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Tracking with an H^∞ Criterion[1]

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Abstract

In this paper we study the problem of tracking a reference signal from the H^∞ point of view. As opposed to general H^∞ problems, where only suboptimal solutions are obtained, we show that for both the full information and measurement feedback tracking problems the H^∞-optimal solutions can be explicitly found. The results also indicate an interesting dichotomy between minimum phase and non-minimum phase plants: for minimum phase plants the best causal tracker performs as well as the best noncausal tracker, whereas for non-minimum phase plants, causal trackers cannot reduce the H^∞ norms from their a priori values. We also mention some remedies for the non-minimum phase case, such as adding more actuators (control inputs) or allowing for some finite delay. For causal tracking of non-minimum phase plants, we show that a delay equal to at least the number of non-minimum phase zeros of the plant is required.

1 Introduction

H^∞ control theory has been introduced as a method for designing controllers that have acceptable performance in the face of model uncertainty on the plant and lack of statistical information on the exogenous signals. The approach may therefore be attractive for the problem of tracking reference signals with unknown statistical properties, where conventional statistical methods, such as H^2, may not be directly applicable.

In this paper we shall study the tracking problem from the H^∞ point of view. We shall study two problems: (i) the full information tracking problem where the controller has direct access to the reference signal, and (ii), the measurement feedback problem where the controller has access only to corrupted measurements of the reference signal. We also consider two separate cases, (a) the reference signal (or it measurement) is known a priori, so that the tracker can be noncausal, and (b), the reference signal (or its measurement) is given on-line, so that the tracker must be causal.

Our study of the tracking problem leads to some surprising results. First, it turns out that we can obtain explicit formulas for the H^∞-optimal norms and the corresponding H^∞-optimal trackers in all of the aforementioned cases. This is in contrast to the general H^∞ control problem where explicit optimal solutions are not available, and where what is given is a certain suboptimal solution. Second, and perhaps more importantly, we gain much more insight into the problem itself and what its fundamental limitations are.

For example, in the noncausal case (case (a) above), it turns out that it is essential to have at least as many control inputs as one has reference signals, and that the underlying plant should have no unit circle zeros. In the causal case (case (b) above), the result is even more interesting: it is essential that the plant be minimum phase, since for minimum phase plants the best causal tracker performs as well as the best noncausal tracker, whereas for non-minimum phase plants, causal trackers cannot reduce the H^∞ norms from their a priori values. Moreover, we show that the fundamental limitation for causal tracking with a finite delay is the number of nominimum zeros of the plant.

The above properties (of not having unit circle zeros, or being minimum phase) are not directly related to H^∞ norms as such, and therefore one may speculate whether the same qualitative results can be obtained using frameworks other than H^∞. By briefly looking at the problem from the H^2 point of view, we see that this is indeed the case, though the results are not as pronounced.

2 Full Information Tracking

2.1 Problem Formulation

Consider the setting of Fig. 1 where $\{r_i\}$ is a given reference signal that we intend to track, $P(z)$ is a known causal and stable transfer matrix, and $K(z)$ is a controller that must be designed. Broadly speaking, the goal in the tracking problem is to design the controller $K(z)$ so that, based upon the reference signal $\{r_i\}$, it

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constructs a control signal $\{u_i\}$ in such a way that the output of the plant $P(z)$, denoted by $\{\hat{r}_i\}$, tracks the reference signal. This problem is referred to as a full information tracking problem since the controller $K(z)$ has full access to the reference signal $\{r_i\}$.

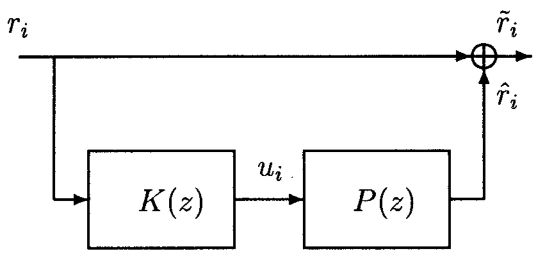

Figure 1: The full information tracking problem.

The deviation of the output of the plant from the desired reference signal is called the tracking error and is defined as,

$$\tilde{r}_i \triangleq r_i - \hat{r}_i. \tag{1}$$

Therefore our goal will be to keep the tracking error, $\{\tilde{r}_i\}$, small. However, in order to guarantee the cost-effectiveness of the final control strategy, it is necessary to try to keep the control signal small as well. Therefore we are left with the twofold objective of designing a controller, $K(z)$, that simultaneously guarantees that the tracking error, $\{\tilde{r}_i\}$, and the control signal, $\{u_i\}$, be small.

One way to achieve the above goal is to define the cost function,

$$J = \|u\|^2 + \|\tilde{r}\|^2 \triangleq \sum_{i=-\infty}^{\infty} u_i^* u_i + \sum_{i=-\infty}^{\infty} \tilde{r}_i^* \tilde{r}_i, \tag{2}$$

where now to make sense of the infinite sums we must assume that $\{u_i\}, \{\tilde{r}_i\} \in l^2$ (the space of square-summable sequences). Moreover, since $\{r_i\}$ may be arbitrary, a better measure is the normalized cost,

$$\frac{\|u\|^2 + \|\tilde{r}\|^2}{\|r\|^2}. \tag{3}$$

Eq. (3) can be regarded as the "energy gain" from the tracking signal $\{r_i\}$ to the control and tracking error signals $\{u_i, \tilde{r}_i\}$. Obviously large gains will correspond to poor tracking and vice-versa. Therefore in this paper our measure of the tracking performance is the following "maximum energy gain",

$$\sup_{\{r_i\}\in l^2} \frac{\|u\|^2 + \|\tilde{r}\|^2}{\|r\|^2}. \tag{4}$$

The above cost function is the essence of the H^∞ approach to tracking. We can also express it in terms of the transfer matrices of Fig. 1. To this end, note that the transfer matrix mapping $\{r_i\}$ to $\left\{\begin{bmatrix} \tilde{r}_i \\ u_i \end{bmatrix}\right\}$ is given by

$$T_K(z) = \begin{bmatrix} I - P(z)K(z) \\ K(z) \end{bmatrix}, \tag{5}$$

so that it is well known that we may write

$$\sup_{\{r_i\}\in l^2} \frac{\|u\|^2 + \|\tilde{r}\|^2}{\|r\|^2} = \|T_K(z)\|_\infty^2. \tag{6}$$

The goal in the H^∞ approach is to choose $K(z)$ so as to minimize (or bound) $\|T_K(z)\|_\infty$. Two distinct cases can be envisioned.

(i) The reference signal is known a priori. *In such cases, the controller $K(z)$ can be noncausal since we have access to future values of the reference signal.*

(ii) The reference signal is given on-line. *In such cases, the controller $K(z)$ is restricted to being causal.* Mathematically, this means that we must have $K(z) \in H^\infty$, *i.e.*, $K(z)$ must be analytic on and outside the unit circle.

We can thus formalize the following problem.

Problem 1 (Full Information Tracking) *Consider the setting of Fig. 1 where the causal and stable plant $P(z)$ is given.*

(a) Find a noncausal controller $K(z)$ that solves

$$\inf_{K(z)} \left\| \begin{bmatrix} I - P(z)K(z) \\ K(z) \end{bmatrix} \right\|_\infty^2 \triangleq \gamma_s^2. \tag{7}$$

(b) Find a causal controller $K(z) \in H^\infty$ that solves

$$\inf_{K(z)\in H^\infty} \left\| \begin{bmatrix} I - P(z)K(z) \\ K(z) \end{bmatrix} \right\|_\infty^2 \triangleq \gamma_c^2. \tag{8}$$

Moreover, find the corresponding minimax energy gains γ_s^2 and γ_c^2.

Remark: Note that in both the above problems $\gamma_s \leq 1$ and $\gamma_c \leq 1$. The reason is that if we do nothing (*i.e.*, $K(z) = 0$) we have $\{u_i\} = 0$ and $\{\tilde{r}_i\} = \{r_i\}$, in which case,

$$\frac{\|u\|^2 + \|\tilde{r}\|^2}{\|r\|^2} = \frac{0 + \|r\|^2}{\|r\|^2} = 1. \tag{9}$$

Therefore how much the optimal values of γ_s and γ_c can be reduced from unity shows how successful we are in the tracking problem. We can therefore write

$$\gamma_s \leq \gamma_c \leq 1, \tag{10}$$

since noncausal trackers have access to more information than causal ones, and should thus perform at least as well.

2.2 Noncausal Solution

Theorem 1 (Noncausal F.I. Tracker) *The minimax energy gain of Problem 1(a) is given by*

$$\gamma_s^2 = \sup_{\omega\in[0,2\pi]} \bar{\sigma}\left[I + P(e^{j\omega})P^*(e^{j\omega})\right]^{-1}, \qquad (11)$$

where $\bar{\sigma}(\cdot)$ denotes the maximum singular value. Moreover, for any $\gamma > \gamma_s$, all controllers that guarantee $\left\|\begin{bmatrix} I - P(z)K(z) \\ K(z)\end{bmatrix}\right\|_\infty^2 \le \gamma^2$ are given by

$$K(z) = \left(I + P^*(z^{-*})P(z)\right)^{-1} P^*(z^{-*}) + S^{-1}(z)Q(z)R(z),$$

where $R(z)$ and $S(z)$ are found from the canonical spectral factorizations,

$$I + P^*(z^{-*})P(z) = R^*(z^{-*})R(z),$$

and

$$\gamma^2 I - \left(I + P(z)P^*(z^{-*})\right)^{-1} = S^*(z^{-*})S(z),$$

and where $Q(z)$ is any contraction, i.e.,

$$Q^*(z^{-*})Q(z) \le I, \quad \forall\, |z| = 1.$$

Remarks:

(i) Suppose $P(z)$ is a $p\times m$ transfer matrix. If $p > m$ (so that there are more signals to track than control inputs), we have $\gamma_s = 1$, since $P(e^{j\omega})P^*(e^{j\omega})$ will be rank deficient at all frequencies. In the square case ($p = m$), if the plant has a unit circle zero then $\gamma_s = 1$ (since $P(e^{j\omega})P^*(e^{j\omega})$ will be rank deficient at the frequency where the zero occurs).

(ii) Recall that $\gamma_s = 1$ implies that we have no improvement over not tracking at all ($K(z) = 0$). This is quite clear when we have a unit circle zero, corresponding to frequency ω_1, say. In this case the plant cannot generate a sinusoid of frequency ω_1, and hence if $\{r_i\}$ is precisely this signal then $P(z)$ cannot track it.

(iii) The choice $Q(z) = 0$ in Theorem 1 yields

$$K_{cen}(z) = \left(I + P^*(z^{-*})P(z)\right)^{-1} P^*(z^{-*}), \quad (12)$$

which is called the *central* solution. This solution coincides with the H^2-optimal noncausal tracker and has the following important property that

$$T^*_{K_{cen}}(e^{j\omega})T^*_{K_{cen}}(e^{j\omega}) \le T^*_K(e^{j\omega})T^*_K(e^{j\omega}), \quad (13)$$

at all frequencies. In other words, $K_{cen}(z)$ outperforms <u>all</u> trackers at <u>all</u> frequencies.

2.3 Causal Solution

The solution of Problem 1(b)is given below. For convenience, we have separated the case of a square plant ($p = m$) from the case of a nonsquare plant ($p \ne m$).

Theorem 2 (Square Plant Case) *Consider the setting of Problem 1(b) where $P(z) = P_0 + P_1 z^{-1} + P_2 z^{-2} + \ldots$ is a causal and stable $p\times p$ transfer matrix.*

(i) If $P(z)$ is minimum phase, i.e., *if $P^{-1}(z)$ is analytic on and outside the unit circle, then the minimax energy gain is given by*

$$\gamma_c^2 = \sup_{\omega\in[0,2\pi]} \bar{\sigma}\left[I + P(e^{j\omega})P^*(e^{j\omega})\right]^{-1}. \quad (14)$$

Moreover, for any $\gamma > \gamma_c$, all causal controllers that guarantee $\left\|\begin{bmatrix} I - P(z)K(z) \\ K(z)\end{bmatrix}\right\|_\infty^2 \le \gamma^2$ are given by

$$K(z) = \left(L_{11}(z) - Q(z)L_{21}(z)\right)^{-1}\left(Q(z)L_{22}(z) - L_{12}(z)\right),$$

where the $L_{ij}(z)$ are given by

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z)\end{bmatrix} = \begin{bmatrix} \frac{X^{-*}}{1-\gamma^2}\cdot\left(P_0^*P(z) - R_\Delta\Delta(z)\right) & -X^{-*}P_0^* \\ \frac{Y}{\sqrt{1-\gamma^2}}\cdot\left(-P(z) + P_0\Delta(z)\right) & Y\sqrt{1-\gamma^2}\end{bmatrix},$$

with the <u>monic</u> transfer matrix $\Delta(z)$ and the matrix R_Δ found from the canonical spectral factorization,

$$\Delta^*(z^{-*})R_\Delta\Delta(z) = \frac{\gamma^2}{1-\gamma^2}\cdot P^*(z^{-*})P(z) - I > 0,$$

and where the constant matrices $\{X, Y\}$ are found from,

$$\begin{cases} X^*X = \frac{1}{1-\gamma^2}\cdot P_0^*P_0 - R_\Delta > 0 \\ Y^*Y = \left[\frac{1}{1-\gamma^2}\cdot P_0 R_\Delta^{-1}P_0^* - I\right]^{-1} > 0 \end{cases},$$

and where $Q(z)$ is any causal contraction contraction, i.e., *$Q(z)$ is analytic on and outside the unit circle and $Q^*(z^{-*})Q(z) \le I$, for all $|z| = 1$.*

(ii) If $P(z)$ is non-minimum phase, i.e., *if $P^{-1}(z)$ is not analytic on and outside the unit circle, then the minimax energy gain is given by $\gamma_c = 1$.*

Remarks:

(i) Note that when $P(z)$ is minimum phase, $\gamma_c = \gamma_s$! This implies that for minimum phase plants causal trackers perform as well as noncausal ones, and that (from an H^∞ point of view) there is no gain in knowing future values of the reference signal $\{r_i\}$.

(ii) However, if $P(z)$ is non-minimum phase, then $\gamma_c = 1$! Thus causal tracking of non-minimum phase plants is not possible, since $\gamma = 1$ is the same bound obtained by not tracking at all ($K(z) = 0$).

(iii) A similar behaviour can be observed had we studied the tracking problem from an H^2 point of view, though the result is not as pronounced. In the scalar case (which for simplicity we shall only consider) the H^2 norm of the H^2-optimal causal tracker is given by

$$d_2 = 1 - \frac{|P_0|^2}{R_e}, \tag{15}$$

where R_e is found from the spectral factorization

$$1 + P^*(z^{-*})P(z) = L^*(z^{-*})R_e L(z), \tag{16}$$

with $L(z)$ monic and minimum phase. It is now easy to show that for all plants that have the same spectrum (*i.e.*, $1 + P_1^*(z^{-*})P_1(z) = 1 + P_2^*(z^{-*})P_2(z)$), so that noncausal trackers have the same performance, $|P_0|^2$ is largest for the minimum phase plant. Thus d_2 is smallest for the minimum phase plant which means that the corresponding tracker has the best H^2 performance. (In fact, it can also be shown that d_2 increases as the number of non-minimum phase zeros increases.)

The most natural choice of the causal contraction of Theorem 2 is $Q(z) = 0$, which corresponds to the central controller

$$K_{cen}(z) = (1-\gamma^2)\left(P_0^* P(z) - R_\Delta \Delta(z)\right)^{-1} P_0^*. \tag{17}$$

The central controller has various other desirable optimality properties, such as being risk-sensitive optimal [Whi90] and maximum entropy [MG90], but we shall not go into these details here.

A less obvious, but nonetheless intriguing, choice is

$$Q(z) = \frac{-1}{\sqrt{1-\gamma^2}} \cdot X^{-*} R_\Delta P_0^{-1} Y^{-1}, \tag{18}$$

a constant matrix, which, when $P(z)$ is minimum phase, can be shown to be a contraction. With this choice, we have

$$K(z) = (1-\gamma^2)P^{-1}(z), \tag{19}$$

i.e., one H^∞ optimal tracker is simply (a scaled version of) the inverse of the plant! [Note that if there were no penalty on the size of the control signal, the inverse of the plant would perfectly reconstruct the reference signal. The scaling factor $1-\gamma^2$, though, is crucial here — if $P(z)$ has zeros close to the unit circle the frequency response of $P(z)$ could get very large at certain frequencies, resulting in large control signals. However, in this case γ_c will be close to one, and thus the factor $1-\gamma^2$ prohibits such large control signals.]

Theorem 3 (Nonsquare Plant Case) *Consider the setting of Problem 1(b) where $P(z)$ is a causal and stable $p \times m$ transfer matrix with $p \neq m$.*

(i) Suppose $p < m$. Then if $P(z)$ has no zeros on or outside the unit circle, $\gamma_c^2 < 1$. Otherwise $\gamma_c = 1$.

(ii) Suppose $p > m$. Then $\gamma_c = 1$.

Remark: Note that when $p < m$, $P(z)$ will generically have no zeros because it will generically have full rank for all z. To be more explicit, suppose that $p = 1$ and $m = 2$, so that

$$P(z) = \begin{bmatrix} P_1(z) & P_2(z) \end{bmatrix}. \tag{20}$$

Now $P(z)$ will have a zero outside the unit circle if, and only if, $P_1(z)$ and $P_2(z)$ *share* some non-minimum phase zero. But of course any two arbitrary rational functions will generically not have common zeros.

2.4 Remedies for Non-Minimum Phase Plants

Theorem 2 indicates that in the square case, causal tracking of non-minimum phase plants is not possible. The above analysis, however, suggests the following remedies for the non-minimum phase case.

- *Add more actuators.* This will result in $p < m$ for which Theorem 3 indicates that we will generically have $\gamma_c^2 < 1$.
- *Allow for some finite delay.*

The second solution essentially means that to track r_i one uses observations of the reference signal from time $-\infty$ to $i + d$, for some finite $d > 0$. Thus, in this framework we will be tracking the reference signal with a delay of d time units.

To begin to study the effect of delay it will be instructive to begin with the following simple example:

$$P(z) = 1 + \alpha z^{-1}, \quad \alpha \in \mathcal{C}. \tag{21}$$

From Theorem 2 we know

$$\gamma_c = \begin{cases} \gamma_s = \frac{1}{1+(1+|\alpha|)^2} & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha \geq 1 \end{cases}. \tag{22}$$

Now it can be shown that if we allow for a delay of $d = 1$, we get

$$\gamma_c = \begin{cases} \gamma_s & \text{if } \alpha < 2 \\ \frac{2}{|\alpha|^2} & \text{if } \alpha \geq 2 \end{cases}. \tag{23}$$

Thus the region for which we perform as good as the noncausal solution (corresponding to $d = \infty$) expands from the unit circle to a circle of radius 2, and moreover, there is no region for which $\gamma_c = 1$. Finally, for $d \geq 2$, we obtain $\gamma_c = \gamma_s$ for all α, so that, for this particular choice of plant, a delay of two units allows the same performance as a noncausal tracker.

Unfortunately the above pattern does not generalize, and analyzing the case of a general plant is much more difficult. Here, however, is a result that indicates the minimum delay necessary to avoid $\gamma_c = 1$.

Theorem 4 (Effect of Delay) *Consider the setting of Problem 1(b) where $P(z)$ is causal and stable a $p \times p$ rational transfer matrix, but now suppose that the causal tracker has to track the delayed reference signal $\{r_{i-d}\}$, for some $d > 0$. Denote by l the number of non-minimum phase zeros of $P(z)$. Then,*

(i) If $d < l$, we have $\gamma_c^2 = 1$.

(ii) If $d \geq l$, we have $\gamma_c < 1$.

Thus the minimum delay required to ensure $\gamma_c < 1$ is the number of non-minimum phase zeros of $P(z)$.

3 Measurement Feedback Tracking

3.1 Problem Formulation

In many applications the reference signal is not directly available and can only be obtained through some measurement process. This situation is depicted in Fig. 2 where $\{r_i\}$ is the reference signal that we intend to track, $P_1(z)$ and $P_2(z)$ are known causal and stable transfer matrices, and $K(z)$ is a controller that must be designed.

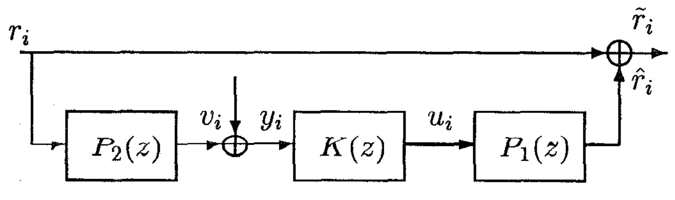

Figure 2: The measurement feedback tracking problem.

In the measurement feedback tracking problem the reference signal $\{r_i\}$ is not known to the controller. What is known is the signal $\{y_i\}$, which can be regarded as a noisy measurement (because of the unknown additive disturbance $\{v_i\}$) of the reference signal $\{r_i\}$ via the (measurement) system $P_2(z)$. The controller must now use the measurement signal $\{y_i\}$ to construct a control signal $\{u_i\}$, such that the output of the plant $P_1(z)$ tracks the reference signal.

As in the full information problem, to ensure a cost-effective control strategy we are confronted with the two-fold task of guaranteeing that the control signal $\{u_i\}$ and that tracking error $\{\tilde{r}_i\}$ be simultaneously small. In the H^∞ framework this leads to the normalized cost function

$$\frac{\|u\|^2 + \|\tilde{r}\|^2}{\|r\|^2 + \|v\|^2}, \tag{24}$$

where we have assumed that the $\{r_i\}, \{v_i\} \in l^2$. The above expression can be regarded as the energy gain from the unknown reference signal $\{r_i\}$ and the unknown additive disturbance $\{v_i\}$ to the control signal $\{u_i\}$ and the tracking error $\{\tilde{r}_i\}$. Therefore in the H^∞ framework, that we are considering, the tracking performance is measured by the worst-case energy gain of (24). Noting that the transfer matrix mapping $\left\{\begin{bmatrix} r_i \\ v_i \end{bmatrix}\right\}$ to $\left\{\begin{bmatrix} \tilde{r}_i \\ u_i \end{bmatrix}\right\}$ is given by

$$T_K(z) = \begin{bmatrix} I - P_1(z)K(z)P_2(z) & -P_1(z)K(z) \\ K(z)P_2(z) & K(z) \end{bmatrix},$$

the desired cost can be written as

$$\sup_{\{r_i\},\{v_i\}\in l^2} \frac{\|u\|^2 + \|\tilde{r}\|^2}{\|r\|^2 + \|v\|^2} = \|T_K(z)\|_\infty^2. \tag{25}$$

As in the full information case, we have the following two problems.

Problem 2 (Measurement Feedback Tracking) *Consider the setting of Fig. 2 where the causal and stable plants $P_1(z)$ and $P_2(z)$ are given.*

(a) Find a noncausal controller $K(z)$ that solves

$$\inf_{K(z)} \left\| \begin{bmatrix} I - P_1(z)K(z)P_2(z) & -P_1(z)K(z) \\ K(z)P_2(z) & K(z) \end{bmatrix} \right\|_\infty^2.$$

(b) Find a causal controller $K(z) \in H^\infty$ that solves

$$\inf_{K(z)\in H^\infty} \left\| \begin{bmatrix} I - P_1(z)K(z)P_2(z) & -P_1(z)K(z) \\ K(z)P_2(z) & K(z) \end{bmatrix} \right\|_\infty^2.$$

Moreover, find the corresponding minimax energy gains, γ_s^2 and γ_c^2.

Remark: Note that, as in the full information case, if we choose $K(z) = 0$ then

$$\frac{\|u\|^2 + \|\tilde{r}\|^2}{\|r\|^2 + \|v\|^2} = \frac{0 + \|r\|^2}{\|r\|^2 + \|v\|^2} \leq 1. \tag{26}$$

We can therefore write $\gamma_s \leq \gamma_c \leq 1$. Once again, how much the optimal values of γ_s and γ_c can be reduced from unity shows how successful we are in the tracking problem.

3.2 Noncausal Solution

Theorem 5 (Noncausal M.F. Tracker) *The minimax energy gain of Problem 2(a) is given by*

$$\gamma_s^2 = \max\left\{ \sup_{\omega\in[0,2\pi]} \bar{\sigma}\left[I + P_1(e^{j\omega})P_1^*(e^{j\omega})\right]^{-1}, \sup_{\omega\in[0,2\pi]} \bar{\sigma}\left[I + P_2^*(e^{j\omega})P_2(e^{j\omega})\right]^{-1} \right\}. \quad (27)$$

Moreover, for any $\gamma > \gamma_s$, all controllers that guarantee $\left\| \begin{bmatrix} I - P_1(z)K(z)P_2(z) & -P_1(z)K(z) \\ K(z)P_2(z) & K(z) \end{bmatrix} \right\|_\infty^2 \leq \gamma^2$ *are given by*

$$K(z) = K_{cen}(z) + S^{-1}(z)Q(z)R(z),$$

where

$$K_{cen}(z) = P_1^*(z^{-*})\left[\gamma^{-2}I - \left(I + P_2^*(z^{-*})P_2(z)\right) \times \left(I + P_1(z)P_1^*(z^{-*})\right)\right]^{-1} P_2^*(z^{-*}),$$

and $R(z)$ and $S(z)$ are found from the canonical spectral factorizations,

$$I + P_2(z)\left[I - \frac{(I + P_1(z)P_1^*(z^{-*}))}{\gamma^2}\right]^{-1} P_2^*(z^{-*}) = R^*(z^{-*})R(z),$$

and

$$I + P_1^*(z^{-*})\left[I - \frac{(I + P_2^*(z^{-*})P_2(z))}{\gamma^2}\right]^{-1} P_1(z) = \gamma^2 S^*(z^{-*})S(z),$$

and where $Q(z)$ is any contraction.

Remark: Note that γ_s^2 is the maximum of

$$\gamma_{s,1}^2 = \sup_{\omega\in[0,2\pi]} \bar{\sigma}\left[I + P_1(e^{j\omega})P_1^*(e^{j\omega})\right]^{-1}, \quad (28)$$

which is the minimax energy gain for the full information tracking problem with plant $P_1(z)$, and

$$\gamma_{s,2}^2 = \sup_{\omega\in[0,2\pi]} \bar{\sigma}\left[I + P_2^*(e^{j\omega})P_2(e^{j\omega})\right]^{-1}, \quad (29)$$

which is the minimax energy gain for the estimation problem of estimating the reference signal $\{r_i\}$ from the measurement signal $\{y_i\}$. Thus the performance of the measurement feedback tracking problem is constrained both by our ability to perform full information tracking (assuming that $\{r_i\}$ is known) and by our ability to estimate the reference signal from the measurements.

3.3 Causal Solution

The solution to 2(b) is given below. For simplicity, we have not given the expressions for the optimal controllers (since they are somewhat involved), and have only given the expressions for γ_c. As can be seen, the crucial distinction is between minimum phase and non-minimum phase plants (for both $P_1(z)$ and $P_2(z)$).

Theorem 6 (Measurement Feedback Tracker) *Consider the setting of Problem 2(b) where $P_1(z)$ and $P_2(z)$ are given causal and stable $p \times m$ and $q \times p$ transfer matrices, respectively.*

(i) Suppose $p = m = q$. If both $P_1(z)$ and $P_2(z)$ are minimum phase, i.e., *if both $P_1^{-1}(z)$ and $P_2^{-1}(z)$ are analytic on and outside the unit circle, then the minimax energy gain is given by $\gamma_c = \gamma_s$, which is the same as in the noncausal case. If, however, either $P_1(z)$ or $P_2(z)$ is non-minimum phase,* i.e., *if either $P_1^{-1}(z)$ or $P_2^{-1}(z)$ is not analytic on and outside the unit circle, then the minimax energy gain is given by $\gamma_c = 1$.*

(ii) Suppose $p > m$ or $p > q$. Then the minimax energy gain is given by $\gamma_c = 1$.

(iii) Suppose $p < m$ and $p < q$. Then if $P_1(z)$ and $P_2(z)$ have no zeros on or outside the unit circle, we have $\gamma_c < 1$, and otherwise, $\gamma_c = 1$.

Note that the comments following Theorem 2, as well as the remedies for non-minimum phase plants, apply here as well. We shall therefore not repeat them.

4 Conclusion

In this paper we studied the tracking problem from the H^∞ point of view and obtained and parametrized all possible H^∞ optimal trackers (for both the full information and measurement feedback tracking problems). We observed a strict dichotomy between minimum phase and non-minimum phase plants, in the sense that for minimum phase plants the best causal tracker performs as well as the best noncausal tracker, whereas for non-minimum phase plants, causal trackers cannot reduce the H^∞ norms from their a priori values. We also showed that the fundamental limitation in the causal tracking of a non-minimum phase plant is its number of non-minimum phase zeros. These results also have various implications to "worst-case controllability" (essentially, the question of whether a given plant is easy to control or not) and to dual problems in estimation (especially the equalization problem) which are currently under investigation.

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