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State-space Structure of Finite Horizon Optimal Mixed H_2/H_∞ Filters*

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Abstract

We show that finite horizon optimal mixed H_2/H_∞ filters are not of fixed order. Moreover, when the underlying systems have state-space models of order n, the optimal finite horizon mixed H_2/H_∞ filter has state-space model of order no greater than $n+J$ where J is the multiplicity of the maximum singular value of $\mathbf{T}_2(\mathbf{K})$ equal to the H_∞ bound γ.

1. Introduction

The mixed H_2/H_∞ estimation problem was introduced as an attempt to capture the benefits of both the pure H_2 and H_∞ estimators. However, unlike the H_2 or H_∞ problems, with readily computable solutions, there is no known "nice" solution for the mixed problem. On the contrary, it has been shown that except for some trivial cases, the infinite horizon mixed problem does not have a bounded order solution [1, 2]. In this paper we show analogous results for the *finite horizon* optimal mixed estimation problem. For the finite horizon problem, given a system of order n, the order of the optimal filter depends on the horizon N and the H_∞ bound γ. Hence, unlike the optimal H_2 filters or the central H_∞ filters, the mixed optimal filters have no fixed order. Moreover, given a certain system order there is no fixed upper bound on the possible filter order as the horizon N increases.

To show the structure of the optimal mixed solution, we first recast the mixed problem as an unconstrained convex optimization problem and derive the optimality conditions using the idea of sub-gradients for convex functions [3, 4]. The derivation of the optimality conditions, exploiting the convexity of the problem, is the key step. From the optimality conditions, we develop an explicit expression for the optimal solution and hence, derive an upper bound on the filter order.

2. The Data Model

A general framework for estimation problems is shown in Fig.1, which subsumes both time variant and time invariant finite horizon estimation problems. We assume that both the *causal* linear systems $\mathbf{H}$ and $\mathbf{L}$ have known state space structures of order n, such that we may write

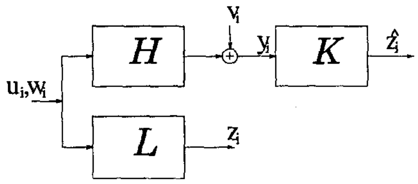

Figure 1: *A general estimation problem.*

$$\begin{cases} x_{i+1} &= A_i x_i + B_i^{(1)} u_i + B_i^{(2)} w_i, \\ y_i &= C_i x_i + D_i^{(1)} u_i + D_i^{(2)} w_i + v_i, \qquad 0 \le i \le N, \\ z_i &= F_i x_i + E_i^{(1)} u_i + E_i^{(2)} w_i \end{cases}$$

where $A_i \in \mathcal{R}^{n\times n}$, $B_i^{(1)}$, $B_i^{(2)}$, C_i, $D_i^{(1)}$, $D_i^{(2)}$, F_i, $E_i^{(1)}$ and $E_i^{(2)}$ are known system matrices of compatible dimensions. This implies that $\mathbf{H} = [\mathbf{H}_1\ \mathbf{H}_2]$, where

$$\mathbf{H}_n = \begin{bmatrix} D_0^{(n)} & & & \\ C_1 B_0^{(n)} & D_1^{(n)} & & \\ \vdots & \vdots & \ddots & \\ C_N A_{N-1} \cdots A_1 B_0^{(n)} & \cdots & \cdots & D_N^{(n)} \end{bmatrix},$$

are the impulse response matrices for $n = 1, 2$. (Similarly, $\mathbf{L} = [\mathbf{L}_1\ \mathbf{L}_2]$.) In what follows we shall denote sequences such as $\{u_i\}$, $i = 0, \ldots, N$ by u, and simply write $z = \mathbf{L}[u^T\ w^T]^T$, to denote that $\mathbf{L}$ maps the input sequence $\{u_i\}$ and $\{w_i\}$ to the output sequence $\{z_i\}$.

The sequences $\{u_i\}$, $\{w_i\}$ and $\{v_i\}$ are assumed to be *unknown*. The sequences $\{u_i\}$ and $\{w_i\}$ may be considered as driving disturbances and $\{v_i\}$ as a measurement noise. The goal is to design a causal filter, i.e., a lower block triangular matrix of compatible dimension, $\mathbf{K}$ that estimates z_i, the output of $\mathbf{L}$, using the observations $\{y_j, j \le i\}$. The estimates are denoted by $\hat{z}_i$ and the estimation errors by $\tilde{z}_i \triangleq z_i - \hat{z}_i$. The estimation errors $\{\tilde{z}_i\}$ and the disturbances $\{u_i\}$,$\{w_i\}$ and $\{v_i\}$ are related as: $\tilde{z} = (\mathbf{L}_1 - \mathbf{K}\mathbf{H}_1)u + (\mathbf{L}_2 - \mathbf{K}\mathbf{H}_2)w - \mathbf{K}v$. Here we are concerned with the pure mixed problem which can be stated as follows.

Mixed H_2/H_∞ Estimation Problem: *Given a $\gamma > 0$, find a lower block triangular matrix $\mathbf{K}$, if possible, that satisfies*

$$\min_{causal\ \mathbf{K}} \frac{1}{2} tr(\mathbf{T}_1(\mathbf{K})\mathbf{T}_1(\mathbf{K})^T), \text{ s.t. } \bar{\sigma}(\mathbf{T}_2(\mathbf{K})) \le \gamma. \quad (1)$$

where $\mathbf{T}_1(\mathbf{K}) = [\mathbf{L}_1 - \mathbf{K}\mathbf{H}_1 \quad -\mathbf{K}]$, and $\mathbf{T}_2(\mathbf{K}) = [\mathbf{L}_2 - \mathbf{K}\mathbf{H}_2 \quad -\mathbf{K}]$ and $\bar{\sigma}(\cdot)$ is the maximum singular value.

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For $\gamma \geq \gamma_2 \triangleq \bar{\sigma}(\mathbf{T}_2(\mathbf{K}^2_{opt}))$, the H_2-optimal filter K^2_{opt} is also the optimal mixed filter. However, for $\gamma_{opt} < \gamma < \gamma_2$, the optimal mixed filter attains the H_∞ bound, i.e., $\bar{\sigma}(\mathbf{T}_{\mathbf{K}^{mix}_{opt}}) = \gamma$ [5]. Hence, for $\gamma_{opt} < \gamma < \bar{\sigma}(\mathbf{T}_{\mathbf{K}^2_{opt}})$, using the Lagrange multiplier technique, we can recast the mixed problem (1) as an unconstrained convex optimization problem:

$$\min_{\text{causal } \mathbf{K}} \frac{1}{2} tr\{\mathbf{T}_1(\mathbf{K})\mathbf{T}_1(\mathbf{K})^T\} + \lambda(\bar{\sigma}(\mathbf{T}_2(\mathbf{K})) - \gamma), \quad (2)$$

where the scalar λ is the Lagrange multiplier.

3. Optimality Condition

Because of the convexity of problem (2), the first order condition is necessary and sufficient for optimality. However, the cost is a non-smooth convex function and so we use the idea of a sub-gradient of a convex function [3, 4] to derive the following optimality condition.

Proposition 1 *$\mathbf{K}^{mix}_{opt}$ is a solution to problem (2), if and only if, (i) $\bar{\sigma}(\mathbf{T}_{\mathbf{K}^{mix}_{opt}}) = \gamma$, and (ii) the matrix $\mathbf{T}_2(\mathbf{K}^{mix}_{opt})$ has unit-norm linearly independent right-singular vectors $\mathbf{q}_j$ and left-singular vectors $\mathbf{p}_j = [\mathbf{p}^T_{1,j}\ \mathbf{p}^T_{2,j}]^T$, $j = 1, \cdots, J$ corresponding to the maximum singular value γ of multiplicity J, such that*

$$\{\mathbf{K}^{mix}_{opt}(\mathbf{I} + \mathbf{H}_1\mathbf{H}_1^T) - \mathbf{L}_1\mathbf{H}_1^T - \lambda \sum_{j=1}^{J} \alpha_j \mathbf{w}_j \mathbf{q}_j^T\}_{lbt} = 0, \quad (3)$$

where $\mathbf{w}_j = (\mathbf{H}_2\mathbf{p}_{1,j} + \mathbf{p}_{2,j})$, and $\{\mathbf{X}\}_{lbt}$ denotes the lower block triangular part of $\mathbf{X}$.

Note that the vectors $\mathbf{q}_j$ and $\mathbf{w}_j$ are non-linear functions of the optimal filter $\mathbf{K}^{mix}_{opt}$ and thus (3) may not be suitable for obtaining a numerical solution for the filter $\mathbf{K}^{mix}_{opt}$. However, relation (3) can be exploited to deduce the finite state-space structure of the optimal mixed filter as shown in the next section.

4. Order of the Optimal Filter

Rearranging the optimality condition (3), we can write the optimal mixed filter as

$$\mathbf{K}^{mix}_{opt} = \mathbf{K}^2_{opt} + \lambda \sum_{j=1}^{J} \alpha_j \{\mathbf{w}_j \bar{\mathbf{q}}_j^T\}_{lbt} \mathbf{M}^{-1}, \quad (4)$$

where $\mathbf{M}$ is the the block-lower triangular factor of $(\mathbf{I} + \mathbf{H}_1\mathbf{H}_1^T)$, i.e., $\mathbf{I} + \mathbf{H}_1\mathbf{H}_1^T = \mathbf{M}\mathbf{M}^T$, $\bar{\mathbf{q}}_j = \mathbf{M}^{-1}\mathbf{q}_j$ and $\mathbf{K}^2_{opt} = \{\mathbf{L}_1\mathbf{H}_1^T\mathbf{M}^{-T}\}_{lbt}\mathbf{M}^{-1}$ is the optimal H_2 filter. Thus, the optimal mixed filter is the sum of the usual H_2 optimal filter and a second filter that depends on the value of γ that ensures the H_∞ bound. As mentioned earlier, for $\gamma \geq \gamma_2$, the Lagrange multiplier is zero ($\lambda = 0$) and the second term vanishes.

Proposition 2 *The optimal mixed filter, if exists, has a time-variant state space model of order no greater than $n + J$, where n is the dimensionality of the underlying systems $\mathbf{H}$ and $\mathbf{L}$ and J is the multiplicity of the maximum singular value ($= \gamma$) of $\mathbf{T}_2(\mathbf{K}^{mix}_{opt})$. Moreover, this bound on the model order is achievable.*

For a given system, the exact value of J has a complex dependence on the γ and N and J is hard to find. However, there are easily computable lower bound for J.

Proposition 3 *Let $\sigma_1 \geq \sigma_2 \geq \sigma_{N\times q}$ be the ordered singular values of $\mathbf{T}_2(\mathbf{K}^2_{opt})$ where q is the size of z_i. If L is the largest index such that $\sigma_L \geq \gamma$. Then $J \geq L$.*

Therefore, unlike the optimal H_2 filter or the central H_∞ filter, the mixed optimal filter has no fixed order. Moreover, for a given system order there is no upper bound on J and hence, the maximum filter order is not restricted by the order of the underlying system. Note that for a stable A the singular values of $\mathbf{T}_2(\mathbf{K}^2_{opt})$ remains bounded as $N \to \infty$, hence, the singular values are closely spaced for larger N. Roughly speaking, for any $\gamma < \gamma_2 (= \sigma_1)$, the number of singular values larger than γ increases with N. As a result, the filter order increases with N. This provides an intuitive explanation for the result that the optimal mixed infinite horizon filters have no bounded order solution whenever $\gamma < \gamma_2$. Besides the order of the optimal filter there are two more interesting facts we would like to stress. First, even if the underlying systems $\mathbf{H}$ and $\mathbf{L}$ are time-invariant the optimal filter is time-variant whenever the second term involving the singular vectors is present. Second, the presence of the additional term makes it impossible for the optimal filter to have a recursive solution.

5. Conclusion

In this paper, we have derived the optimality condition and an upper bound for the finite horizon mixed H_2/H_∞ estimation problem. Using the optimality condition, we derived an upper bound on the order of the optimal mixed H_2/H_∞ filter. Unlike the optimal H_2 filter or the central H_∞ filter the optimal mixed filter has no fixed order. Moreover, for a given system order there is no upper bound on the maximum possible filter order, a result which is consistent with the results obtained for infinite horizon mixed H_2/H_∞ filters.

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