^{\dagger}(z)$. Thus, for arbitrary z, $\tilde{\boldsymbol{F}}(z) = \boldsymbol{F}_*^T(z^{-1})$ and for functions with real coefficients, $\tilde{\boldsymbol{F}}(z) = \boldsymbol{F}^T(z^{-1})$.

A transfer matrix $\boldsymbol{T}(z)$ is said to be lossless if it is stable and satisfies

$$\boldsymbol{T}^{\dagger}(e^{j\omega})\,\boldsymbol{T}(e^{j\omega}) = \boldsymbol{I} \tag{2a}$$

for all ω. The above equation is equivalent, by analytic continuation, to the equality

$$\tilde{\boldsymbol{T}}(z)\,\boldsymbol{T}(z) = \boldsymbol{I} \tag{2b}$$

for all z. The property (2b) is called the "paraunitary property" in analogy with similar behavior of scattering matrices of continuous-time lossless systems [29]. For a stable scalar function $T(z)$, losslessness is equivalent to the allpass property.

II. Necessary and Sufficient Conditions for Alias Cancellation

Any discrete time transfer function $H_k(z)$ can be expressed uniquely in terms of M transfer functions $E_{kl}(z)$ in the form [3], [22]

$$H_k(z) = \sum_{l=0}^{M-1} z^{-l} E_{kl}(z^M), \tag{3}$$

where $E_{kl}(z)$ are called the polyphase components of $H_k(z)$. Let us assume that each analysis filter $H_k(z)$, $0 \le k \le M-1$ has been expressed as in (3). Similarly, each synthesis filter $F_k(z)$ can be uniquely expressed as

$$F_k(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} R_{lk}(z^M) \tag{4}$$

where $R_{lk}(z)$ are called the polyphase components of $F_k(z)$. Accordingly, the QMF bank of Fig. 1 can be drawn as in Fig. 2, where the $M \times M$ transfer matrices $\boldsymbol{E}(z)$ and $\boldsymbol{R}(z)$ are defined according to

$$\boldsymbol{E}(z) = [E_{kl}(z)], \qquad \boldsymbol{R}(z) = [R_{lk}(z)]. \tag{5}$$

The transfer matrix $\boldsymbol{E}(z^M)$ can be moved past the decimators by replacing z^M with z. Similarly, $\boldsymbol{R}(z^M)$ can be moved past the interpolators [3]. The result is Fig. 3, which is equivalent to Fig. 1, where the $M \times M$ matrix $\boldsymbol{P}(z)$ is defined as

$$\boldsymbol{P}(z) = \boldsymbol{R}(z)\,\boldsymbol{E}(z). \tag{6}$$

Since Fig. 3 is equivalent to Fig. 1, we shall express various properties pertaining to the QMF bank (such as alias cancellation, distortion function, and so on) in terms of $\boldsymbol{P}(z)$. In a recent paper [14], necessary and sufficient conditions for perfect reconstruction were expressed in terms of $\boldsymbol{P}(z)$. In this section we shall derive a set of necessary and sufficient conditions for *alias cancellation* in terms of $\boldsymbol{P}(z)$.

The signal $a_l(n)$ in Fig. 3, which is the decimated version of $x(n-l)$, can be expressed in the transform domain as [3]

$$A_l(z) = \frac{1}{M}\sum_{k=0}^{M-1} (z^{1/M}W^k)^{-l}\, X(z^{1/M}W^k), \qquad 0 \le l \le M-1 \tag{7}$$

whereas the transform of $b_s(n)$ is given by

$$B_s(z) = \frac{1}{M}\sum_{l=0}^{M-1} P_{s,l}(z) \sum_{k=0}^{M-1} (z^{1/M}W^k)^{-l} \cdot X(z^{1/M}W^k), \qquad 0 \le s \le M-1. \tag{8}$$

The reconstructed signal $\hat{X}(z)$, which can be expressed as

$$\hat{X}(z) = \sum_{s=0}^{M-1} z^{-(M-1-s)} B_s(z^M), \tag{9}$$

therefore simplifies to

$$\hat{X}(z) = \frac{1}{M} z^{-(M-1)} \sum_{k=0}^{M-1} X(zW^k) \cdot \sum_{l=0}^{M-1} W^{-kl} \sum_{s=0}^{M-1} P_{s,l}(z^M)\, z^{-(l-s)}. \tag{10}$$

This expression is free from aliasing [for arbitrary $x(n)$] if and only if

$$\sum_{l=0}^{M-1} W^{-kl} \sum_{s=0}^{M-1} P_{s,l}(z^M)\, z^{-(l-s)} = 0, \quad k \ne 0, \tag{11}$$

i.e., if and only if the M-point sequence in l, defined by $V_l(z) = \sum_{s=0}^{M-1} P_{s,l}(z^M)\, z^{-(l-s)}$ is constant with respect

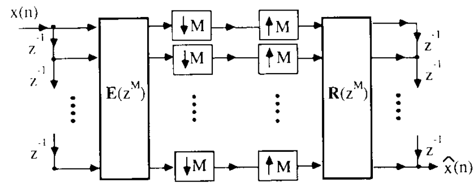

Fig. 2. Redrawing of Fig. 1 in terms of the polyphase component matrices.

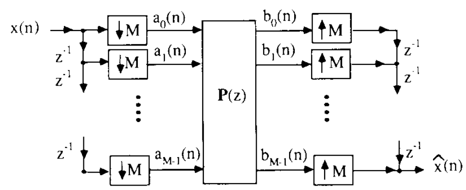

Fig. 3. An equivalent representation of Fig. 1.

to l, i.e.,

$$\sum_{s=0}^{M-1} P_{s,l}(z^M)\, z^{-(l-s)} = V(z) \tag{12}$$

for all l. If we let

$$V(z) = \sum_{n=0}^{\infty} v(n) z^{-n},$$

$$P_{s,l}(z) = \sum_{n=0}^{\infty} p_{s,l}(n) z^{-n} \tag{13}$$

then (12) is equivalent to saying

$$p_{s,l}(n) = v(nM + l - s) \tag{14}$$

for all n, and $0 \le s,\ l \le M - 1$. In other words, $P_{s,l}(z)$ depends only on $l - s$, and hence, $\boldsymbol{P}(z)$ is a Toeplitz matrix:

$$\boldsymbol{P}(z) = \begin{bmatrix} P_0(z) & P_1(z) & \cdots & \cdots\ P_{M-1}(z) \\ P_{-1}(z) & P_0(z) & P_1(z) & \cdots\ P_{M-2}(z) \\ \vdots & & & \vdots \\ P_{-(M-1)}(z) & P_{-(M-2)}(z) & P_{-(M-3)}(z) & \cdots\ P_0(z) \end{bmatrix}. \tag{15}$$

From (14) we also see that

$$\begin{aligned} p_{k,0}(n) &= v(nM - k) \\ &= v[(n-1)M + M - k] \\ &= p_{0,M-k}(n-1) \end{aligned} \tag{16a}$$

whence

$$P_{k,0}(z) = z^{-1} P_{0,M-k}(z). \tag{16b}$$

For example, with $M = 3$, the matrix in (15) becomes with (16b),

$$\boldsymbol{P}(z) = \begin{bmatrix} P_0(z) & P_1(z) & P_2(z) \\ z^{-1}P_2(z) & P_0(z) & P_1(z) \\ z^{-1}P_1(z) & z^{-1}P_2(z) & P_0(z) \end{bmatrix} \tag{17}$$

which is a circulant matrix, with the exception that the elements below the diagonal are multiplied by z^{-1}. With $z = 1$, (17) represents a circulant, and with $z = -1$, it becomes a skew circulant [26].

Definition: Let $\boldsymbol{P}(z)$ be an $M \times M$ Toeplitz matrix, and let $P_{k,0}(z) = z^{-1}P_{0,M-k}(z)$, $1 \le k \le M - 1$. Then $\boldsymbol{P}(z)$ is called a pseudocirculant transfer matrix. The kth row of $\boldsymbol{P}(z)$ is obtained from the 0th row by performing k units of pseudocirculant right-shift[1] of the 0th row.

The above derivations lead us to the following theorem.

Theorem 2.1: Consider the maximally decimated QMF bank of Fig. 1, and let $\boldsymbol{E}(z)$, $\boldsymbol{R}(z)$, and $\boldsymbol{P}(z)$ be defined by (5) and (6). Then the reconstructed signal $\hat{x}(n)$ is free from aliasing for any $x(n)$ if and only if $\boldsymbol{P}(z)$ is a pseudocirculant transfer matrix.

Thus, according to the theorem, any QMF bank which is free from aliasing is such that $\boldsymbol{P}(z) = \boldsymbol{R}(z)\,\boldsymbol{E}(z)$ is pseudocirculant, and vice versa. Notice that the pseudocirculant property basically implies that each row of $\boldsymbol{P}(z)$ can be expressed in terms of the 0th row as

$$P_{k,l}(z) = \begin{cases} P_{0,l-k}(z) & 0 \le k \le l \\ z^{-1}P_{0,l-k+M}(z) & l < k \le M-1. \end{cases} \tag{18}$$

Equivalently, each column can be expressed in terms of the rightmost (i.e., $(M-1)$th) column as

$$P_{k,l}(z) = \begin{cases} P_{M-1+k-l,M-1}(z) & 0 \le k \le l \\ z^{-1}P_{k-l-1,M-1}(z) & l < k \le M-1. \end{cases} \tag{19}$$

Either (18) or (19) can be taken as a definition of the pseudocirculant property.

Example 1: Consider the alias-free 2-channel QMF bank [1]–[5] with

$$H_1(z) = H_0(-z),\ F_0(z) = H_0(z),$$
$$\text{and } F_1(z) = -H_1(z). \tag{20a}$$

[1] A pseudocirculant shift is essentially a circular shift, followed by the multiplication of z^{-1} to those elements which have "spilled over."

Letting

$$H_0(z) = E_{00}(z^2) + z^{-1}E_{01}(z^2), \tag{20b}$$

we have $H_1(z) = E_{00}(z^2) - z^{-1}E_{01}(z^2)$, whence

$$E(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) \\ E_{00}(z) & -E_{01}(z) \end{bmatrix},$$

$$R(z) = \begin{bmatrix} E_{01}(z) & E_{01}(z) \\ E_{00}(z) & -E_{00}(z) \end{bmatrix} \tag{21}$$

so that

$$P(z) = R(z)\,E(z) = 2\begin{bmatrix} E_{00}(z)\,E_{01}(z) & 0 \\ 0 & E_{00}(z)\,E_{01}(z) \end{bmatrix} \tag{22}$$

which is pseudocirculant. This type of QMF bank can be found in [1]–[5] (FIR case) and in [10], [27], and [28] (IIR case).

Example 2: Consider the FIR perfect-reconstruction structures reported in [6] and [14] where $E(z)$ is unitary on the unit circle and where $R(z)$ is chosen to be $R(z) = z^{-r}E_*^T(z^{-1})$ so that $P(z) = z^{-r}E_*^T(z^{-1})\,E(z) = z^{-r}I$ which is a simple pseudocirculant form.

Example 3: For the M-channel alias-free QMF banks reported in [10] (uniform DFT banks and so on), the matrix $E(z)$ is

$$E(z) = MW^{-1}\Lambda(z) \tag{23}$$

where $\Lambda(z)$ is a diagonal matrix with diagonal elements $\Lambda_{ll}(z) = E_{0l}(z)$, so that the analysis filter $H_0(z) = \Sigma_{l=0}^{M-1} z^{-l}\Lambda_{ll}(z^M)$. The synthesis filters here are chosen such that $R(z)$ is given by

$$R(z) = \frac{1}{M}\Lambda'(z)W \tag{24}$$

where $\Lambda'(z)$ is such that $\Lambda'(z)\,\Lambda(z) = S(z)\cdot I$. Thus, $P(z) = R(z)\,E(z) = S(z)\cdot I$ which is pseudocirculant.

Curiously enough, all the practically significant alias-free QMF banks known today are such that $P(z)$ is a diagonal matrix, even though this is not a *necessary* condition for alias cancellation. Examples of nondiagonal pseudocirculant $P(z)$ have, however, been published in the past. For example, see [14, p. 483, eqn. (60), and Fig. 11]. Such examples can also be constructed by using the results in [38]. An obvious example of nondiagonal $P(z)$ can be constructed by letting $H_0(z) = 1$, $H_1(z) = z^{-1}$, $F_0(z) = z^{-2}$, $F_1(z) = z^{-1}$, which gives

$$E(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R(z) = \begin{bmatrix} 0 & 1 \\ z^{-1} & 0 \end{bmatrix},$$

$$P(z) = \begin{bmatrix} 0 & 1 \\ z^{-1} & 0 \end{bmatrix}.$$

Nondiagonal pseudocirculants contribute to the generality of the family of alias-free banks, but their advantage, if any, over diagonal $P(z)$'s remains to be explored.

III. Necessary and Sufficient Conditions for Absence of Amplitude Distortion and/or Phase Distortion

Suppose the analysis and synthesis filters are chosen such that $P(z)$ is pseudocirculant, so that aliasing is cancelled. The QMF system is now time invariant. Absence of aliasing implies (11), whence, from (10) we have,

$$\frac{\hat{X}(z)}{X(z)} = T(z) = \frac{1}{M}z^{-(M-1)} \cdot \sum_{l=0}^{M-1}\sum_{s=0}^{M-1} P_{s,l}(z^M)\,z^{-(l-s)}. \tag{25}$$

In other words, the QMF system can be described by a scalar transfer function $T(z)$ as in (1b). By using the pseudocirculant property of $P(z)$, we can express $P_{s,l}(z)$ in terms of elements of the 0th row of $P(z)$ and simplify (25) to the form (see Appendix B)

$$T(z) = z^{-(M-1)}\sum_{k=0}^{M-1} z^{-k}P_{0,k}(z^M). \tag{26}$$

The reconstructed signal is free from amplitude distortion, if and only if $T(z)$ is allpass. It is of interest to know how this allpass property can be expressed in terms of the elements of $P(z)$. We shall now prove the following result.

Theorem 3.1: In the alias-free QMF bank, the distortion function $T(z)$ is allpass if and only if the pseudocirculant transfer matrix $P(z)$ is lossless.

By definition, losslessness means stability and paraunitariness. Since $F_k(z)$ and $H_k(z)$ are stable, stability of $P(z)$ and $T(z)$ will be taken for granted in all our discussions. Accordingly, losslessness and paraunitariness will mean the same property.

Proof of Theorem 3.1: For simplicity, let us define

$$S(z) = \sum_{k=0}^{M-1} z^{-k}P_{0,k}(z^M) \tag{27}$$

so that the allpass property of $T(z)$ is equivalent to that of $S(z)$. Defining

$$e(z) \triangleq [1\,z^{-1}z^{-2}\cdots z^{-(M-1)}]^T \tag{28}$$

we have

$$S(z) = [P_{0,0}(z^M)\,P_{0,1}(z^M)\cdots P_{0,M-1}(z^M)]\,e(z) \tag{29}$$

whence

$$z^{-1}S(z) = [z^{-M}P_{0,M-1}(z^M)\;\; P_{00}(z^M)\;\; P_{01}(z^M)\cdots P_{0,M-2}(z^M)]\,e(z). \tag{30}$$

The row vector in (30) is a circularly right-shifted version of that in (29), with an additional z^{-M} multiplying the endaround element. If we proceed in this manner and write $z^{-2}S(z)$, $z^{-3}S(z)$, $\cdots$, and so on, we therefore get the M equations

$$S(z)\,\mathbf{e}(z) = \mathbf{P}(z^M)\,\mathbf{e}(z) \tag{31}$$

where $\mathbf{P}(z)$ is the pseudocirculant matrix $\mathbf{R}(z)\,\mathbf{E}(z)$. Thus,

$$\tilde{S}(z)\,S(z)\cdot\tilde{\mathbf{e}}(z)\,\mathbf{e}(z) = \tilde{\mathbf{e}}(z)\,\tilde{\mathbf{P}}(z^M)\,\mathbf{P}(z^M)\,\mathbf{e}(z) \tag{32}$$

(the tilde notation is defined in Section I). Since $\tilde{\mathbf{e}}(z)\,\mathbf{e}(z) = M$, (32) simplifies to

$$M\tilde{S}(z)\,S(z) = \tilde{\mathbf{e}}(z)\,\tilde{\mathbf{P}}(z^M)\,\mathbf{P}(z^M)\,\mathbf{e}(z). \tag{33}$$

If $\mathbf{P}(z)$ is lossless, then $\tilde{\mathbf{P}}(z)\,\mathbf{P}(z) = \mathbf{I}$, whence (33) reduces to

$$\tilde{S}(z)\,S(z) = 1 \tag{34}$$

which is the allpass property of $S(z)$.

Conversely, assuming that $S(z)$ is allpass, we would like to show that $\mathbf{P}(z)$ is lossless. This proof is somewhat more involved. Note that (31) also implies

$$S(zW^k)\,\mathbf{e}_k(z) = \mathbf{P}(z^M)\,\mathbf{e}_k(z), \qquad 0 \le k \le M-1 \tag{35}$$

where $\mathbf{e}_k(z) = [1 \quad z^{-1}W^{-k} \quad z^{-2}W^{-2k} \quad \cdots \quad z^{-(M-1)}W^{-(M-1)k}]^T$. The set of M equations described in (35) can be rearranged as

$$\begin{bmatrix} 1 & & & 0 \\ & z^{-1} & & \\ & & \ddots & \\ 0 & & & z^{-(M-1)} \end{bmatrix} \begin{bmatrix} S(zW^k) \\ W^{-k}S(zW^k) \\ \vdots \\ W^{-(M-1)k}S(zW^k) \end{bmatrix} = \mathbf{P}(z^M) \begin{bmatrix} 1 & & & 0 \\ & z^{-1} & & \\ & & \ddots & \\ 0 & & & z^{-(M-1)} \end{bmatrix} \begin{bmatrix} 1 \\ W^{-k} \\ \vdots \\ W^{-(M-1)k} \end{bmatrix}. \tag{36}$$

Since (36) holds for $0 \le k \le M-1$, we get M sets of M equations as in (36), all of which can be compactly expressed as

$$\mathbf{\Lambda}(z)\,\mathbf{W}^\dagger\mathbf{Q}(z) = \mathbf{P}(z^M)\,\mathbf{\Lambda}(z)\,\mathbf{W}^\dagger \tag{37}$$

where $\mathbf{\Lambda}(z)$ is an $M \times M$ diagonal matrix with diagonal elements $[\mathbf{\Lambda}(z)]_{kk} = z^{-k}$, $\mathbf{Q}(z)$ is an $M \times M$ diagonal matrix with diagonal elements $[\mathbf{Q}(z)]_{kk} = S(zW^k)$, and $\mathbf{W}$ is the $M \times M$ DFT matrix. From (37) we have

$$\mathbf{\Lambda}(z)\,\mathbf{W}^\dagger\mathbf{Q}(z)\,\tilde{\mathbf{Q}}(z)\,\mathbf{W}\tilde{\mathbf{\Lambda}}(z) = \mathbf{P}(z^M)\,\mathbf{\Lambda}(z)\,\mathbf{W}^\dagger\mathbf{W}\tilde{\mathbf{\Lambda}}(z)\,\tilde{\mathbf{P}}(z^M). \tag{38}$$

Since $S(z)$ is allpass, $S(zW^k)$ is allpass, hence, $\mathbf{Q}(z)$ is lossless. Moreover, by definition, $\mathbf{\Lambda}(z)$ is obviously lossless. Using the identity $\mathbf{W}^\dagger\mathbf{W} = M\mathbf{I}$, (38) therefore reduces to

$$\mathbf{I} = \mathbf{P}(z^M)\,\tilde{\mathbf{P}}(z^M) \tag{39}$$

proving that $\mathbf{P}(z)$ is paraunitary, i.e., lossless.

Example 4: A simple two-band example of an alias-free system with lossless $\mathbf{P}(z)$ can be found in [28] and [30] where the analysis and synthesis filters are as in (20a) with the additional stipulation that the polyphase components $E_{00}(z)$, $E_{01}(z)$ be allpass. The resulting $\mathbf{P}(z)$ matrix [which is (22)] is therefore allpass. One way to design such a QMF system is to take $H_0(z)$ to be a half-band elliptic filter [10], [35]–[37] of odd order. It is well known [28] that such filters can be expressed as in (20b) with allpass polyphase components.

Freedom from Phase Distortion: The alias-free QMF bank, characterized by transfer function $T(z)$, as in (26), is free from phase distortion if and only if $S(z)$ in (27) has linear phase. Since a (causal and) stable linear phase rational transfer function cannot be IIR, $S(z)$ has to be FIR. Let $N-1$ denote its order so that $S(z) = \Sigma_{n=0}^{N-1} s(n)z^{-n}$. Assuming $s(n)$ is real, the sequence $\{s(n)\}$ has to be either symmetric or antisymmetric; if it is antisymmetric, then $S(e^{j0}) = 0$, and there is severe amplitude distortion at $\omega = 0$. Accordingly, $s(n)$ should be restricted to be a symmetric sequence. It is shown in [31] that an FIR impulse response $s(n)$ is symmetric if and only if its polyphase components $P_{0,k}(z)$ satisfy

$$P_{0,k}(z) = \begin{cases} z^{-m_1}P_{0,m_0-k}(z^{-1}) & 0 \le k \le m_0 \\ z^{-(m_1-1)}P_{0,M+m_0-k}(z^{-1}) & m_0 < k \le M-1 \end{cases} \tag{40}$$

where m_0 and m_1 are unique integers such that $N-1 = m_0 + m_1M$, $0 \le m_0 \le M-1$. It can be verified that $P_{0,k}(z)$ has order $\le m_1$, for $0 \le k \le m_0$ and order $\le m_1 - 1$ for $m_0 < k \le M-1$. The property (40) says that the elements $P_{0,k}(z)$ and $P_{0,m_0-k}(z)$ (or $P_{0,M+m_0-k}(z)$ as the case may be) of the pseudocirculant matrix $\mathbf{P}(z)$ must have impulse responses which are mirror images of each other. This then gives us a set of necessary and sufficient conditions in terms of $\mathbf{P}(z)$ to eliminate phase distortion.

Comment on Existence of Solutions: From examples 1-3 of Section II and the references therein, it is evident that there exist QMF banks which satisfy the pseudocirculant conditions. From Example 4 above (and from [31]) it is clear that there exist practical schemes which meet the pseudocirculant conditions, and are at the same time free from amplitude distortion (and phase distortion, respectively). Finally, the results in [14]–[16] establish the existence of pseudocirculant QMF systems which are simultaneously free from amplitude and phase distortions. All the references mentioned in this paragraph also present constructive design methods in order to satisfy the appropriate sufficient conditions of interest.

IV. Block Digital Filters, QMF Banks, and Periodically Time Varying Systems

The technique of block digital filtering [17]–[21] has been introduced and used in the past as a means of increasing the parallelism of computation, so as to achieve a higher filtering throughput. The relation between multirate filtering, periodically time varying systems, and block filtering has also been pointed out [21]. The concept of block-shift invariance introduced in [20] can be used to relate the "block-filter" formalism to circulants; in particular, see equations (18)–(20b) in [20]. In this section we review this connection by showing that the condition for alias cancellation in a QMF bank is directly related to the "blocking formalism."

A. Review of the Block Filtering Framework

Given a "scalar filter" with transfer function $H(z)$, let us designate its input and output sequences by $x(n)$ and $y(n)$, respectively. The "blocked versions" of these sequences, with block length M, are defined to be vector sequences of the form

$$\boldsymbol{x}_B(n) = \begin{bmatrix} x(nM + M - 1) \\ x(nM + M - 2) \\ \vdots \\ x(nM) \end{bmatrix},$$

$$\boldsymbol{y}_B(n) = \begin{bmatrix} y(nM + M - 1) \\ y(nM + M - 2) \\ \vdots \\ y(nM) \end{bmatrix}. \quad (41)$$

If $\boldsymbol{X}_B(z) \triangleq \Sigma_n \boldsymbol{x}_B(n) z^{-n}$ and $\boldsymbol{Y}_B(z) \triangleq \Sigma_n \boldsymbol{y}_B(n) z^{-n}$ denote the z-transforms of these blocked versions, then they are related by a transfer function $\boldsymbol{H}_B(z)$ [i.e., $\boldsymbol{Y}_B(z) = \boldsymbol{H}_B(z) \boldsymbol{X}_B(z)$], which can be written in terms of $H(z)$ using well-known techniques [19]. $\boldsymbol{H}_B(z)$ is called the "block transfer matrix" corresponding to $H(z)$. If we express the scalar quantities $X(z)$ and $Y(z)$ in terms of polyphase components $X_k(z)$ and $Y_k(z)$, i.e.,

$$X(z) = X_0(z^M) + z^{-1}X_1(z^M) + \cdots + z^{-(M-1)}X_{M-1}(z^M) \quad (42)$$

$$Y(z) = Y_0(z^M) + z^{-1}Y_1(z^M) + \cdots + z^{-(M-1)}Y_{M-1}(z^M), \quad (43)$$

then $\boldsymbol{x}_B(n)$ and $\boldsymbol{y}_B(n)$ are precisely the quantities

$$\boldsymbol{x}_B(n) = [x_{M-1}(n)\, x_{M-2}(n) \cdots x_0(n)]^T,$$

$$\boldsymbol{y}_B(n) = [y_{M-1}(n)\, y_{M-2}(n) \cdots y_0(n)]^T. \quad (44)$$

With $H_{k,l}(z)$ denoting the elements of $\boldsymbol{H}_B(z)$, we now have

$$\begin{bmatrix} Y_{M-1}(z) \\ Y_{M-2}(z) \\ \vdots \\ Y_0(z) \end{bmatrix} = \begin{bmatrix} H_{00}(z) & \cdots & H_{0,M-1}(z) \\ H_{1,0}(z) & \cdots & H_{1,M-1}(z) \\ \vdots & & \vdots \\ H_{M-1,0}(z) & \cdots & H_{M-1,M-1}(z) \end{bmatrix} \cdot \begin{bmatrix} X_{M-1}(z) \\ X_{M-2}(z) \\ \vdots \\ X_0(z) \end{bmatrix}. \quad (45)$$

An important property of the $M \times M$ block transfer matrix $\boldsymbol{H}_B(z)$ in (45) will now be derived. If we apply a unit pulse input $x(n) = \delta(n)$ to the scalar transfer function $H(z)$, then $X(z) = 1$, whence the "blocked version" has input $\boldsymbol{X}_B(z) = [0\ 0 \cdots 1]^T$. The corresponding output is

$$\boldsymbol{Y}_B(z) = \begin{bmatrix} Y_{M-1}(z) \\ Y_{M-2}(z) \\ \vdots \\ Y_0(z) \end{bmatrix} \triangleq \begin{bmatrix} H_{0,M-1}(z) \\ H_{1,M-1}(z) \\ \vdots \\ H_{M-1,M-1}(z) \end{bmatrix}. \quad (46)$$

If we, however, apply a shifted unit pulse input $x(n) = \delta(n - k)$, $0 \le k \le M - 1$, then $X(z) = z^{-k}$, so that

$$X_l(z) = \begin{cases} 0 & l \ne k, \quad 0 \le l \le M - 1 \\ 1 & l = k. \end{cases} \quad (47)$$

Accordingly, $\boldsymbol{X}_B(z) = [0 \cdots 0\ 1\ 0 \cdots 0]^T$ where the "1" is in the $(M - 1 - k)$th position. The corresponding output is then

$$\boldsymbol{Y}'_B(z) = \begin{bmatrix} H_{0,M-1-k}(z) \\ H_{1,M-1-k}(z) \\ \vdots \\ H_{M-1,M-1-k}(z) \end{bmatrix}. \quad (48)$$

Since $H(z)$ represents a shift invariant system, the "unblocked outputs" $Y(z)$ and $Y'(z)$ corresponding to $\boldsymbol{Y}_B(z)$ and $\boldsymbol{Y}'_B(z)$ are related by $Y'(z) = z^{-k}Y(z)$, so from (43)

$$Y'(z) = z^{-k}Y_0(z^M) + z^{-(k+1)}Y_1(z^M) + \cdots + z^{-M}Y_{M-k}(z^M) + \cdots + z^{-(M-1+k)}Y_{M-1}(z^M). \quad (49)$$

Thus, $\boldsymbol{Y}_B'(z)$ is also given by

$$\boldsymbol{Y}_B'(z) = \begin{bmatrix} Y_{M-k-1}(z) \\ \vdots \\ Y_0(z) \\ z^{-1}Y_{M-1}(z) \\ \vdots \\ z^{-1}Y_{M-k}(z) \end{bmatrix} = \begin{bmatrix} H_{k,M-1}(z) \\ \vdots \\ H_{M-1,M-1}(z) \\ z^{-1}H_{0,M-1}(z) \\ \vdots \\ z^{-1}H_{k-1,M-1}(z) \end{bmatrix}. \tag{50}$$

From (48) and (50), we therefore arrive at the relation,

$$H_{l,M-1-k}(z) = \begin{cases} H_{k+l,M-1} & 0 \le l \le M-1-k \\ z^{-1}H_{k+l-M,M-1} & M-1-k < l \le M-1. \end{cases} \tag{51}$$

By making the change of variables $l' = M - 1 - k$, (51) reduces to the form (19) which is the definition of a pseudocirculant. We therefore conclude that when a scalar transfer function $H(z)$ is "blocked," the resulting block transfer matrix $\boldsymbol{H}_B(z)$ is pseudocirculant. Since (46) is obtained in response to an impulse, it is also clear that

$$H(z) = H_{M-1,M-1}(z^M) + z^{-1}H_{M-2,M-1}(z^M) + \cdots + z^{-(M-1)}H_{0,M-1}(z^M) \tag{52}$$

which, because of the pseudocirculant property, becomes

$$H(z) = H_{0,0}(z^M) + z^{-1}H_{0,1}(z^M) + \cdots + z^{-(M-1)}H_{0,M-1}(z^M). \tag{53}$$

As a converse to the above result, it can be verified that any pseudocirculant matrix $\boldsymbol{H}_B(z)$ is the "blocked version" of a scalar transfer function $H(z)$ given by (53).

B. The Polyphase Formulation and Block Filtering

Let us redraw Fig. 3 as in Fig. 4, by introducing an advance operator z^{M-1} for convenience of discussion. Define the signals $x(n)$, $\hat{x}(n)$, $(= y(n))$, $x_k(n)$, $y_k(n)$ as in Fig. 4. Clearly, we have

$$X(z) = \sum_{k=0}^{M-1} z^{-k}X_k(z^M),$$

$$Y(z) = \sum_{k=0}^{M-1} z^{-k}Y_k(z^M) \tag{54}$$

which is the well-known polyphase representation [3], [22]. Let us define the vectors $\boldsymbol{x}(n) = \boldsymbol{x}_B(n)$ and $\boldsymbol{y}(n) = \boldsymbol{y}_B(n)$, which are the blocked versions of the input $x(n)$ and the reconstructed signal $y(n)$, respectively. Assume that $\boldsymbol{P}(z)$ has been chosen to cancel aliasing. We then have

$$Y(z) = S(z)\,X(z), \qquad \boldsymbol{Y}(z) = \boldsymbol{P}(z)\,\boldsymbol{X}(z). \tag{55}$$

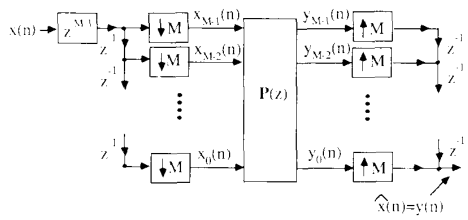

Fig. 4. Pertaining to the block-filtering formalism.

Thus, $S(z)$ relates the signals $X(z)$ and $Y(z)$, whereas $\boldsymbol{P}(z)$ relates the corresponding blocked versions (i.e., $\boldsymbol{P}(z)$ is a blocked version of $S(z)$, with block length equal to M).

Given any scalar transfer function $S(z)$, we know that its blocked version $\boldsymbol{P}(z)$ is necessarily pseudocirculant, and conversely, any pseudocirculant transfer matrix $\boldsymbol{P}(z)$ is the blocked version of some scalar transfer function $S(z)$. On the other hand, we also know from Section II that the QMF bank is alias free if and only if $\boldsymbol{P}(z)$ is pseudocirculant. Notice that the expression for $S(z)$ in (27) is nothing but the expression for an unblocked transfer function in terms of the elements of the blocked version. We can combine these observations in the form of the following theorem.

Theorem 4.1: Consider the QMF bank of Fig. 1 and define $\boldsymbol{E}(z)$, $\boldsymbol{P}(z)$, and $\boldsymbol{R}(z)$ as in (5) and (6). Then $\hat{x}(n)$ is free from aliasing if and only if $\boldsymbol{P}(z)$ is the blocked version (with block length M) of a scalar transfer function $S(z)$. Moreover, the unblocked version $S(z)$ of $\boldsymbol{P}(z)$, given by (27), is related to the distortion function $T(z) = \hat{X}(z)/X(z)$ of such an alias-free QMF bank, by $T(z) = z^{-(M-1)}S(z)$.

In Theorem 3.1 we showed that $T(z)$ is allpass if and only if $\boldsymbol{P}(z)$ is lossless. By combining this result with Theorem 4.1, we have the following corollary.

Corollary: A scalar transfer function $T(z)$ is allpass if and only if its blocked version $\boldsymbol{P}(z)$ (for any block length M) is lossless.

It can be verified that this corollary also follows from simple energy balance properties [25] of lossless functions and matrices.

C. Periodically Time Varying Systems in the Polyphase Context

Based on the polyphase representation of transfer functions, it is possible to represent linear periodically time varying (LPTV) systems in an elegant fashion. This representation, essentially implicit in [21], can be used in conjunction with the results of Section II to obtain several properties of LPTV (such as invertibility conditions and conditions for time-*invariance*).

Consider a discrete-time linear system, whose coefficients vary periodically with time, with period M. For $0 \le k \le M-1$, let $A_k(z)$ represent the "transfer function" if all the coefficients of the system were frozen at their value at time $-k$. Let $A_k(z) = \sum_{l=0}^{M-1} z^{-l}G_{kl}(z^M)$,

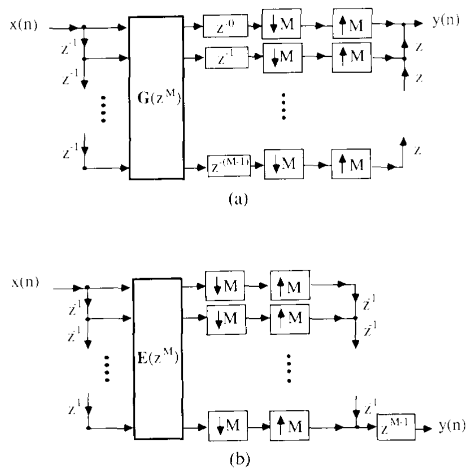

Fig. 5. Representation of an arbitrary linear periodically time varying system using the polyphase framework.

i.e., let $G_{kl}(z)$ by the polyphase components of $A_k(z)$. With the matrix $G(z)$ defined as $G(z) = [G_{kl}(z)]$, we can represent any LPTV system schematically as in Fig. 5(a). This representation can be equivalently redrawn as in Fig. 5(b). It is easily verified that the elements of $E(z)$ and $G(z)$ are related as

$$E_{kl}(z) = \begin{cases} z^{-1}G_{k,M-k+l}(z), & 0 \le l \le k-1 \\ G_{k,l-k}(z), & k \le l \le M-1. \end{cases}$$

Thus, the kth row of $E(z)$, in terms of $G_{k,l}(z)$, is

$$[z^{-1}G_{k,M-k}(z) \quad z^{-1}G_{k,M-k+1}(z) \cdots \cdot z^{-1}G_{k,M-1}(z) \quad G_{k,0}(z) \cdots G_{k,M-k-1}(z)]$$

which is a pseudocircularly right-shifted version of the kth row of $G(z)$.

When does the LPTV system of Fig. 5(a) reduce to an LTI (i.e., time invariant) system? The obvious requirement is $A_k(z) = A_0(z)$, for $0 \le k \le M-1$. The matrix $G(z)$ then has identical rows (hence rank $= 1$) so that $E(z)$ becomes a pseudocirculant! In other words, any LPTV system (period M) can be represented as in Fig. 5(b), and reduces to an LTI system if and only if $E(z)$ is pseudocirculant.

Next, when and how can an LPTV system be "inverted," to get back the signal $x(n)$? This problem is equivalent to inserting a compensator $R(z)$ into Fig. 5(b) so that the system reduces to that of Fig. 2. If (and only if) the product $R(z)E(z)$ is forced to be pseudocirculant, the system is converted into an LTI system. On the other hand, the system is converted into an identity system [except for a scale factor and a delay, i.e., $y(n) = cx(n - n_0)$], if and only if $R(z)E(z)$ has the form given in [14, equation (60)]. In particular, if $R(z)E(z)$ is proportional to the identity matrix, this condition of [14] is satisfied.

V. Concluding Remarks

The main purpose of this paper has been to explore the theoretical relationship between the block-digital filtering framework, alias-free QMF banks, and lossless transfer functions and matrices, based on a polyphase setting.

In Section II we developed a set of necessary and sufficient conditions for alias cancellation in terms of the matrix $P(z)$. An equivalent set of conditions can be developed directly in terms of the analysis and synthesis filters $H_k(z)$, $F_k(z)$, $0 \le k \le M-1$. From (1a) we can write down the conditions for alias cancellation in matrix notation as

$$H(z)f(z) = v(z) \tag{56}$$

where $H(z)$ is the well-known alias component matrix (AC-matrix) first defined by Smith and Barnwell in [7]. This has been subsequently used in [14] and [39]. The quantities $f(z)$ and $v(z)$ are M-vectors defined as

$$f(z) = [F_0(z)\, F_1(z) \cdots F_{M-1}(z)]^T,$$
$$v(z) = [T(z)\, 0\, 0 \cdots 0]^T \tag{57}$$

and $H(z)$ is $M \times M$, given by

$$H(z) = \begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW^{-1}) & H_1(zW^{-1}) & \cdots & H_{M-1}(zW^{-1}) \\ \vdots & & & \vdots \\ H_0(zW^{-M+1}) & H_1(zW^{-M+1}) & \cdots & H_{M-1}(zW^{-M+1}) \end{bmatrix}. \tag{58}$$

By using manipulations similar to those in [14, equations (38)–(42)], it can be shown that (56) holds if and only if

$$H(z)\,F(z) = T(z) \tag{59}$$

where

$$F^T(z) = \begin{bmatrix} F_0(z) & F_1(z) & \cdots & F_{M-1}(z) \\ F_0(zW^{-1}) & F_1(zW^{-1}) & \cdots & F_{M-1}(zW^{-1}) \\ \vdots & & & \vdots \\ F_0(zW^{-M+1}) & F_1(zW^{-M+1}) & \cdots & F_{M-1}(zW^{-M+1}) \end{bmatrix}, \tag{60}$$

and $T(z)$ is an $M \times M$ diagonal matrix of the form

$$T(z) = \begin{bmatrix} T(z) & & & O \\ & T(zW^{-1}) & & \\ & & \ddots & \\ O & & & T(zW^{-M+1}) \end{bmatrix}. \quad (61)$$

A special form of this result for the case of perfect-reconstruction appears in [14, equation (41)]. This result can also be inferred from [10] and [38, equations (25) and (51)]. Notice that $F(z)$ is structurally similar to the transpose of the AC-matrix $H(z)$ [except that $H_k(z)$ are replaced with $F_k(z)$]. With (56) written in the form (59), a number of useful conclusions can now be drawn. First, aliasing is cancelled if and only if the product $H(z)\,F(z)$ is of the form (61). This says that the kth row of $H(z)$ (which represents the analysis bank filters or uniformly frequency-shifted versions) must be orthogonal to the lth column of $F(z)$ (which represents the synthesis bank filters *or uniformly* shifted versions), when $k \neq l$. Second, once aliasing is cancelled, the distortion function $T(z)$ is allpass if and only if $H(z)\,F(z)$ is lossless. Next, if $H(z)$ is lossless, we can force $T(z)$ to be allpass only by choosing $F_k(z)$ such that $F(z)$ is lossless.

If $H(z)$ is lossless, we can see from (56) that $f(z)$ is given by $f(z) = H_*^T(z^{-1})\,v(z)$, hence, $F_k(z) = H_{k,*}(z^{-1})\,T(z)$. In other words, if an alias-free system has a lossless AC-matrix, then the synthesis filters are related to corresponding analysis filters in a simple way. (The poles of $H_{k,*}(z^{-1})$, which are outside the unit circle, are cancelled by the zeros of $T(z)$.)

A number of authors have used the "bifrequency approach" in the past [32]–[34] for the study of linear time-varying systems. Perhaps a natural question of interest here is how to describe the QMF bank in terms of the bifrequency approach. Based on standard definitions [3], it can be verified that the "transmission function" or the "bifrequency system function" of the QMF bank of Fig. 1 is given by

$$K(e^{j\omega'}, e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} F_k(e^{j\omega'}) \cdot \sum_{l=-\infty}^{\infty} \delta\left(\omega - \omega' + \frac{2\pi}{M} l\right) H_k(e^{j\omega}) \quad (62)$$

so that

$$\hat{X}(e^{j\omega'}) = \int_{-\pi}^{\pi} K(e^{j\omega'}, e^{j\omega})\, X(e^{j\omega})\, d\omega \quad (63)$$

which reduces to (1a) with $z = e^{j\omega'}$. In other words, the bifrequency approach brings us to the starting point from which most other conclusions of the paper have been drawn.

Appendix A

Let $P(z)$ be an $M \times M$ pseudocirculant matrix (as defined in Section II). All its elements can be expressed in terms of its 0th row or $(M-1)$th column as in (18) and (19). A compact way to express (18) is

$$P_{k,l}(z) = P_{0,((l-k))}\, z^{-[((l-k))-(l-k)]/M}, \quad 0 \le k, l \le M-1 \quad (A.1)$$

where $((\cdot))$ stands for modulo-M operation. Equation (A.1) can be taken as an equivalent definition of a pseudocirculant matrix. $P(z)$ can be expressed in the form

$$P(z) = \Lambda(z)\, R(z)\, \Lambda^{-1}(z) \quad (A.2)$$

where $\Lambda(z)$ is a diagonal matrix with $[\Lambda(z)]_{ii} = z^{-i/M}$ and where $R(z)$ is an ordinary circulant matrix [26], i.e.,

$$R_{ij}(z) = R_{0,((j-i))}(z), \quad 0 \le i, j \le M-1. \quad (A.3)$$

Conversely, any matrix of the form $\Lambda(z)\,R(z)\,\Lambda^{-1}(z)$ is pseudocirculant as long as $R(z)$ is circulant. These results can be verified by explicit substitution. The top rows of $P(z)$ and $R(z)$ are related as $P_{0,l}(z) = z^{l/M} R_{0,l}(z)$. Now, it is well known [26] that any $M \times M$ circulant $R(z)$ can be diagonalized by the $M \times M$ DFT matrix W, i.e.,

$$R(z) = W\Lambda_p(z)\, W^{-1} \quad (A.4)$$

where $\Lambda_p(z)$ is a diagonal matrix of eigenvalues of $R(z)$ and $W = [W^{kl}]$ with $W = e^{-j2\pi/M}$. Notice that these eigenvalues are the M DFT-coefficients of the M-point sequence defined by the first row of the circulant $R(z)$. Substituting (A.4) in (A.2) we obtain

$$P(z) = \Lambda(z)\, W\Lambda_p(z)\, W^{-1}\Lambda^{-1}(z) \quad (A.5)$$

so that

$$P(z)\,\Lambda(z)\,W = \Lambda(z)\, W\Lambda_p(z). \quad (A.6)$$

Equation (A.6) says that $P(z)$ can be diagonalized by the similarity transformation $\Lambda(z)W$. The eigenvalues of $P(z)$ are the diagonal elements of $\Lambda_p(z)$, and the eigenvectors are the columns of $\Lambda(z)W$. For example, with $M = 3$ we have

$$\Lambda(z)W = \begin{bmatrix} 1 & 1 & 1 \\ z^{-1/3} & z^{-1/3}W & z^{-1/3}W^2 \\ z^{-2/3} & z^{-2/3}W^2 & z^{-2/3}W^4 \end{bmatrix}. \quad (A.7)$$

The diagonalization property (A.5) of a pseudocirculant matrix can be used to derive a number of results, which are listed below. The fractional powers of z in (A.7) need not bother us because the decomposition (A.5) is only a theoretical tool and does not enter into actual implementations.

1) If $P_1(z)$ and $P_2(z)$ are $M \times M$ pseudocirculants, then $P_1(z)\,P_2(z) = P_2(z)\,P_1(z)$, and moreover, this product is pseudocirculant.

2) If $P(z)$ is pseudocirculant, then so is $\tilde{P}(z)$.

3) As a result of Properties 1 and 2, $\tilde{P}(z)\,P(z) = P(z)$

$\tilde{P}(z)$. This implies $P^{\dagger}(e^{j\omega})\, P(e^{j\omega}) = P(e^{j\omega})\, P^{\dagger}(e^{j\omega})$, i.e., $P(z)$ is a normal matrix on the unit circle. The property

$$\tilde{P}(z)\, P(z) = P(z)\, \tilde{P}(z) \qquad \text{for all } z, \tag{A.8}$$

which is an analytic continuation of this normal property, is called the "paranormal" property. Pseudocirculants are, therefore, paranormal.

4) The eigenvalues of an $M \times M$ pseudocirculant $P(z)$ are given by

$$\lambda_l(z) = \sum_{k=0}^{M-1} R_{0,k}(z) W^{kl} = \sum_{k=0}^{M-1} z^{-k/M} P_{0,k}(z) W^{kl}. \tag{A.9}$$

In particular, notice that

$$\lambda_0(z^M) = S(z) = \sum_{k=0}^{M-1} z^{-k} P_{0,k}(z^M). \tag{A.10}$$

Here $\lambda_0(z^M)$ is nothing but the "unblocked transfer function" corresponding to $P(z)$ [see (27)].

Appendix B

From (25) we have

$$T(z) = \frac{1}{M} z^{-(M-1)} \sum_{l=0}^{M-1} \left[\sum_{s=0}^{l} P_{sl}(z^M)\, z^{-(l-s)} + \sum_{s=l+1}^{M-1} P_{sl}(z^M)\, z^{-(l-s)} \right]. \tag{A.11}$$

By using the pseudocirculant property (18), this reduces to

$$T(z) = \frac{1}{M} z^{-(M-1)} \sum_{l=0}^{M-1} \left[\sum_{s=0}^{l} P_{0,l-s}(z^M)\, z^{-(l-s)} + \sum_{s=l+1}^{M-1} P_{0,l-s+M}(z^M)\, z^{-(l-s+M)} \right] \tag{A.12}$$

which can be simplified to yield

$$T(z) = \frac{1}{M} z^{-(M-1)} \sum_{l=0}^{M-1} \sum_{k=0}^{M-1} P_{0,k}(z^M) z^{-k} \tag{A.13}$$

and this is precisely (26).

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P. P. Vaidyanathan (S'80–M'83), for a photograph and biography, see p. 94 of the January 1988 issue of this TRANSACTIONS.

S. K. Mitra (S'59–M'63–SM'69–F'74), for a photograph and biography, see this issue, p. 380.