Fully-Diverse Multi-Antenna Space-Time Codes Based on $Sp(2)$

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Abstract

Fully-diverse *constellations, i.e., a set of unitary matrices whose pairwise differences are nonsingular, are useful in multi-antenna communications, especially in multi-antenna differential modulation, since they have good pairwise error properties. Recently, group theoretic ideas, especially* fixed-point-free (fpf) *groups, have been used to design fully-diverse constellations of unitary matrices. Here we construct four-transmit-antenna constellations appropriate for differential modulation based on the symplectic group $Sp(2)$. These can be regarded as extensions of Alamouti's celebrated two-transmit-antenna orthogonal design which can be constructed from the group $Sp(1)$. We further show that the structure of the code lends itself to efficient maximum likelihood (ML) decoding via the sphere decoding algorithm. Finally, the performance of the code is compared with existing methods including Alamouti's scheme, Cayley differential unitary space-time codes and group-based codes.*

1 Introduction

It is well known in theory that multiple antennas can greatly increase the data rate and the reliability of a wireless communication link in a fading environment. In practice, however, one needs to devise effective space-time transmission schemes. This is particularly challenging when the propagation environment is unknown to the sender and the receiver, which is often the case for mobile applications when the channel changes rapidly.

A differential transmission scheme called *differential unitary space-time modulation* was proposed in [1, 2, 3], which is well-tailored for unknown continuously varying Rayleigh flat-fading channels. The signals transmitted are unitary matrices. In this scheme the probability of error of mistaking one signal S_i for another $S_{i'}$, at high SNR, is proved to be inversely proportional to $|\det(S_i - S_{i'})|$. Therefore the quality of the code is measured by its *diversity product*

$$\xi_C = \frac{1}{2} \min_{S_i \neq S_{i'} \in C} |\det(S_i - S_{i'})|^{\frac{1}{M}} \tag{1}$$

where M is the number of transmit antennas and C is the set of all possible signals. We therefore say that a code is *fully-diverse* or has *full diversity* if the determinants of the pairwise differences are all nonzero. The design problem is thus the following: "Given the number of transmitter antennas, M, and the transmission rate, R, find a set C of $L = 2^{MR}$ $M \times M$ unitary matrices, such that the minimum of the absolute value of the determinant of their pairwise differences is as large as possible."

The design problem, as just stated, appears to be intractable since first the signal set and the cost function are non-convex and second, the size of the problem can be huge, especially at high data rates. Therefore, in [4, 5, 6], it was proposed to enforce a group structure on the constellation. This has several advantages that are discussed in [4, 5, 6]. Moreover, it is shown that a constellation is fully-diverse iff the corresponding group is fixed-point-free (fpf), i.e. all non-identity matrices have no eigenvalue at one. In [5], all finite fully-diverse constellations that form a group are classified. And also, in [6], it is proved that the only fpf infinite Lie groups are $U(1)$, the group of unit-modulus scalars, and $SU(2)$, the group of unit-determinant 2×2 unitary matrices.

However, finite fpf groups are few and far between and no good constellations are obtained for very high rates from the finite fpf groups classified in [5], and constellations based on $U(1)$ and $SU(2)$ are constrained to one and two-transmit-antenna systems. In this paper, to get high rate constellations which work for 4-transmit-antenna systems, we relax the fpf condition by considering Lie groups with non-identity elements having no more than $k > 0$ unit

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eigenvalues instead of no unit eigenvalues. It can be shown that if a Lie group has rank n, then it has at least one element with $n-1$ eigenvalues at 1. (The *rank* of a Lie group equals to the maximum number of commuting basis elements of its Lie algebra and it can be shown that fpf groups have rank 1. See [6].) The lower the rank, the more possible it is to get a subset with no unit eigenvalue elements, that is, the more possible for us to find a fully-diverse subset of it. There are only three compact, simply connected, simple Lie groups of rank 2, the 4×4 unitary, symplectic group $Sp(2)$, the group of 3×3 unit-determinant unitary matrices, $SU(3)$ and one of the five exceptional group G_2. In this paper, we focus on $Sp(2)$. The codes designed based on it are fully-diverse, can be used in four transmit antenna and any number of receive antenna systems, exist for almost any rate and lend themselves to polynomial-time ML decoding via the sphere decoder.

1.1 Differential Unitary Space-Time Modulation

Consider a wireless communication system with M transmit antennas and N receive antennas. The channel is used in blocks of M transmissions (for more on this model, see [7, 8]). the system equations of the τth block can be written as:

$$X_\tau = \sqrt{\rho} S_\tau H_\tau + V_\tau$$

Here, S denotes the $M \times M$ transmitted signal with s_{tm} the signal sent by the mth transmit antenna at time t. H is the $M \times N$ complex-valued propagation matrix , which is unknown to both the transmitter and the receiver, and h_{mn} is the propagation coefficient between the mth transmit antenna and the nth receive antenna and has a $\mathcal{CN}(0,1)$ distribution independent of all other entries of H. V is the $M \times N$ noise matrix with v_{tn}, the noise at the nth receive antenna at time t, iid $\mathcal{CN}(0,1)$ distribution. X is the $M \times N$ received signal matrix with x_{tn} the received value by the nth receive antenna at time t. The transmitted power constraint is $\sum_{m=1}^{M} \mathrm{E}\,|s_{tm}|^2 = 1, \quad t = 1, ..., M$, so ρ represents the expected SNR at each receive antenna.

One way to communicate with unknown channel is the multiple-antenna differential modulation, which can be seen as a natural extension of standard differential phase shift keying (DPSK) commonly used in single-antenna unknown-channel systems. In differential modulation, the transmitted matrix S_τ at block τ equals to the product of the previously transmitted matrix and a unitary data matrix V_{z_τ} taken from our signal set $\mathcal{C}$. In other words, $S_\tau = V_{z_\tau} S_{\tau-1}$ where $S_0 = I_M$. We immediately see the advantage in practice to have our code form a group under matrix multiplication: all the transmitted signal matrices also belong to the signal set when it forms a group. Having V_{z_τ} unitary assures that our transmitted signal will not vanish or blow up to infinity. Since the channel is used M times, the transmission rate is $R = \frac{1}{M}\log_2 L$, where L indicates the cardinality of our code. Further assume that the propagation environment is approximately constant for $2M$ consecutive channel uses, that is, $H_\tau \approx H_{\tau-1}$, we may get the fundamental differential receiver equations [9]

$$X_\tau = V_{z_\tau} X_{\tau-1} + W'_\tau \qquad (2)$$

where $W'_\tau = W_\tau - V_{z_\tau} W_{\tau-1}$. We can see that the channel matrix H does not appear in (2). This implies that, as long as the channel is approximately constant for $2M$ channel uses, differential transmission permits decoding without knowing the channel information. The ML decoder of z_τ is given by

$$\hat{z}_\tau = \arg \max_{l=0,...,L-1} \|X_\tau - V_l X_{\tau-1}\| \qquad (3)$$

It is shown in [1, 3] that, at high SNR, the pairwise probability of error (of transmitting V_l and erroneously decoding $V_{l'}$) has an upper bound

$$\Pr \le \frac{1}{2}\left(\frac{8}{\rho}\right)^{MN} \frac{1}{|\det(V_l - V_{l'})|^{2N}}$$

which is inversely proportional to the diversity product of the code. Therefore, most design schemes [1, 3, 5] have focused on finding a constellation $\mathcal{V} = \{V_0, ..., V_L\}$ of $L = 2^{MR}$ unitary $M \times M$ matrices that maximizes $\xi_{\mathcal{C}}$ defined in (1).

2 Math Fundamentals

Definition 1 (Fixed-point-free Group). *[6] A group $\mathcal{G}$ is called* **fixed-point-free (fpf)** *iff it has a faithful representation as unitary matrices with the property that the representation of each non-unit element of the group has no eigenvalue at unity.*

It can be proved easily that constellations that form a group are fully-diverse iff the group is fpf. In [5], all finite fpf groups, are classified. These finite fpf groups are few and far between although there exists an infinite number of them. Although these yield very good constellations at low to moderate rates, no good constellations are obtained for very high rates from them. This motivates the search for *infinite* fpf groups, in particular, their most interesting case, Lie groups.

Definition 2 (Lie Group). *[10] A* **Lie group** *is a differential manifold which is also a group such that the group multiplication and inversion map are differential maps.*

Here are some examples of Lie groups. $GL(n, C)$ is the group of nonsingular $n \times n$ complex matrices. $SL(n, C)$ is the group of unit-determinant nonsingular $n \times n$ complex matrices. $U(n)$ is the group of $n \times n$ complex unitary matrices. $SU(n)$ is the group of unit-determinant $n \times n$ unitary matrices and $Sp(n)$ is the group of $2n \times 2n$ symplectic, unitary matrices. The following result shows that the groups of interest to us are compact semi-simple Lie groups.

Theorem 1 (Lie groups with Unitary Representations). *[6] A Lie group has a representation as unitary matrices iff it is a compact semi-simple group or the direct sum of $U(1)$ and a compact semi-simple group.*

It is proved in [6], that the only fpf infinite Lie groups are $U(1)$, the group of unit modulus scalars, and $SU(2)$, the group of 2×2 unitary matrices. Due to their dimensions, constellations based on the two Lie groups are constrained to one and two-transmit-antenna systems. To obtain a four-transmit-antenna constellation, we relax the fpf condition and consider compact semi-simple Lie groups whose non-identity elements have no more than $k > 0$ unit eigenvalues ($k = 0$ corresponds to fpf groups.) In designing a constellation of finite size, we need to sample the Lie group's underlying manifold. When k is small, there is a good chance that, sampling appropriately, the resulting code is fully-diverse. In general, it does not seem that there is a straightforward way to analyze the number of unit eigenvalues of a matrix element of any given Lie group. However, it is possible to relate the number of unit eigenvalues to the rank of the group, where by the rank of a Lie group we mean the maximum number of commuting basis elements of its Lie algebra.

Claim 1. *If a matrix Lie group $\mathcal{G}$ has rank r, then it has at least one non-identity element with $r - 1$ unit eigenvalues.*

Therefore, instead of exploring Lie groups whose non-identity elements have no more than k unit eigenvalues, we study compact semi-simple Lie groups with rank no more than $k + 1$ and design codes that are fully-diverse subsets of it.

Since semi-simple Lie groups can be written as direct sums of simple Lie groups, we first consider simple, simply connected, compact Lie groups instead of semi-simple ones with rank 2. As mentioned in the introduction, there are three of them: the Lie group of unit-determinant 3×3 unitary matrices $SU(3)$, the Lie group of unit-determinant 4×4 unitary, symplectic matrices $Sp(2)$ and one of the exceptional groups $\mathcal{G}_2$. Since $Sp(1) = SU(2)$, and $SU(2)$ constitutes the orthogonal design of Alamouti [11], the symplectic group $Sp(2)$ can be regarded as a generalization of orthogonal designs.

Definition 3 (Symplectic Group). *[12] $Sp(n)$, the nth order symptectic group, is the set of complex $2n \times 2n$ matrices S obeying*

1. *Unitary condition: $S^*S = SS^* = I_{2n}$*
2. *Symplectic condition: $S^tJS = J$.*

where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, S^t denotes the transpose of S and S^* denotes its conjugate transpose. I_n indicates the $n \times n$ identity matrix.

$Sp(n)$ has dimension $n(2n + 1)$ and rank n. We are most interested in the case of $n = 2$. Actually, it is readily shown that the maximum number of unit eigenvalues of any non-identity element in $Sp(2)$ is 2.

3 $Sp(2)$ Fully-Diverse Code Design

From the two conditions in Definition 3, it is easy to get that for any $2n \times 2n$ matrix $S \in Sp(n)$,

$$JS = \bar{S}J$$

is true. By partitioning the matrix S into a 2×2 block of $n \times n$ matrices, it is easy to get that S has the form

$$\begin{bmatrix} A & B \\ -\bar{B} & \bar{A} \end{bmatrix}$$

for some complex $n \times n$ matrices A and B. The group can be identified as the subgroup of unitary matrices with a structure that is similar to Alamouti's 2-dimensional orthogonal design [11], but here each entry is an $n \times n$ matrix instead of a scalar. Using the unitary condition of S and the singular value decompositions of A and B, we can get that A and B can be diagonalized simultaneously by some unitary matrices U and V. The following theorem can be proved.

Theorem 2 (Parametrization of $Sp(n)$). *Any matrix S belongs to $Sp(n)$ iff it can be written as*

$$S = \begin{bmatrix} U\Sigma_A V & U\Sigma_B \bar{V} \\ -\bar{U}\Sigma_B V & \bar{U}\Sigma_A V \end{bmatrix}$$

where U and V are any $n \times n$ unitary matrices, and $\Sigma_A =$ diag $(\cos\theta_1, \ldots \cos\theta_n)$, $\Sigma_B =$ diag $(\sin\theta_1, \ldots \sin\theta_n)$ for some real angles $\theta_1, \ldots, \theta_n$. $\bar{U}$ and $\bar{V}$ denote the conjugates of U and V.

Since any $n \times n$ unitary matrix has dimension n^2, there are all together $2n^2$ degrees of freedom in the unitary matrices U and V. Together with the n real angles, θ_i, the dimension of S is, therefore, $n(2n + 1)$, which is exactly the same as that of $Sp(n)$. Based on Theorem 2, the matrices in $Sp(n)$ can be parameterized by U, V and the θ_is.

Now, let us look at the easiest case of $n = 2$. For simplicity, we first let $\Sigma_A = \Sigma_B = \frac{1}{\sqrt{2}} I_2$, by which 2 degrees of freedom are lost. We further choose U and V as orthogonal designs with entries of U in the set of P-PSK signals, $\{1, e^{j\frac{2\pi}{P}}, \cdots, e^{j\frac{2\pi(P-1)}{P}}\}$, and entries of V chosen from the set of Q-PSK signals shifted by an angle θ, $\{e^{j\theta}, e^{j(\frac{2\pi}{Q}+\theta)}, \cdots, e^{j(\frac{2\pi(Q-1)}{Q}+\theta)}\}$. The following code is obtained.

$$\mathcal{C}_{P,Q,\theta} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} UV & U\bar{V} \\ -\bar{U}V & \bar{U}\bar{V} \end{bmatrix} \right\} \tag{4}$$

where $U = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{j\frac{2\pi k}{P}} & e^{j\frac{2\pi l}{P}} \\ -e^{-j\frac{2\pi l}{P}} & e^{-j\frac{2\pi k}{P}} \end{bmatrix}$, and $V = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{j(\frac{2\pi m}{Q}+\theta)} & e^{j(\frac{2\pi n}{Q}+\theta)} \\ -e^{-j(\frac{2\pi n}{Q}+\theta)} & e^{-j(\frac{2\pi m}{Q}+\theta)} \end{bmatrix}$ for $0 \leq k, l < P, 0 \leq m, n < Q$ and P and Q are integers. $\theta \in [0, 2\pi)$ is an angle to be chosen later. The rate of the code is therefore $\frac{1}{2}(\log_2 P + \log_2 Q)$. The angle θ, an extra degree of freedom added to the code to gain diversity product, is crucial in the proof of the full diversity of the code although simulation result indicates that the full diversity of the code is not affected by the value of θ.

Since the U and V in our code have an orthogonal design structure, it is not difficult to calculate the determinant of the difference of any two signals in the code directly. Using this calculation, we can prove the following theorem.

Theorem 3 (Condition for full diversity). *There exists a θ such that the code $\mathcal{C}_{P,Q,\theta}$ in (4) is fully-diverse iff P and Q are relatively prime.*

To get codes at higher rates, we can add one of the two degrees of freedom in the diagonal matrices Σ_A and Σ_B in by letting $\Sigma_A = \cos\gamma_i I_2, \Sigma_B = \sin\gamma_i I_2$ for $\gamma_i \in \Gamma$. The full diversity of the modified codes can be proved similarly when θ and the set Γ are properly chosen.

4 Decoding of the $Sp(2)$ Code

One of the most prominent properties of our $Sp(2)$ code is that it can be seen as a generalization of orthogonal designs. This property can be used to get linear decoding, which means that the receiver can be made to form a system of linear equations in the unknowns.

From (3), the ML decoder can be written as,

$$\arg\max_{U,V} \left\| X_\tau - \frac{1}{\sqrt{2}} \begin{bmatrix} U & 0 \\ 0 & \bar{U} \end{bmatrix} \begin{bmatrix} I_2 & I_2 \\ -I_2 & I_2 \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & \bar{V} \end{bmatrix} X_{\tau-1} \right\|_F^2$$

which is equivalent to,

$$\arg\max_{U,V} \left\| \begin{bmatrix} U^* & 0 \\ 0 & U^t \end{bmatrix} X_\tau - \frac{1}{\sqrt{2}} \begin{bmatrix} I_2 & I_2 \\ -I_2 & I_2 \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & \bar{V} \end{bmatrix} X_{\tau-1} \right\|_F^2$$

Note that formula is quadratic in the entries of U and V. Using the fact that U and V are orthogonal designs, it can be shown that the ML decoder reduces to,

$$\arg\max_{0\leq k,l<P, 0\leq m,n<Q} \left\| \begin{bmatrix} \mathcal{A} & -\mathcal{C} \\ \mathcal{B} & -\mathcal{D} \end{bmatrix} \begin{bmatrix} \cos\frac{2\pi k}{P} \\ \sin\frac{2\pi k}{P} \\ \cos\frac{2\pi l}{P} \\ \sin\frac{2\pi l}{P} \\ \cos(\frac{2\pi m}{Q}+\theta) \\ \sin(\frac{2\pi m}{Q}+\theta) \\ \cos(\frac{2\pi n}{Q}+\theta) \\ \sin(\frac{2\pi n}{Q}+\theta) \end{bmatrix} \right\|_F^2 \tag{5}$$

where $\mathcal{A}, \mathcal{B}$ are 4×4 real matrices which depend only on X_τ and $\mathcal{C}, \mathcal{D}$ are 4×4 real matrices which depend only on $X_{\tau-1}$.

We can see form formula (5) that the decoding criterion is quadratic in the sine and cosine of the unknowns. Thus, it can be solved using the sphere decoder algorithm [13]. By choosing P odd, the map $f : \theta \rightarrow \sin\theta$ for $\theta \in \{0, \frac{2\pi}{P}, \cdots, \frac{2(P-1)\pi}{P}\}$ is a one-to-one and onto map. Therefore, we can equivalently regard $\sin\frac{2\pi k}{P}$ and $\sin\frac{2\pi l}{P}$ to be our unknowns instead of k and l. And the same for m and n. Also notice that there are actually 4 independent unknowns instead of 8 in (5). We combine the $2i$-th components (of the form $\cos x$) and the $(2i+1)$-th component (of the form $\sin x$) together in the sphere decoding.

5 Simulation Results

In this section, the performances of the $Sp(2)$ codes are compared with other codes, including the Alamouti's orthogonal designs for two transmit antenna systems, a Cayley differential unitary space-time code [9] and also the group-based codes. The block error rate (bler), which corresponds to errors in decoding the 4×4 transmitted matrices, is demonstrated as the error event of interest.

In Fig 1, we compare our $Sp(2)$ code of $P = 5, Q = 3$ and rate $R = 1.95$ with rate 2 orthogonal design and a Cayley differential code at rate 1.75. The number of receive antenna is 1. At a bler of 10^{-3}, the $Sp(2)$ code is 2dB better than the Cayley differential code, even though it has a lower rate, and 4dB better than the orthogonal design.

In Fig 2, we compare our $Sp(2)$ code with a group-based diagonal code and the fpf $K_{1,1,-1}$ code [5] at rate 1.98. The number of receive antenna is 1. At a bler of 10^{-3}, 2dB improvement is obtained by using the $Sp(2)$ code instead of a diagonal code, but the $Sp(2)$ code is

1.5dB worse than the $K_{1,1,-1}$ group code. However, decoding the $K_{1,1,-1}$ code requires an exhaustive search over the entire constellation.

In Fig 3, the comparison of the $Sp(2)$ code of $P = 23, Q = 11$ and rate $R = 3.99$ with the rate 4 orthogonal design is shown. The number of receive antenna is 1. We can see that the $Sp(2)$ code get a better performance than the orthogonal design at high SNR.

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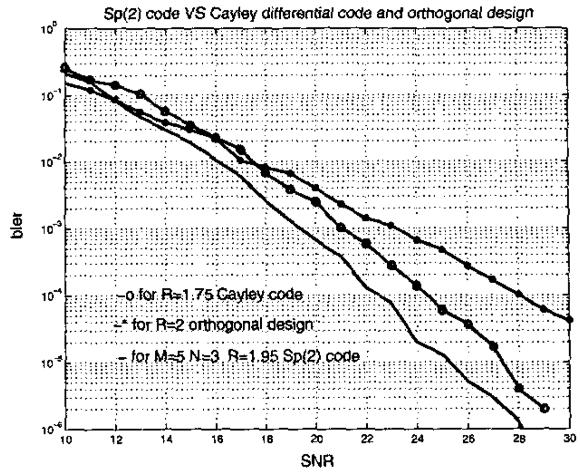

Figure 1: Comparison of the $Sp(2)$ code with a Cayley differential code and orthogonal design.

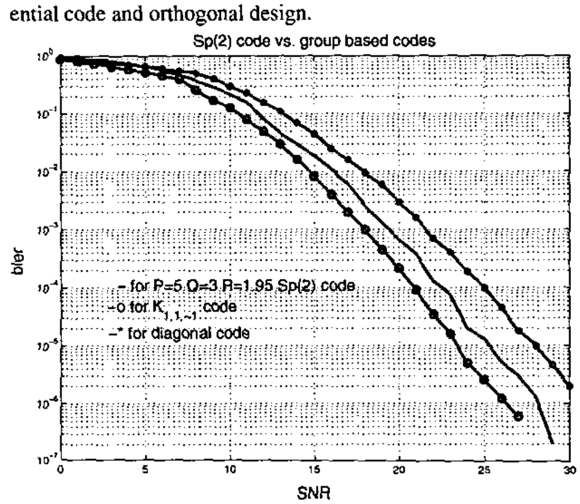

Figure 2: Comparison of the $Sp(2)$ code with a group-based diagonal code and the $K_{1,1,-1}$ code.

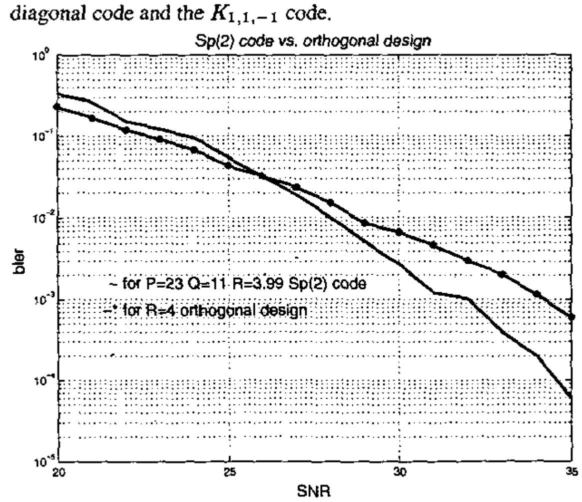

Figure 3: Comparision of the $Sp(2)$ code with orthogonal design.