On the capacity region of broadcast over wireless erasure networks

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Abstract

In this paper, we consider a special class of wireless networks, called wireless erasure networks. In these networks, each node is connected to a set of nodes by independent erasure channels. The network model incorporates the broadcast nature of the wireless environment in that each node sends out the same signal on its outgoing channels. However, we assume there is no interference in reception. In this paper we first look at the single source single destination unicast problem. We obtain the capacity under the assumption that erasure locations on all the links of the network are provided to the destination. It turns out that the capacity has a nice max-flow min-cut interpretation. The definition of cut-capacity in these network is such that it incorporates the broadcast property of the wireless medium. In the second part of the paper, a time-sharing scheme for broadcast problems over these networks is proposed and its achievable region is analyzed. We show that for some special cases, this time-sharing scheme is optimal.

1 Introduction

Determining the capacity region for general multi-terminal networks has been a long-standing open problem. An outer bound for the capacity region is proposed in [1]. This outer bound has a nice min-cut interpretation: The rate of flow of information across any cut (a cut is a partition of the network into two parts) is less than the cut-capacity corresponding to that cut, where the cut-capacity is defined as the maximum rate that can be achieved if the nodes on each side of the cut can fully cooperate and also use their inputs as side-information.

The difficulty in multi-terminal information theory is that this outer bound is not necessarily tight. For instance, for single relay channels introduced in [2], no scheme is known that can achieve the min-cut outer bound of [1].

However, for a class of network problems called multicast problems in *wireline* networks, it is shown that the max-flow min-cut outer bound can be achieved [3, 4, 5].

In a wireless setup, however, the problem of finding the capacity region is more complicated. The main reason is that unlike wireline networks in which communication between

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different nodes is done using separated media, in a wireless system the communication medium is shared. The capacity (region) of many information-theoretic channels that capture this effect is not known. The capacity of broadcast channels in general is an unsolved problem [6].

In this paper we look at a special class of wireless networks which only incorporates the broadcast feature of wireless networks. We assume that each communication channel in the network can be modeled as an independent and memoryless erasure channel. We assume that each node sends out the same signal on each outgoing link. However, for reception we assume a multiple access model without interference, i.e., messages coming in to a node from different incoming links do not interfere. In general, this is not true for a wireless system. However, this can be realized through some time/frequency/code division multiple access scheme.

Finally, we assume that the side-information regarding erasure locations on each link is available to the destination(s). If we assume that the erasure network operates on long packets, i.e., packets are either erased or received exactly on each link, then this assumption can be justified by using headers in the packets to convey erasure locations or by sending a number of extra packets containing this information. By making the packets very long the overhead of transmitting the erasure locations can be made negligible compared to the packet length. We should remark that provided that the side-information is available to the destinations, all the results in this paper hold for any packet length.

In this paper, we first show that with a suitable definition of cut-capacity, a max-flow min-cut type of result holds for single source/single destination problems in wireless erasure networks under the assumptions mentioned above. Then we look at a class of network problems, called broadcast problems. In these problems, there is one source and a number of destinations. Each destination demands an independent information from the source. Unlike the wireline setup, we will show that in wireless erasure networks, max-flow min-cut outer bounds can be loose. We then analyze achievable rates for broadcast problems using time-sharing schemes and show that in some cases time-sharing achieves all the rates in the capacity region.

This paper is organized as follows. We introduce the network model in Section 2 and the problem setup in Section 3. Section 4 considers a class of network problems, called broadcast problems . We first briefly mention the capacity result for single source/single destination problem. Next we find an achievable rate region based on time-sharing schemes and show its optimality in some cases. We mention future directions of our work and conclude in Section 5.

2 Notations and Network Model

Notations

Throughout this paper, upper case letters (e.g., X, Y, Z) are usually used to denote random variables and lower case letters (e.g., x, y, z) denote the values they take. Underlined letters (e.g., $\underline{x}$) are used to denote vectors. Sets are denoted by calligraphic alphabet (e.g. $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$). The complement of a set $\mathcal{A}$ is shown by $\mathcal{A}^c$. Subscripts are usually used to denote different nodes, edges, inputs, outputs and time.

Consider a sequence of numbers $x_1, x_2, x_3, \ldots$. We use notation x^n to denote the sequence $x_1, x_2, \ldots, x_n$. We also use notation $(x_i,\ i \in \mathcal{I})$ to denote the ordered tuple specified by index set $\mathcal{I}$. Finally, the cardinality of set $\mathcal{X}$ is denoted by $|\mathcal{X}|$.

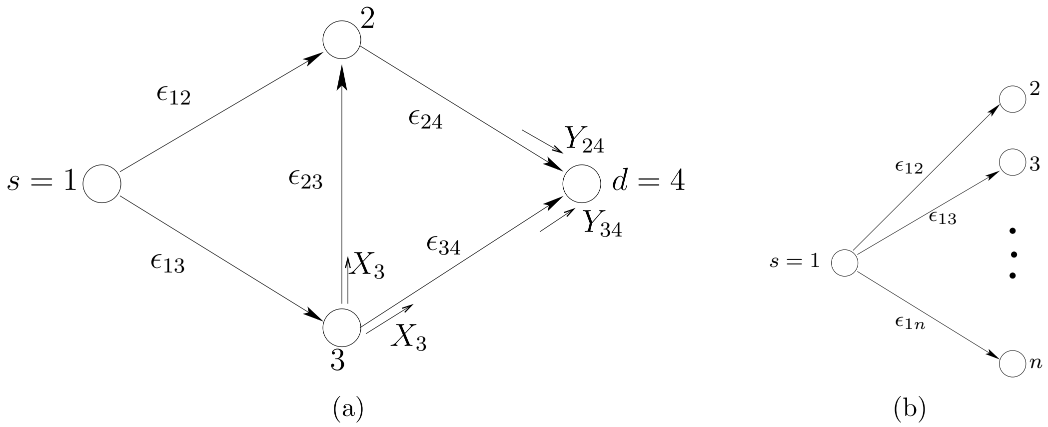

Figure 1: (a): An example of an erasure wireless network. The probability of erasure of each link is written on it. Each node (e.g. node 3) sends out the same signal (X_3) across its outgoing channels. Also because of the interference-free property, node 4 receives both signals Y_{24} and Y_{34} completely. (b): Erasure broadcast channel with $n-1$ receivers

Wireless Packet Erasure Network

We model the wireless packet[1] erasure network by a directed acyclic graph $\mathcal{G} = (\mathcal{V} = \{1, \ldots, |\mathcal{V}|\}, \mathcal{E})$. Each edge $(i, j) \in \mathcal{E}$ represents an independent memoryless packet erasure channel from node i to node j. A packet sent across this channel is erased with probability of erasure ϵ_{ij} or is received without error. We denote the input alphabet (the set of possible packets) of the erasure channel by $\mathcal{X}$.[2]

Let $S_{ij,t}$ be a random variable indicating erasure occurrence across channel (i, j) at time t. $S_{ij,t}$ has a Bernoulli distribution with parameter ϵ_{ij}.

We assume that transmission on each channel experiences one unit of time delay. The input of the all the channels originating from node i is denoted by X_i chosen from the input alphabet $\mathcal{X}$. Note that with this definition we have insisted that each node transmit the same symbol on all its outgoing edges, i.e., all channels corresponding to edges in $\mathcal{N}_O(i) = \{(i, j)|(i, j) \in \mathcal{E}\}$ have the (same) input X_i (See Fig. 1). This constraint incorporates broadcast in our network model. The output of the communication channel corresponding to edge $(i, j) \in \mathcal{E}$ is denoted by Y_{ij} chosen from output alphabet $\mathcal{Y} = \mathcal{X} \cup \{e\}$, where e denotes erasure symbol. We also assume that the outputs of all the channels corresponding to edges in $\mathcal{N}_I(i) = \{(j, i)|(j, i) \in \mathcal{E}\}$ are available at node i. This condition is equivalent to having no interference in receptions in the network. Having this, let $Y_i = (Y_{ji}, (j, i) \in \mathcal{N}_I(i))$ be the symbols that are received at node i from all its incoming channels. We have $Y_i \in \prod_{j:(j,i)\in\mathcal{E}} \mathcal{Y}$. The relation between the Y_is and X_is would define a coding scheme for the network.

Cut-capacity Definition

For directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and any two nodes $s, d \in \mathcal{V}$, an $s-d$ cut is a partition of nodes in two subsets one containing s and the other d. Clearly any s-d cut, can be specified completely by the source set (i.e., the set containing node s). For $s-d$ cut specified by source set $\mathcal{V}_s$, the cut-set is defined as the set of edges from source set to the

[1]Throughout this paper a packet can be of any length, for the case when the length of packets is one, the channel is a binary erasure channel.

[2]For simplicity and without loss of generality we consider $\mathcal{X} = \{0, 1\}$ in our analysis and proofs, however we should remark that all the results and analysis holds for input alphabet of arbitrary length.

destination set, i.e., $[\mathcal{V}_s, \mathcal{V}_s^c] = \{(i,j)|i \in \mathcal{V}_s, j \in \mathcal{V}_s^c, (i,j) \in \mathcal{E}\}$.
We also define $X(\mathcal{V}_s)$ and $Y(\mathcal{V}_s)$ as

$$X(\mathcal{V}_s) = \{X_i | i \in \mathcal{V}_s^*\} \qquad\qquad Y(\mathcal{V}_s) = \{Y_{ij} | (i,j) \in [\mathcal{V}_s, \mathcal{V}_s^c]\}. \tag{1}$$

At the end of this section, we define the cut-capacity for wireless erasure networks. In wireline networks, the value of cut-capacity is the sum of the capacities of the edges in the cut-set [5]. Such a definition of cut-capacity in wireline networks makes sense because the nodes can send out different signals across their outgoing edges. However, this is not the case for wireless erasure networks where the nodes should broadcast their signal. The following definition of cut-capacity is different from wireline network settings and it incorporates the broadcast nature of transmission in our network.

Definition Consider an erasure wireless network represented by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and probabilities of erasure ϵ_{ij} as described in Section 2. The cut-capacity corresponding to any $s - d_l$ cut represented by *s-set*, $\mathcal{V}_s$, is denoted by $C(\mathcal{V}_s)$ and is equal to

$$C(\mathcal{V}_s) = \sum_{i \in \mathcal{V}_s^*} \left(1 - \prod_{j:\,(i,j) \in [\mathcal{V}_s, \mathcal{V}_{d_l}]} \epsilon_{ij}\right) \tag{2}$$

Example 2.1. Consider the network shown in Fig. 1. For the $s - d$ cut specified by the s-set $\mathcal{V}_s = \{1,3\}$, the cut-capacity is $C(\mathcal{V}_s) = 1 - \epsilon_{12} + 1 - \epsilon_{32}\epsilon_{34}$. Looking at this example, we see that all the edges in the cut-set that originate from a common node, i.e. edges $(3,2)$ and $(3,4)$, are grouped together and they contribute a value of one minus the product of their erasure probabilities, i.e., $1 - \epsilon_{32}\epsilon_{34}$ to the cut-capacity. This observation holds in general for wireless erasure networks.

3 Broadcast Problem Statement

In this paper we will look at a class of network problems called broadcast problems. In these problems, there is one source and multiple destinations. Each of the destinations request an independent information from the source. Let us define the class of block codes considered in this paper. Let $\mathcal{D} = \{d_1, \ldots, d_{|\mathcal{D}|}\}$ denote the set of destination nodes and s be the source node.

A $(\lceil 2^{nR_1} \rceil, \ldots, \lceil 2^{nR_{|\mathcal{D}|}} \rceil, n)$ code for the broadcast problem in a wireless erasure network described in previous sections, consists of the following components:

- A set of integers $\mathcal{W}^{(d_i)} = \{1, 2, \ldots, \lceil 2^{nR_i} \rceil\}$ that represent the message indices corresponding to information source that is intended for destination node $d_i \in \mathcal{D}$. We assume that all the messages are equally likely. All the information sources are available at the source node indexed by $s \in \mathcal{V}$.

- An encoding function for the source node s: $f_s : \prod_{d \in \mathcal{D}} \mathcal{W}^{(d)} \to \mathcal{X}^n$.

- A set of encoding functions $\{f_{i,t}\}_{t=1}^n$ for each node $i \neq s \in \mathcal{V}$, where $x_{i,t} = f_{i,t}(y_i^{t-1})$ is the signal transmitted by node i at time t. Note that $x_{i,t}$ is a function of all the received symbols from all its incoming channels up to time $t-1$.

- A decoding function g_{d_i} at destination node $d_i \in \mathcal{D}$, $g_{d_i} : \mathcal{Y}_{d_i}^n \times \{0,1\}^{n|\mathcal{E}|} \to \mathcal{W}^{(d_i)}$ such that

  $$\hat{w}^{(d_i)} = g_{d_i}(y_{d_i}^n, (s_{ij,t}\,, (i,j) \in \mathcal{E},\, 1 \le t \le n)) \tag{3}$$

  where $\hat{w}^{(d_i)}$ is the estimate of the message sent from source s based on received signals at d_i, and also the erasure occurrences on all the links of the network in the current block.

Note that X_i, Y_{ij} and Y_i all depend on the message vector $\underline{w} = (w^{(d_i)},\ d_i \in \mathcal{D})$, that is being transmitted. Therefore we will write them as $X_i(\underline{w})$,$Y_{ij}(\underline{w})$ and $Y_i(\underline{w})$ to specify what specific set of messages in transmitted.

We define the probability of error as the probability that the decoded message at one of the destinations is not equal to the transmitted message, i.e.,

$$P_{err} = \Pr\left(\exists d_i \in \mathcal{D} : \hat{W}^{(d_i)} \neq W^{(d_i)}\right) \tag{4}$$

The set of rates $(R_i, \quad 1 \leq i \leq |\mathcal{D}|)$ is said to be achievable if there exist a sequence of $(\lceil 2^{nR_1} \rceil, \ldots, \lceil 2^{nR_{|\mathcal{D}|}} \rceil, n)$ codes such that $P_{err} \to 0$ as $n \to \infty$. The capacity region is the set closure of the set of achievable rates.

4 Results

It is shown in [5] that for wireline networks the capacity region for broadcast problems satisfies a max-flow min-cut interpretation. One important question is whether these results hold for the wireless erasure networks considered in this paper. In this section, we will look at this problem.

Before looking at the general problem, we consider a special case of these problems, namely when there is only one destination node. We briefly mention the capacity result for a single source/single destination problem in wireless erasure networks. The main result is that using the cut-capacity definition of the previous section, it can be proved that the maximum achievable rate in the single source/single destination problem is given by the value of the source/destination cut with the minimum cut-capacity. We have stated the result formally in the following theorem.

Theorem 1. *Consider a single source/ single destination wireless erasure network described by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and assumptions of Section 2. Let $s \in \mathcal{V}$ and $d \in \mathcal{V}$ denote the source and destination respectively. Then the capacity of the network with side-information at the destination is given by the value of the minimum value s-d cut. More precisely, we have*

$$C = \min_{\mathcal{V}_s:\text{an } s\text{-}d \text{ cut}} C(\mathcal{V}_s). \tag{5}$$

Proof: The achievability is proved by a random coding argument. The converse part can be easily shown by allowing for cooperation among sub-set of nodes. We omit the proof for brevity. For proof refer to [7]. □

Erasure Broadcast Channel

Now let's look at the broadcast problem. We start with a very simple example of wireless erasure networks. Consider an erasure broadcast channel shown in Fig. 1.(b). The capacity region of broadcast channels in general is not known [6]. However, the capacity region for a special class of broadcast channels called degraded channels is known [8],[9]. Fortunately, erasure broadcast channels are degraded and therefore their capacity region is known. It is shown in [10] that the all the rates in the capacity region of a two user broadcast channel can be achieved by time-sharing. Using the same approach and the expression for the capacity region of a degraded broadcast channel with more than two users [8], the optimality of time-sharing for erasure broadcast networks with any number of receivers can be proved. The proof is provided in the Appendix. We state the above result in the following theorem.

Theorem 2. *The capacity region of the erasure broadcast channel shown in Fig. 1.(b) is*

$$\{(R_2,\ldots,R_n)|\forall i \quad R_i \geq 0, \quad \sum_{i=2}^{n} \frac{R_i}{1-\epsilon_{1i}} \leq 1\}.$$

In other words, the capacity region is achieved by time-sharing scheme.

It is important to observe that unlike the wireline case the max-flow min-cut upper bound of [1] for the rate of information transmission can be loose.

Time-Sharing Scheme

In the remaining part of this section we will look at the performance of time-sharing schemes for wireless networks. Suppose that each node performs time-sharing between the destinations. In other words, each node $i \in \mathcal{V}$ allocates a fraction $\alpha_{id}, \quad d \in \mathcal{D}$ of its block length to transmit to destination $d \in \mathcal{D}$. These fractions may not be the same for different nodes. In fact as we will see later, in some cases, the optimal fractions are unequal.

Before analyzing the achievable rate region of time-sharing scheme for broadcast problems, let us look back at the coding scheme definition in Section 3 for the single destination case. There, it is supposed that all the nodes are using blocks of the same length, n, to perform encoding for the destination. However, if we assume that the block lengths are not equal across the network, a similar max-flow min-cut result will still hold. The only difference is in the definition of the cut-capacity. We have stated the result without proof as the following lemma.

Lemma 1. *Consider an erasure wireless network with single source and single destination d. Furthermore suppose that node $i \in \mathcal{V}$ uses a block code of length $\lceil \alpha_{id} n \rceil, \quad \alpha_{id} \leq 1$ to perform encoding. Then the capacity of the network with side-information at the destination and under this coding scheme is given by the minimum of the cut-capacities over all the s-d cuts, where the cut-capacity of s-d cut $\mathcal{V}_s$ is defined as*

$$C(\mathcal{V}_s, \{\alpha_{id}\}_{i\in\mathcal{V}}) = \sum_{i\in\mathcal{V}_s^*} \alpha_{id}(1 - \prod_{j:(i,j)\in[\mathcal{V}_s,\mathcal{V}_s^c]} \epsilon_{ij}). \tag{6}$$

Now consider our broadcast problem in erasure wireless networks. We represent any admissible time-sharing policy by $\underline{\alpha} = (\alpha_{id}, \quad i \in \mathcal{V}, d \in \mathcal{D})$ where as mentioned earlier α_{id} specifies the fraction of the block length allocated by node i for transmission to destination d. It is clear that for any addmissible time-sharing $\underline{\alpha}$ we should have $\alpha_{id} \geq 0$ and $\sum_{d\in\mathcal{D}} \alpha_{id} = 1$ for any $i \in \mathcal{V}$ and $d \in \mathcal{D}$. According to the previous lemma, the achievable rate region using a fixed time-sharing scheme given by $\underline{\alpha}$ will be

$$C_{TS}(\underline{\alpha}) = \{(R_d, \quad d \in \mathcal{D})|\forall d \in \mathcal{D}, \quad R_d \leq \min_{\mathcal{V}_s:\text{s-d cut}} C(\mathcal{V}_s, \{\alpha_{id}\}_{i\in\mathcal{V}})\}, \tag{7}$$

and therefore the achievable rate region using time-sharing in the network is

$$C_{TS} = \bigcup_{\underline{\alpha}} C_{TS}(\underline{\alpha}). \tag{8}$$

where union is taken over all admissible $\underline{\alpha}$'s and subscript TS is used to refer to the time-sharing scheme.

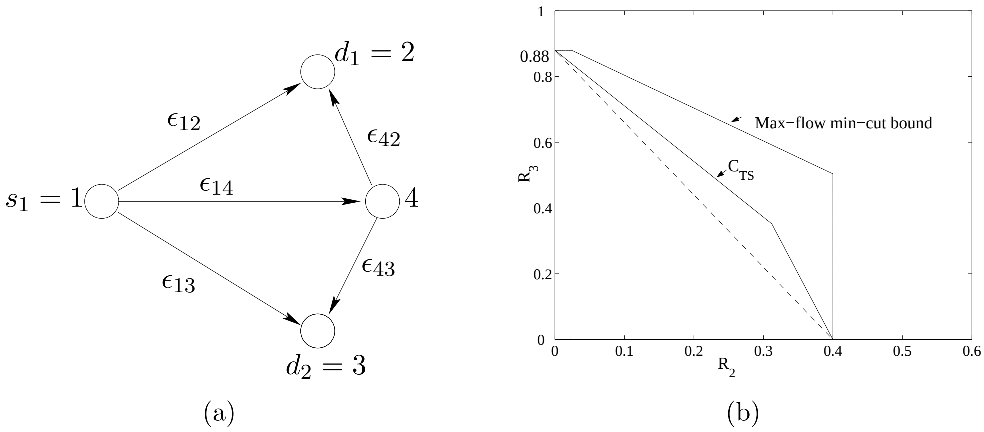

Figure 2: (a): A wireless erasure network with one source and two destination nodes. Nodes 2 and 3 request independent information from the source. (b): Different achievable regions for the network in part (a) with $(\epsilon_{12}, \epsilon_{13}, \epsilon_{14}, \epsilon_{42}, \epsilon_{43}) = (0.8, 0.2, 0.6, 0.8, 0.2)$. The dashed line shows the boundary of the region achieved by time-sharing with $\alpha_{42} = \alpha_{12}$.

Example 4.1. Consider the wireless erasure network shown in Figure 2, with one source, one relay node and two destination nodes. Based on the above argument, the achievable rate region using time-sharing is

$$C_{TS} = \cup_{(\alpha,\beta)\in[0,1]^2} \left\{ (R_2, R_3) \middle| \begin{array}{c} R_2 \leq \min\{\alpha(1-\epsilon_{12}\epsilon_{14}), \alpha(1-\epsilon_{12}) + \beta(1-\epsilon_{42})\} \\ R_3 \leq \min\{(1-\alpha)(1-\epsilon_{13}\epsilon_{14}), (1-\alpha)(1-\epsilon_{13}) + (1-\beta)(1-\epsilon_{43})\} \end{array} \right\}$$

where α (resp. β) is the fraction of the block length that node 1 (resp. 4) allocates to transmit to destination 2. In Figure 2.(a), we have plotted the achievable rate region using the time-sharing scheme described above for $(\epsilon_{12}, \epsilon_{13}, \epsilon_{14}, \epsilon_{42}, \epsilon_{43}) = (0.8, 0.2, 0.6, 0.8, 0.2)$. The dashed line specify boundaries of the region achievable by time-sharing between source and one of the destination sub-networks (this corresponds to the case when $\alpha = \beta$). As we can observe, the optimal time-sharing is not achieved by equal fractions α and β. The max-flow min-cut upper bound for this network is also plotted.

So far we have found an achievability region for broadcast problems in wireless erasure networks using time-sharing schemes. Determining the optimal scheme is a work under progress. However, for some special cases, we can show that the capacity region is achieved by time-sharing. For instance, if the sub-networks obtained by considering all the paths from the source to each of the destinations do not have any common node (except the source node), we can show that the time-sharing scheme gives all the points in the capacity region. Note that in this case since the intermediate nodes of the sub-networks are distinct, only the source node needs to perform time-sharing. We prove our claim in the following theorem.

Theorem 3. *Consider broadcast problem for a wireless erasure network represented by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with one source node $s \in \mathcal{V}$ and destination set $\mathcal{D}$. Assume that each of the destination nodes request independent information from the source. If the sub-graphs induced by considering all the nodes connected (by at least a path) to each of the destination has only the source node in common, then capacity region $\mathcal{C}$ is given by*

$$\mathcal{C} = C_{TS}$$

and it is achieved by the source node time-sharing its block length between different sub-graphs and all the other nodes allocating their whole block length to their corresponding destination. C_{TS} is also defined in (8).

Proof: The achievability part follows directly from previous discussions. We state the proof of the converse part here. For simplicity of notation we consider a network with two destinations. A similar proof goes through when the number of destinations is more than two. Denote the message intended to be decoded at destination d_1 by W_1 and the message intended to be decoded at destination d_2 by W_2. By assumption of the theorem, we can decompose our graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ into two sub-graphs $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ containing d_1 and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ containing d_2 such that $\mathcal{V}_2 \cap \mathcal{V}_1 = \{s\}$ and $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$.

We know that W_1 can be decoded at destination with small probability of error. Hence using Fano's inequality we have $H(W_1|Y^n_{d_1}, S^n) \leq n\epsilon_n$, where $\epsilon_n \to 0$ as $n \to \infty$. Consider an $s - d_1$ cut described by d_1-set $\mathcal{V}_{d_1} \in \mathcal{V}_1$. Then by a sequence of inequalities we can show that,

$$\begin{aligned} nR_1 = H(W_1) &= I(W_1; Y^n_{d_1}, S^n) + H(W_1|Y^n_{d_1}, S^n) \\ &\leq I(W_1; Y^n_{d_1}, S^n) + n\epsilon^{(1)}_n \\ &\leq I(W_1; Y^n(\mathcal{V}^c_{d_1}), S^n) + n\epsilon^{(1)}_n \\ &= I(W_1; Y^n_{\mathcal{A}}) + I(X^n(\mathcal{B}); Y^n_{\mathcal{B}}|S^n) + n\epsilon^{(1)}_n \end{aligned}$$

where we have grouped $Y(\mathcal{V}^c_{d_1})$ into two sets as follows: $\mathcal{A} = \{(s,j)|(s,j) \in [\mathcal{V}^c_{d_1}, \mathcal{V}_{d_1}]\}$ is the set of edges in the cut-set that are connected to the source and $\mathcal{B} = \{(i,j)|i \neq s, (i,j) \in [\mathcal{V}^c_{d_1}, \mathcal{V}_{d_1}])\}$ is the set of edges not connected to the source node. $X^n(\mathcal{B})$ is also defined as $X^n(\mathcal{B}) = (X^n_i, \quad i \neq s, i \in {\mathcal{V}^c}^*_{d_1})$. Here $Y_{\mathcal{I}} \triangleq \{Y_i | i \in \mathcal{I}\}$, for a set of indices $\mathcal{I}$.

Next step is to bound each of the terms appearing in the last line of (9). Using memoryless and independence of the channels it can be verified that

$$I(Y^n_{\mathcal{B}}; X^n(\mathcal{B})|S^n) \leq n \sum_{s \neq i \in {\mathcal{V}^c}^*_{d_1}} (1 - \prod_{j:(i,j) \in [\mathcal{V}^c_{d_1}, \mathcal{V}_{d_1}]} \epsilon_{ij}).$$

Combining the above equation with (9), we get

$$nR_1 \leq I(W_1; Y^n_{\mathcal{A}}) + n \sum_{s \neq i \in {\mathcal{V}^c}^*_{d_1}} (1 - \prod_{j:(i,j) \in [\mathcal{V}^c_{d_1}, \mathcal{V}_{d_1}]} \epsilon_{ij}) + n\epsilon^{(1)}_n. \quad (9)$$

Note that using a similar argument as above, for any $s - d_2$ cut specified by d_2-set $\mathcal{V}_{d_2} \in \mathcal{V}_2$, we have

$$nR_2 \leq I(W_2; Y^n_{\mathcal{A}'}) + n \sum_{s \neq i \in {\mathcal{V}^c}^*_{d_2}} (1 - \prod_{j:(i,j) \in [\mathcal{V}^c_{d_2}, \mathcal{V}_{d_2}]} \epsilon_{ij}) + n\epsilon^{(2)}_n \quad (10)$$

where $\mathcal{A}' = \{(s,j)|(s,j) \in [\mathcal{V}^c_{d_2}, \mathcal{V}_{d_2}]\}$. Note that since the two subgraph are disjoint $\mathcal{A}' \cap \mathcal{A} = \emptyset$. The tricky part is finding a tight bound for the terms appearing in (9) and (10) Now consider a two user discrete memoryless broadcast channel, induced from our network as follows: The transmitter is node $s \in \mathcal{V}$ with input alphabet $\mathcal{X}$. Receiver one has access to outputs of the channels on edges in $\mathcal{A}$. The second receiver has access to the output of the channels on edges in $\mathcal{A}'$. The probability transition matrix is also induced from the original network by considering only the edges in $\mathcal{A}$ and $\mathcal{A}'$ (which all originate from the source node s),

$$P(Y_{\mathcal{A}}, Y_{\mathcal{A}'}|X) = \prod_{j:(s,j) \in \mathcal{A}} P_{sj}(Y_{sj}|X) \prod_{l:(s,l) \in \mathcal{A}'} P_{sl}(Y_{sl}|X) \quad (11)$$

where $P_{si}(.|.)$ denotes the transition matrix of the erasure channel on link (s,i). The key point is that this broadcast channel belong to the *less noisy* class in the sense defined in [11] and the capacity region of these class of channels is known [11].

Definition 1. *A discrete memoryless channel with transition matrix* $p(y,z|x)$, $x \in \mathcal{X}$ *and* $(y,z) \in \mathcal{Y} \times \mathcal{Z}$ *belongs to the* less noisy *class if* $I(U;Y) - I(U;Z)$ *does not change sign over all probability distributions of form* $p(u,x,y,z) = p(u)p(x|u)p(y,z|x)$.

We will show that the channel considered here belongs to the *less noisy* case. It can be shown that for erasure channels and any probability distribution of the form defined in Definition 2 we have

$$I(U;Y_{\mathcal{A}}) - I(U;Y_{\mathcal{A}'}) = (\prod_{(s,j)\in\mathcal{A}'} \epsilon_{sj} - \prod_{(s,j)\in\mathcal{A}} \epsilon_{sj})(H(X) - H(X|U)). \tag{12}$$

Therefore, depending on the sign of $a = \prod_{(s,j)\in\mathcal{A}'} \epsilon_{sj} - \prod_{(s,j)\in\mathcal{A}} \epsilon_{sj}$, the above expression does not change sign ever. That suggests that this channel belongs to the *less noisy* class. Suppose that a is positive. Then based on the result of [11], we will have

$$\frac{1}{n}I(W_1;Y^n_{\mathcal{A}'}) \leq I(U;Y_{\mathcal{A}'}) \qquad \frac{1}{n}I(W_2;Y^n_{\mathcal{A}}) \leq I(X;Y_{\mathcal{A}}|U)$$

for some auxiliary random variable U with a joint distribution $p(u,x,y_{\mathcal{A}},y_{\mathcal{A}'}) = p(u)p(x|u)p(y_{\mathcal{A}}|x)$ $p(y_{\mathcal{A}'}|x)$. It can be shown that for erasure channels the left hand side can be rewritten

$$\begin{aligned}
\frac{1}{n}I(W_1;Y^n_{\mathcal{A}'}) &\leq (1 - \prod_{(s,j)\in\mathcal{A}'} \epsilon_{sj})\underbrace{(H(X) - H(X|U))}_{\alpha} \\
\frac{1}{n}I(W_2;Y^n_{\mathcal{A}}) &\leq (1 - \prod_{(s,j)\in\mathcal{A}} \epsilon_{sj})\underbrace{H(X|U)}_{\beta}
\end{aligned}$$

By noticing that $\alpha + \beta = H(X) \leq 1$ and using the above relations in (9) and (10), we get

$$\begin{aligned}
R_1 &\leq (1-\alpha)(1 - \prod_{(s,j)\in[\mathcal{V}^c_{d_1},\mathcal{V}_{d_1}]} \epsilon_{sj}) + \sum_{s\neq i\in\mathcal{V}^{c*}_{d_1}} (1 - \prod_{j:(i,j)\in[\mathcal{V}^c_{d_1},\mathcal{V}_{d_1}]} \epsilon_{ij}) \\
R_2 &\leq \alpha(1 - \prod_{(s,j)\in[\mathcal{V}^c_{d_2},\mathcal{V}_{d_2}]} \epsilon_{sj}) + \sum_{s\neq i\in\mathcal{V}^{c*}_{d_2}} (1 - \prod_{j:(i,j)\in[\mathcal{V}^c_{d_2},\mathcal{V}_{d_2}]} \epsilon_{ij})
\end{aligned} \tag{13}$$

for some $\alpha \in [0,1]$. This suggests that the achievable rates are in the intersection of this regions. From the above equation it can be shown that all the rates above can be achieved by time-sharing scheme if the source node time-share and all the other nodes allocate their whole block to their corresponding destination. □

An immediate corollary of the above theorem is that we can achieve the capacity region of networks described by a tree structure with time-sharing scheme described in this section.

5 Conclusion and Further Work

We have obtained the capacity for the class of wireless erasure networks with broadcast and no interference at reception. We have shown that for single source/single destination wireless erasure networks, capacity has a max-flow min-cut interpretation. We also looked at broadcast problems in these networks and showed that unlike wireline networks, the max-flow min-cut outerbounds can be loose. Then we give an achievable rate region based on time-sharing schemes for broadcast problems in wireless erasure networks and prove its optimality for a special class of these networks.

Many problems related to wireless networks remain open. Finding the optimal coding scheme for broadcast problems in general network graphs is an interesting problem. It will be also interesting to see if similar results can be obtained for other types of networks such as erasure wireless networks in which interference is incorporated in the reception model.

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Appendix

Since the erasure broadcast channel is degraded, for an ordering of the nodes, the rates in the capacity region satisfy[8]

$$\forall k \in \{2, \ldots, n\} \quad 0 \leq R_k \leq I(Y_{1k}; U_k | U_2, \ldots, U_{k-1})$$

for auxiliary random variables $(U_2, \ldots, U_n) - X_1 - (Y_{12}, \ldots, Y_{1n})$. Now using similar technique to the proof of Theorem 2 we can write the above inequality as

$$\forall k \in \{2, \ldots, n\} \quad 0 \leq R_k \leq (1 - \epsilon_{1k}) I(X_1; U_k | U_2, \ldots, U_{k-1})$$

Therefore,

$$\sum_{k=2}^{n} \frac{R_k}{1 - \epsilon_{1k}} \leq \sum_{k=2}^{n} I(X_1; U_k | U_2, \ldots, U_{k-1}) = I(X_1; U_2, \ldots, U_n) \leq H(X_1)$$

Hence we have

$$\sum_{k=2}^{n} \frac{R_k}{1 - \epsilon_{1k}} \leq 1.$$