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First-order Formalism and Odd-derivative Actions

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Abstract

In this pedagogical note, we discuss obstacles to the usual Palatini formulations of gauge and gravity theories in presence of odd-derivative order, Chern-Simons, terms.

1 Introduction

Writing second order actions in first order form is a particular case of the general Ostrogradski'i procedure of lowering derivative order by adding new variables. This method has proven particularly illuminating in gauge theories, as exemplified by the Palatini formulation of general relativity, keeping metric and connection as independent field variables. Indeed, it has become a major activity for generalized gravity models, but it has long been well-understood for standard GR, where the primary distinction between the two formulations consists of relative matter contact terms when spinors are present [1]. In this note, we consider a more fundamental distinction, quite apart from matter couplings, in presence of odd-derivative order, Chern–Simons terms, particularly in D=3 where they play a direct kinematical role. We shall see that major changes take place, in part because gauge invariance can be lost in the Ostrogradski'i process. We will deal first with the simplest, abelian vector, topologically massive model (TME), then with the full nonlinear gravitational one (TMG); both have been exhaustively analyzed in their second order avatars in [2], whose notations we follow.

2 Vectors

The equivalent Maxwell Lagrangians are

$$M_2(A) = -\tfrac{1}{4}\, f^2_{\mu\nu}(A) \quad , \quad f_{\mu\nu}(A) \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \; , \tag{1a}$$

$$M_1(A,F) = -\tfrac{1}{2}\, F^{\mu\nu} f_{\mu\nu}(A) + \tfrac{1}{4}\, F^2_{\mu\nu} \; . \tag{1b}$$

Here $F_{\mu\nu}$ is an independent gauge-invariant variable. The Chern–Simons (CS) contributions can also be written in two ways,

$$C_1(A) = -\tfrac{m}{2}\, \varepsilon^{\mu\nu\alpha} f_{\mu\nu}(A) A_\alpha \tag{2a}$$

$$C_0(A,F) = -\tfrac{m}{2}\,\varepsilon^{\mu\nu\alpha} F_{\mu\nu} A_\alpha \ . \tag{2b}$$

The "$0^{\rm th}$ order" form (2b) is no longer gauge invariant (unlike the integral of (2a)). [A modification of (1b, 2b) that we will not consider here, replaces $F_{\mu\nu}$ by the combination $f_{\mu\nu}(B)$, where B_μ is a second potential; this retains gauge invariance, at the price of keeping derivative order.] The combination (1a + 2a) is of course the original TME with a single massive excitation. This same model is reproduced by taking (1b + 2a); the CS term does not affect the $F_{\mu\nu} = f_{\mu\nu}(A)$ field equation. However, (1a + 2b) represents a drastic modification: we learn that A_μ, and consequently $F_{\mu\nu}$, both vanish; (loss of gauge invariance of (2b) forces vanishing of A_μ). Finally, the doubly Palatini variant (1b + 2b) has the field equations

$$\partial_\mu F^{\mu\nu} + m\,{}^*F^\nu = 0 \ \ , \quad {}^*F^\alpha \equiv \tfrac{1}{2}\,\varepsilon^{\mu\nu\alpha} F_{\mu\nu} \ . \tag{3a}$$

$$F_{\mu\nu} = f_{\mu\nu}(A) + m\,\varepsilon_{\mu\nu}{}^{\alpha} A_\alpha \ . \tag{3b}$$

Upon eliminating $F_{\mu\nu}$, we find a combined TME/Proca equation,

$$\partial_\mu f^{\mu\alpha} + m\,\varepsilon^{\mu\nu\alpha} f_{\mu\nu} - m^2 A^\alpha = 0 \ . \tag{4}$$

This is known [3] to describe a topological-ordinary mass mix, with two massive excitations, of masses $m(\sqrt{2} \pm 1)$, as against the value m for the single gauge invariant one of TME.

In summary, the differences in kinematic content of the four candidate "TMEs" are extreme, ranging from ordinary TME (in two cases) to no excitation to two massive ones, depending on where $f_{\mu\nu}(A) \to F_{\mu\nu}$ is inserted.

3 Gravity

Let us first list the candidate Lagrangians in the (fully nonlinear) gravitational 2+1 TMG of [2].

$$E_2(g) = \sqrt{-g}\; g^{\mu\nu}\; R_{\mu\nu}(g) \tag{5a}$$

$$E_1(g,\Gamma) = \sqrt{-g}\; g^{\mu\nu}\; R_{\mu\nu}(\Gamma) \ . \tag{5b}$$

Here $R_{\mu\nu}(g)$ is the usual metric Ricci tensor, while $R_{\mu\nu}(\Gamma)$ is the purely affine version in terms of the (symmetric) connection $\Gamma^\alpha_{\mu\nu}$. Although $R_{\mu\nu}(\Gamma)$ is not symmetric (because $\partial_\mu\, \Gamma^\alpha_{\nu\alpha} \neq \partial_\nu\, \Gamma^\alpha_{\mu\alpha}$), only its symmetric projection survives in (5b). The equivalence of (5a) and (5b) rests on the Palatini identity,

$$\delta\, R_{\mu\nu}(\Gamma) \equiv D_\alpha(\Gamma)\, \delta\, \Gamma^\alpha_{\mu\nu} - D_\mu(\Gamma)\, \delta\, \Gamma^\alpha_{\nu\alpha} \tag{6}$$

where $D_\mu(\Gamma)$ is the (affine) covariant derivative and $\delta\Gamma$, being the difference of two affinities at a point, transforms as a tensor. Varying the affinity in (5b) then implies that $D_\alpha(\Gamma) g^{\mu\nu} = 0$, determining $\Gamma^\alpha_{\mu\nu}$ to be the purely metric Christoffel symbol $\{{}^{\,\alpha}_{\mu\nu}\}$, at least if $g^{\mu\nu}$ is invertible [4]. Now we adjoin to (5a, 5b) the two versions of the gravitational CS term. They also differ from vector CS in that the purely affine version is both gauge invariant and depends only on the Γ and not at all on the metric:

$$C_1(\Gamma) = -\tfrac{1}{2}\,\mu^{-1}\varepsilon^{\lambda\mu\nu}\,\Gamma^{\rho}_{\lambda\sigma}\Big(\partial_\mu\Gamma^{\sigma}_{\rho\nu} + \tfrac{2}{3}\,\Gamma^{\sigma}_{\mu\beta}\Gamma^{\beta}_{\nu\rho}\Big) \tag{7a}$$

$$C_3(g) \equiv C_1(\Gamma^{\alpha}_{\mu\nu} = \{{}^{\,\alpha}_{\mu\nu}\})\ . \tag{7b}$$

The C_1 version is of first derivative order (in Γ) while the metric version (7b) involves three derivatives. The usual [2] TMG is the sum (5a + 7b), with a single gauge excitation of mass μ. The combination (5a + 7a), metric Einstein plus "Palatini" CS, is–surprisingly–equivalent to pure Einstein gravity, $R_{\mu\nu}(g) = 0$, plus a totally decoupled and rather empty connection sector, with

$$\varepsilon^{\mu\nu\alpha}\,R^{\lambda}_{\sigma\mu\nu}(\Gamma) + (\alpha\lambda) = 0 \tag{8}$$

where $R^{\lambda}_{\sigma\mu\nu}(\Gamma)$ is the Riemann tensor of Γ. This states essentially that Γ is integrable (but still not metric). The opposite version of Palatini Einstein and metric CS, (5b + 7b), is in fact TMG in disguise: varying the affinity, that only appears in (5b), tells us that Γ is $\Gamma(g)$, whereupon the δg equation states that

$$\sqrt{-g}\,G^{\mu\nu}(g) + \mu^{-1}C^{\mu\nu}(g) = 0\ \ ,\quad C^{\mu\nu}(g) \equiv \varepsilon^{\mu\alpha\beta}g_{\beta\sigma}D_\alpha(R^{\sigma\nu} - \tfrac{1}{4}\,g^{\sigma\nu}R)\ . \tag{9}$$

So far, then, we have two separate versions of TMG and one of pure Einstein. The final, and more difficult, combination is the fully Palatini model of (5b + 7a). Since the metric only appears in (5b), its variation says that $R_{(\mu\nu)}(\Gamma) = 0$. Varying Γ gives two terms:

$$D_\alpha(\Gamma)(g^{\mu\nu}\sqrt{-g}\,) + \Big\{\varepsilon^{\lambda\sigma\nu}R^{\mu}_{\alpha\lambda\sigma}(\Gamma) + (\nu\mu)\Big\} = 0\ . \tag{10}$$

This means that the affinity is no longer a metric one, but rather obeys a far more complicated relation, one we have been unable to solve. [Of course, a consistent solution is to assume each term in (10) to vanish separately, which would reduce the system to pure Einstein.] Unfortunately, the special fact of D=3 metric geometry, that metric Riemann and Ricci tensors are equivalent

$$\tfrac{1}{4}\,\varepsilon^{\mu\alpha\beta}\,R_{\alpha\beta\lambda\sigma}(g)\,\varepsilon^{\lambda\sigma\nu} = \det g\,G^{\mu\nu}(g)\ , \tag{11}$$

does not at all hold at purely affine level.

In summary, we have extended the TME/TMG vector/tensor D=3 models to allow for independence of affinity and potential. In each case, we have seen quite dramatic differences. For vectors, making the higher derivative (Maxwell term) Palatini keeps the TME structure, while the opposite choice forbids any excitations. Double Palatini correspond, to TME + Proca, with two excitations of different mass.

The gravitational situation is different. If we let the "highest-derivative", CS-term only become Palatini, we find pure metric Einstein gravity. The other case, in which CS remains metric just reproduces standard TMG, in analogy to the vector effect. Finally, double Palatini here becomes quite involved, although it does allow for pure Einstein as a consistent solution.

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References

[1] H. Weyl, Phys. Rev. **77**, 699 (1950); S. Deser and B. Zumino, Phys. Lett. **62B**, 335 (1976).

[2] S. Deser, R. Jackiw and S. Templeton, Ann. Phys. **140**, 372 (1982).

[3] S. Deser and B. Tekin, hep-th/0203273, Class. Quant. Grav. **19**, L97 (2002) and references therein.

[4] S. Deser, gr-qc/0603125, Class. Quant. Grav. (to appear).