3rd AIAA Flow Control Conference
5 - 8 June 2006, San Francisco, California

AIAA 2006-3695

Stabilising Control Laws for the Incompressible Navier-Stokes Equations using Sector Stability Theory

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A method for nonlinear global stabilisation of the incompressible Navier-Stokes equations is presented and used to eliminate transient growth in linearly stable Poiseuille flow for the case of full-field actuation and sensing. In the absence of complete velocity field sensing and full actuation the controller synthesis procedure gives a controller that minimises the the attainable perturbation energy over all disturbances and thus maximises the disturbance threshold for transition to occur. The control laws are found using the theory of positive real systems, originating in the control systems community. It is found that a control law making the linearised part of the perturbed Navier-Stokes equations positive real, provides nonlinear global stability. A state-space synthesis procedure is presented that results in two game-theoretic algebraic Riccati equations.

I. Introduction

Recent advances in micro-fabrication techniques have encouraged visions of reactive, intelligent flow surfaces, where MEMS sensors and actuators are combined with control algorithms to facilitate the active control of turbulent flows. Often missing in these visions is a detailed idea of what that control algorithm may be. The nonlinearity and the high-dimensional nature of the flow dynamics are the essential challenges in developing a theory of flow control, which is a necessary precursor to the design of flow control feedback systems. The complexity of fluid flow makes control difficult, but success could bring very significant benefits such as reducing the drag of aircraft and cars, better mixing in chemical reactions, or even improving blood circulation and the flow of air in human airways. In the case of commercial aircraft environmentally important and economically significant drag reductions are required to meet ACARE efficiency targets.[1]

Successful feedback control requires identification of problems receptive to control inputs and control laws that are capable of dealing with the nonlinearity of the Navier-Stokes equations, model uncertainty and the inevitable exogenous disturbances that arise in practical flows (vibration, free-stream disturbances, etc.). Effective, practical control requires our understanding of the basic physical processes to be expressed mathematically (as a finite-dimensional model). Current modelling methods (such as numerical simulation) require very many states; to be implemented in real-time, the control design benefits from reduced-order models of the important system dynamics.[2,3] For this reason model reduction strategies will inevitably be required.

In the last two decades, considerable research effort in control systems theory has focused on these kinds of problem. The $\mathcal{H}_\infty$ theory has been notably successful in providing good control performance for systems with a large class of uncertainties, nonlinearity and exogenous disturbances.[2,4] In cases where system nonlinearity can be sector bounded the $\mathcal{H}_\infty$ theory can be generalised to cope with the nonlinearity.[5–7]

This paper aims to use these advances to form a control-theoretic approach, that considers the nonlinear nature of Navier-Stokes, to a problem that is receptive to practical flow control due to the importance of near-wall mechanisms: delay of so-called "bypass" transition of linearly stable Poiseuille flow.

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Transition to turbulence in wall-bounded flows is typically analysed by considering the growth of small perturbations on a known, laminar solution to the Navier-Stokes equations. For small amplitude perturbations, linearisation is appropriate and subsequent classical eigenmode analysis results in a Reynolds number above which an instability occurs (although note that pipe flow remains linearly stable). Tollmien-Schlichting waves grow exponentially above this critical Reynolds number (in the linear regime) and transition occurs via a secondary instability resulting from non-linear effects. However for many flows transition is observed below the critical Reynolds number derived from the classical analysis. Even when the linearised model is stable, it can exhibit large transient growth of the perturbation energy before eventually returning to equilibrium.[8,9] Physically, the growth is understood to be fed by the transport of energy from the steady flow to the perturbations, typically via streamwise vortices (in channel flow) developing from streaks of spanwise vorticity at the wall, a feature also seen in the turbulent boundary layer.[10] As a result of this energy exchange, perturbations may grow in accordance with the linear model at sub-critical Reynolds number, potentially becoming large enough for the nonlinearities to become significant. Transition then occurs via secondary instability, 'bypassing' the classical mechanism.[11] The transient behaviour is therefore considered as important as the asymptotic behaviour of the linear system (described by eigenvalue analysis), in this case. It is also understood that very particular types of linear mechanism are in fact are essential to sustaining turbulence.[11,12]

Ideas from modern control theory have been applied to transition control before. These typically extrapolate linear control strategies to the nonlinear flow regime. The optimal control approach was tried first.[13] A simple controller designed using classical methods has been used with some success in the stabilization of infinitesimal and finite-amplitude disturbances.[14]

$\mathcal{H}_2$ ('optimal') and $\mathcal{H}_\infty$ ('robust') designs have also been applied to the linearised problem for particular wavenumber pairs.[15] It has been demonstrated that linear feedback control can be used to increase the threshold perturbation amplitudes for transition to occur.[16] The idea is that the linear control strategies will prevent the flow leaving the regime of small perturbations in the first place. This seems unrealistic for many applications, because the linear approximation is valid only for very small perturbations. Likely large-magnitude exogenous disturbances may result in the terminal loss of control, as the assumption of linearity fails.

In this paper we show that the nonlinearity in the system equations can be characterised as positive real, allowing the application of the body of control theory designed to work in the presence of such a nonlinearity.[5,6] A consequence of this is that the control methods derived using this strategy, will work for flow disturbances of any size and not just those small enough to permit linear approximation.[17]

Although many results for finite-dimensional systems in the control literature have analogues in the infinite-dimensional setting, the finite-dimensional theory is usually simpler and computationally tractable. When the system equations are discretised, as is typically done for practical control problems, the finite-dimensional control theory provides a controller synthesis procedure. The procedure outlined in the paper results in two game-theoretic algebraic Riccati equations, relating to the measurement and control problems. Whether solutions exist to these Riccati equations depends on the control forcing and measurement information available to the control algorithm. Because the conditions for existence of the equations are well understood and easily checked, this problem formulation can inform the flow control system designer about the suitability of proposed sensor and actuation arrangements.

In the first instance, full-field volume forcing and measurement of wall-normal velocity is used. This physically impractical arrangement is chosen to guarantee a solution to the control problem and to provide a benchmark for further control studies. It also illustrates that the nonlinearity has been handled correctly. Examining the resulting control action can also be informative at the experiment design stage.

The chosen method of solution involves transforming the positive real problem into the related small-gain problem, which requires a controller to make the product of the maximum signal gain (γ) of the linearised closed-loop and the nonlinearity less than one. In our transformed problem the nonlinearity has $\gamma = 1$. For the likely case where actuation or sensing is insufficient to provide the level of control required for this problem, a γ-minimisation approach can be taken. This provides an index on control system performance in relation to the nonlinearity.

II. Preliminaries

In this section we establish notation, definitions and useful results that are used later in the paper.

A. Notation

x, y, z refer to cartesian co-ordinates in $\mathbb{R}^3$.

x is sometimes also used as a point in $\mathbb{R}^3$.

Throughout, Ω is an open subset of $\mathbb{R}^3$, with boundary $\partial\Omega$.

a^* is the complex conjugate (complex conjugate transpose) of the scalar (vector or matrix) a.

A signal is a Lebesgue measurable function that maps the real numbers $\mathbb{R}$ to $\mathbb{R}^n$.

a_t means signal a truncated on $[0, t]$.

For signals a and b, we define the inner product via integration over the time period $[0, T]$,

$$\langle a, b \rangle = \int_0^T b^*(t) \cdot a(t)\ dt \qquad T \geq 0. \tag{1}$$

For divergence-free, time-dependent vector fields, we define the inner product analogously via integration over spatial domain Ω and the time period $[0, T]$,

$$\langle a, b \rangle = \int_0^T \int_{x \in \Omega} b^*(x, t) \cdot a(x, t)\ dx\ dt \qquad T \geq 0. \tag{2}$$

B. Results from Control Systems Theory

We require our controlled system to have the properties of stability and robustness with respect to noise.

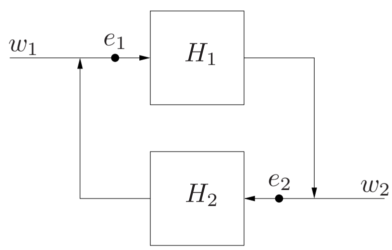

Figure 1. A generalised feedback loop

This is achieved by application of two well-known results[2] which give quite general, open loop conditions for closed loop stability of feedback loops made up of two elements H_1 and H_2, as in Figure 1.

In summary, the first states that if the open loop gain is less than one, then the closed loop is bounded. The second states that if the open loop can be factored into two positive relations, one of which is strictly positive and has finite gain, then the closed loop is bounded. These results will be described briefly below. It is important to realise that the two stability results are equivalent because one can be derived from the other by a suitable transformation.

In the description we will make use of an abstract object called a relation. A relation H can be interpreted as a mapping from a space of input functions into a space of output functions. These spaces contain both bounded and unbounded functions. We only consider relations that map the zero element (o) to the zero element, so $o = Ho$. In other words, H has no bias or offset.

1. The Small-Gain Problem

The gain $\gamma(H)$ of a relation H is

$$\gamma(H) = \sup \frac{\|(Ha)_t\|}{\|a_t\|}$$

where the supremum is over all a in the domain of H and all Ha in the range of H and all t in $[0, T]$ for which $a_t \neq 0$.

If $\gamma(H_1)\gamma(H_2) < 1$, then the closed loop relations from w_1 and w_2 to e_1 and e_2 shown in Figure 1 are bounded. This can be understood in terms of the contraction mapping principle.[2]

2. The Positive Real Problem

A relation H is called positive if

$$\langle a_t, Ha_t \rangle \geq 0$$

for all a in the domain of H and all t in $[0, T]$. In the case of strict inequality, we call H strictly positive.

The following can be derived from the small-gain theorem.[2]

If H_1 is positive and $-H_2$ is strictly positive with finite gain, then the closed loop relations from w_1 and w_2 to e_1 and e_2 shown in Figure 1 are bounded.[18]

If $H(s)$ is a transfer function, then it is positive real if and only if $H(s) + H^*(s) \geq 0, \forall\ \mathrm{Re}(s) > 0$.[2,6]

III. Control of the Navier-Stokes Equations

We begin by writing down the equations for three-dimensional incompressible fluid flow evolving in domain Ω, over a time period $[0, T]$.

The state of the flow at an instant in time t is fully described by a time-dependent velocity vector field $V(x, t)$ and a scalar pressure field $P(x, t)$.

The flow is governed by the incompressible Navier-Stokes equations at Reynolds number Re. A control $f(x, t)$ and an exogenous disturbance $d(x, t)$ are also introduced. The control is restricted by the linear operator B, representing physical limitations on the actuation, whose range is the volume forcings arising from all possible control actions. Assume that both Bf and d are divergence-free.

The equations of motion are then

$$\begin{aligned} \dot{V}(x,t) = &- V(x,t) \cdot \bigtriangledown V(x,t) - \bigtriangledown P(x,t) + \frac{1}{Re} \bigtriangledown^2 V(x,t) + Bf(x,t) + d(x,t) \\ & x \in \Omega,\ t \in [0,T] \end{aligned} \tag{3}$$

$$\bigtriangledown \cdot V(x,t) = 0 \qquad x \in \Omega,\ t \in [0,T]. \tag{4}$$

The flow also obeys prescribed boundary conditions

$$V(x,t) = V_\partial(x,t) \quad x \in \partial\Omega,\ t \in [0,T]. \tag{5}$$

In the case of boundary transpiration control, $f = 0$ and $V_\partial(x, t)$ is prescribed. For the volume forcing case presented here, $V_\partial(x, t) = 0$ at the walls.

We consider perturbations $v(x, t)$ around an assumed steady solution $\bar{v}(x)$, that corresponds to the uncontrolled, disturbance-free situation. This steady solution may or may not be stable. This gives the net velocity vector field

$$V = \bar{v} + v. \tag{6}$$

The steady pressure $\bar{p}(x)$ is similarly perturbed by $p(x, t)$.

Substitution into (3) gives the perturbation equations

$$\begin{aligned}\dot{v}(x,t) =& -\bar{v}(x,t)\cdot\triangledown v(x,t) - v(x,t)\cdot\triangledown\bar{v}(x,t) - n(x,t) - \triangledown p(x,t) + \frac{1}{Re}\triangledown^2 v(x,t) + Bf(x,t) + d(x,t),\\ n(x,t) =& v(x,t)\cdot\triangledown v(x,t),\\ 0 =& \triangledown\cdot v(x,t),\\ & x\in\Omega,\ t\in[0,T].\end{aligned} \tag{7}$$

A substitution has been made for the nonlinear part, giving coupled linear and nonlinear equations. We do not make the assumption of small perturbations.

Let $y(x,t)$ be some the measurements made at time t. These are described by

$$y(x,t) = Cv(x,t)$$

where C is a linear operator whose range covers possible measurements.

The aim is to find a control such that $v(x,t)\to 0$ as $t\to\infty$, given the measurements $y(x,t)$. The controlled system should have the properties described in Section B.

The equations (7) include pressure. The pressure term can be eliminated along with the divergence equation by projecting the equations onto the space of divergence-free functions. The projector Π that achieves this is called the Leray projector.[19] Using $\Pi(\triangledown p) = 0$, $\Pi(v) = v$ and the fact that Bf and d are already divergence-free gives

$$\begin{aligned}\dot{v}(x,t) =& -\Pi\left(\bar{v}(x,t)\cdot\triangledown v(x,t) - v(x,t)\cdot\triangledown\bar{v}(x,t) + \frac{1}{Re}\triangledown^2 v(x,t) - n(x,t)\right) + Bf(x,t) + d(x,t),\\ n(x,t) =& \mathcal{N}(v) = v(x,t)\cdot\triangledown v(x,t),\\ y(x,t) =& Cv(x,t), \quad x\in\Omega,\ t\in[0,T].\end{aligned} \tag{8}$$

The perturbation equations (8) are then described as the feedback interconnection between a linear part and the nonlinear part. We achieve this by writing the system equations (8) in operator form as

$$\begin{aligned}\dot{v}(x,t) &= Av(x,t) + Bf(x,t) - n(x,t) + d(x,t) \qquad t>0\\ y(x,t) &= Cv(x,t)\\ n(x,t) &= \mathcal{N}(v(x,t))\end{aligned} \tag{9}$$

Let $e = d - n$ and G be the relation from n, d and f to v and y, given by (9). Further, define K as the relation generating the control action f from measurements y (the controller). If the equations are discretised, a linearised state-space model of the flow results, with n as a nonlinear disturbance. The arrangement can be represented graphically as in Figure 2.

The nonlinearity is positive real if $\langle v, n\rangle \geq 0$ or equivalently,

$$\langle v, Nv\rangle = \int_0^T \int_{x\in\Omega} v^*(x,t)\cdot(v(x,t)\cdot\triangledown v(x,t))\ dx\ dt \geq 0 \quad \forall T. \tag{10}$$

Applying the divergence theorem and (4), the inner integral is equivalent to an integral over the boundary,

$$\int_{x\in\partial\Omega} (v(x,t)\cdot v(x,t))v(x,t)\cdot\hat{\xi}\ dx \geq 0 \tag{11}$$

where $\hat{\xi}$ is the outward-facing unit vector perpendicular to the boundary $\partial\Omega$. Physically interpreted, (11) quantifies the net flux of disturbance energy out of the domain through the boundary per unit time.

The contribution from volume forcing, or forcing at the boundary in a domain with periodic boundary conditions, is necessarily zero. However, in an open domain, the flux of disturbance energy through the inlet

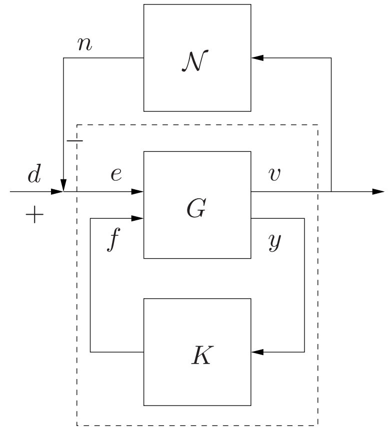

Figure 2. Feedback loop for controlled Navier-Stokes with control

and outlet boundaries and the net flux of disturbance energy from any boundary control both contribute. Where there is such a contribution, (11) enters as a nonlinear constraint on the control law. For the open-domain case where the inlet conditions are relatively undisturbed, there will be a net flux out of the domain of the disturbance energy. In these cases, the nonlinearity has a stabilising influence in the domain of study.

Define Q as the relation with e as input and v as output corresponding to the controlled system inside the dashed box in Figure 2.

By the result in Section 2, if $\mathcal{N}$ is positive real and Q is strictly positive real, then the closed loop in Figure 2 (representing the controlled Navier-Stokes equations) is internally stable and is also strictly positive. In other words, $\langle v, d\rangle > 0$. This is simply verified; from Figure 2 and by the strict positivity of Q and positivity of $\mathcal{N}$,

$$\begin{aligned}\langle v, d\rangle &= \langle v, n+e\rangle \\ &= \langle v, n\rangle + \langle v, e\rangle > 0.\end{aligned} \tag{12}$$

Note that if the uncontrolled, linearised plant is already passive, no control is required, as v is already bounded[a]. The expression $\langle v, d\rangle$ quantifies the flow perturbation energy due to the disturbance. Its positivity implies the system does not itself feed the perturbations.

IV. The Controller Synthesis Problem

For the flow control problem to be solved by the method outlined above, the following conditions are sufficient:

1. A desired steady flow solution must be known explicitly;
2. A linearised, state space model of the behaviour of perturbations away from that equilibrium, incorporating suitable actuation and sensing, must be found;
3. The state-space model must also have the perturbation energy-weighted state and energy-weighted forcing on the state as an input-output pair (for the nonlinearity to act as a positive real disturbance);
4. The actuation and sensing should be sufficient to ensure a solution to the algebraic Riccati equations arising from the positive real synthesis problem.

If the last condition is not satisfied, a γ-minimisation approach can be used obtain the best control given the actuation and sensing limitations, as described below.

[a]For instance, this is the case for the Stokes operator in a (xyz)-periodic domain

Since the nonlinearity can be characterised in terms of a positive real system, the classical results can be applied to guarantee robust stability for the entire nonlinear system if the controller renders the closed loop with the linearised system strictly positive real.

There are two approaches to solving the discrete problem. One is to solve the problem directly, using state-space methods. This synthesis problem is solved in the control literature.[6] Necessary and sufficient conditions for the existence of a solution are given, which relate to the solutions of two algebraic Riccati equations (or inequalities). One Riccati equation relates to the measurement problem, and one to the control problem. It is therefore easy to see whether a solution exists and if it does not, whether this is due to insufficient control or to insufficient measurement.

Alternatively, one may convert the positive real problem to the equivalent small-gain problem,[5] which may then be solved using standard loop-shifting techniques[2,20] and Riccati-based state space methods.[2,4]

The method described by Doyle[4] in fact solves the suboptimal $\mathcal{H}_\infty$ problem for a given γ (see section 1). A $\gamma < 1$ solves the small gain problem and therefore the original positive real control problem. This γ-optimisation approach is useful in the under-actuated or under-sensed case, where the conditions for existence of a solution to the positive real control problem are not met. In this case, we choose a controller to reduce γ on the transformed system to get the original problem as close as possible to positive real, under the constraints of limited actuation and sensing. The guaranteed nonlinear stability is lost, but the control optimally limits the turbulent energy production given the sensing and actuation available. This is useful in the case where there are physical or design constraints on the available measurement and actuation and is still preferable to a naive application of linear control theory.

A solution of the small-gain problem ($\gamma < 1$) results in no transient growth at all. The γ-optimisation approach however optimises the *worst-case* perturbation energy production associated with all possible disturbances given a zero initial condition. Recall from (12) that the nonlinear terms do not directly contribute to the perturbation energy growth.

This is seen from the following argument.

Suppose we transform the problem of making a transfer function $G(s)$ strictly positive real into an equivalent problem of making H strictly bounded real, *i.e.* $\|H\|_\infty < 1$. However say we actually know $\|H\|_\infty < \gamma$. Then for $\mathrm{Re}(s) > 0$

$$\det[I - \gamma^{-1}H(s)] \neq 0, \mathrm{Re}(s) > 0.$$

From the relationship

$$H(s) = (G(s) - I)(G(s) + I)^{-1}$$

it is straightforward to show that

$$H(s)H^*(s) = (G(s) - I)(G(s) + I)^{-1}(G^*(s) + I)^{-1}(G^*(s) - I) < \gamma^2 I.$$

Rearrangement gives

$$G(s) + G^*(s) > \frac{1-\gamma^2}{1+\gamma^2}(G^*(s)G(s) + I).$$

As $\gamma \to 1$, $G(s)$ becomes positive real. Bounding the right hand side by $-\alpha$,

$$-\alpha = \inf_{s=j\omega}\left[\frac{1-\gamma^2}{1+\gamma^2}(G^*(s)G(s) + I)\right]$$

means $\alpha \geq 0$ (since $\gamma \geq 1$).

If $z = Gw$, then it is straightforward to show

$$\langle z, w\rangle > -\frac{\alpha}{2}\langle w, w\rangle \;\; \forall w.$$

If $\|z\|_2$ is the perturbation energy, then this bounds the perturbation energy growth produced by any disturbance w and optimising γ optimises this bound.

This expression is not equivalent to the 'optimal' in the sense of Butler and Farrel,[24] but is arguably more pertinent in characterising the sensitivity of the nonlinear flow to disturbances, and thus the potential for turbulent energy production.

V. Application to Channel Flow

This application of the theory above considers three dimensional perturbations to canonical plane Poiseuille flow. The flow domain is the space between two plates parallel in the xz plane, at $y = \pm 1$. Periodicity is assumed for the streamwise (x) and spanwise (z) directions with period 2π. A pressure gradient $\frac{\partial P}{\partial x}$ is applied in the streamwise direction.

We consider here actuation provided by volume forcing as a precursor to other types of forcing. The controller senses the wall-normal velocity only in the whole domain.

The actuation and sensing requirement is clearly unphysical but the design procedure is general enough to allow further studies with more restrictive sensing capabilities. Thus the volume forcing situation may be considered a precursor to other types of forcing.

The geometry allows Fourier transform of the linearised problem in the x and z directions which converts the (xz)-continuous problem into a number of decoupled y-continuous problems at particular Fourier wavenumber pairs. Truncation at suitably high wavenumber ensures an (xz)-discrete problem with sufficient resolution. Chebyshev pseudo-spectral discretisation in the y direction results in a number of linear time invariant state-space control problems. Fortunately these problems are decoupled at the (linear) synthesis stage (a block diagonal A matrix in the state-space formulation) and only coupled (via the nonlinearity) at the full simulation stage. The controller synthesis problem is further simplified because control is only necessary at those wavenumber pairs where transient growth is possible, with attendant implications for sensing and actuation.

The Orr-Sommerfeld-Squire formulation is used to describe the linearised behaviour of perturbations to plane Poiseuille flow. The Navier-Stokes equations are linearised about the parabolic steady flow solution, and projected onto a divergence-free basis with the resulting fourth-order PDEs discretised using a pseudo-spectral method and rearranged into state-space form. The techniques presented are known in the literature and are presented here together to clearly specify the problem as it relates to our approach and for completeness. The methods used avoid the spurious eigenvalues associated with other methods.[15,21]

The Orr-Sommerfeld-Squire equations can be written in state-space form as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \hat{v} \\ \hat{\omega} \end{bmatrix} = A \begin{bmatrix} \hat{v} \\ \hat{\omega} \end{bmatrix}, \tag{13}$$

with

$$A = \begin{bmatrix} L_{OS} & 0 \\ L_C & L_{SQ} \end{bmatrix} \tag{14}$$

where v and ω are the wall-normal velocity and vorticity respectively and $\hat{\cdot}$ denotes a Fourier transformed variable. The (steady) laminar profile is $U(y)$. L_{OS}, L_{SQ}, and L_C represent the Orr-Sommerfeld, Squire and coupling operators respectively which are

$$L_{OS} = [\nabla^2]^{-1}\left\{-ik_xU\nabla^2 + ik_x\frac{\mathrm{d}^2U}{\mathrm{d}y^2} + \frac{\nabla^2\nabla^2}{Re}\right\} \tag{15}$$

$$L_{SQ} = -ik_xU + \frac{\nabla^2}{Re} \tag{16}$$

$$L_C = -ik_z\frac{\mathrm{d}U}{\mathrm{d}y} \tag{17}$$

where k_x and k_z are the wavenumbers in their respective directions. The Laplacian is given by

$$\nabla^2 = \frac{\partial^2}{\partial y^2} - k_x^2 - k_z^2$$

.

The periodic xz boundary conditions are naturally enforced by the Fourier transform. Further, any wall transpiration is necessarily divergence-free as it is expressed in a divergence-free basis. This is anyway enforced by the shapes of the Fourier modes.

In the y direction, the following boundary conditions must be enforced:

Firstly there are the homogenous Dirichlet boundary conditions for the wall-normal velocity and vorticity at the wall,

$$v(y=1) = v(y=-1) = 0, \tag{18}$$

$$\omega_y(y=\pm 1) = 0. \tag{19}$$

In addition there are the homogenous Neumann boundary conditions for the wall-normal velocity,

$$\frac{\partial v}{\partial y}(y=\pm 1) = 0. \tag{20}$$

Discretisation in the y direction is achieved using the Chebyshev transformation evaluated at the Chebyshev points. The homogenous boundary conditions are enforced automatically by using the method described in.[22]

A. State-Space Formulation

Discretised, the Orr-Sommerfeld-Squire equations have the form

$$\dot{x} = Ax + B_1 n + B_2 f \tag{21}$$

$$y_1 = C_1 x \tag{22}$$

$$y_2 = C_2 x \tag{23}$$

where x is the vector of v or η and ω evaluated at the discretisation points.

The perturbation energy at a wavenumber pair is given by[15]

$$E(t) = \frac{1}{8}\int_{-1}^{1} \hat{v}^*\hat{v} + \frac{1}{k_x^2+k_z^2}\left(\frac{\partial \hat{v}^*}{\partial y}\frac{\partial \hat{v}}{\partial y} + \hat{\omega}^*\hat{\omega}\right) \mathrm{d}y. \tag{24}$$

For the discretised state the perturbation energy is approximated by the inner product on the positive-definite matrix Q so that

$$E(t) \simeq x(t)^* Q x(t),$$

achieving equality in the limit. For $E(t) \simeq y_1(t)^* y_1$, we require simply $C_1^* C_1 = Q$. The input matrix B_1 associated with the forcing from the nonlinearity n, is simply the inverse of C_1.

We write the system equations above in compact notation

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{array}\right]. \tag{25}$$

Define $Q : n \mapsto y_1$ as the finite dimensional approximation of the closed-loop transfer function from e to v, as in Figure 2. Q is then the closed-loop of G and the controller K to be found. We wish to find a controller K such that $Q = \mathcal{F}_l(G, K)$ is positive real. Equivalently, $Q(s) + Q^*(s) > 0$ or $\langle n, y_1 \rangle_{[0,T]} > 0, \; \forall T$.

The controller that satisfies this requirement also solves the small-gain problem[5]

$$\|\tilde{Q}\|_\infty < \gamma$$

with $\gamma = 1$ and $\tilde{P}$ given by the closed-loop of $\tilde{G}$ and K. $\tilde{G}$ is given by

$$\tilde{G} = \left[\begin{array}{c|cc} A - B_1 C_1 & B_1 & B_2 \\ \hline -2C_1 & I & 0 \\ C_2 & 0 & 0 \end{array}\right].$$

To avoid a singular control problem and unbounded control signals, a penalty can be introduced on the control, and sensor noise can be introduced. These are made orthogonal to the dynamics, by augmenting $\tilde{G}$ (to give $\tilde{G}^{+}$),

$$\tilde{G}^{+} = \left[\begin{array}{c|c|c} A - B_1 C_1 & \begin{bmatrix} B_1 & 0 \end{bmatrix} & B_2 \\ \hline \begin{bmatrix} -2C_1 \\ 0 \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ \epsilon I \end{bmatrix} \\ C_2 & \begin{bmatrix} 0 & \epsilon I \end{bmatrix} & 0 \end{array}\right].$$

We call ϵ the penalty weighting.

K is to be designed to minimise γ. In the case that the optimal γ is less than unity, the positive real control problem is solved by K. The γ optimisation is done using loop-shifting techniques[2,20] and the standard $\mathcal{H}_\infty$ synthesis theory.[2,4,23]

VI. Results

As a proof of principle, the method above was applied to the 'optimal' transient growth case (in the sense of[24]) in channel flow for $Re = 5000$ (just below the Reynolds number for linear instability) using 95 Chebyshev modes. This corresponds to the initial condition that gains the most perturbation energy in the linear problem, at this Reynolds number. The peak growth occurs for $k_x = 0$ and $k_z = 2.044$ at a dimensionless time, $t = 379$. Two controllers were synthesised using full-field sensing and actuation of the wall-normal velocity only, which was found to be sufficient to eliminate transient growth. For the first controller, the control action penalty is set small $\epsilon = 10^{-8}$, allowing impractically large control signals. The second controller has a substantial control penalty $\epsilon = 0.1$. The control signals are reduced by a comparable order.

In the longer term the linear mode decays more with the high-penalty controller. Here we see in effect the engineering compromise that an optimal controller (corresponding to the low-penalty case) represses desirable characteristics of the plant in an effort to achieve the optimal condition, because the optimal controller conservatively caters for the disturbance causing most perturbation energy production. The optimal controller is designed to avoid the nonlinearity pushing the flow state back into a configuration where it can experience more perturbation growth, by precluding any possible growth. The more gentle suboptimal (high-penalty) controller may then be preferable if you can bound the expected disturbance and the controlled growth is below the expected transition threshold.

The perturbation energy is shown in Figure 3 for the controlled and uncontrolled case. The uncontrolled case has a peak energy of 4897, the controlled case either greatly reduces or eliminates the perturbation energy.

The low-penalty controller completely eliminates the 'streaks', effectively freezing the initial condition.

With the higher control penalty, the perturbation energy growth is greatly limited but no longer eliminated. The control forcing however is much smaller. The evolution of the flow is similar to the uncontrolled case, but the magnitude of the streaks is greatly reduced.

Figures 4 to 7 show cross sections of the flow field at $t = 0$ and $t = 379$.

VII. Conclusion

A new characterisation of stabilising feedback laws for incompressible Navier-Stokes flows in an open domain has been presented in terms of positive real systems theory. For the discretised case, the ensuing finite-dimensional positive real synthesis problem results in two game-theoretic algebraic Riccati equations. When these Riccati equations have solutions, a nonlinearly stabilising linear controller can be synthesised. By converting the problem to a γ-optimisation $\mathcal{H}_\infty$ problem, the under-actuated or under-sensed case can be tackled. Optimising γ also optimises bounds on the perturbation energy growth.

Application of the procedure to the maximally-growing transient wavenumber pair in channel flow (with full-field volume forcing) leads to elimination of transient growth, or (if a penalty is introduced to limit the

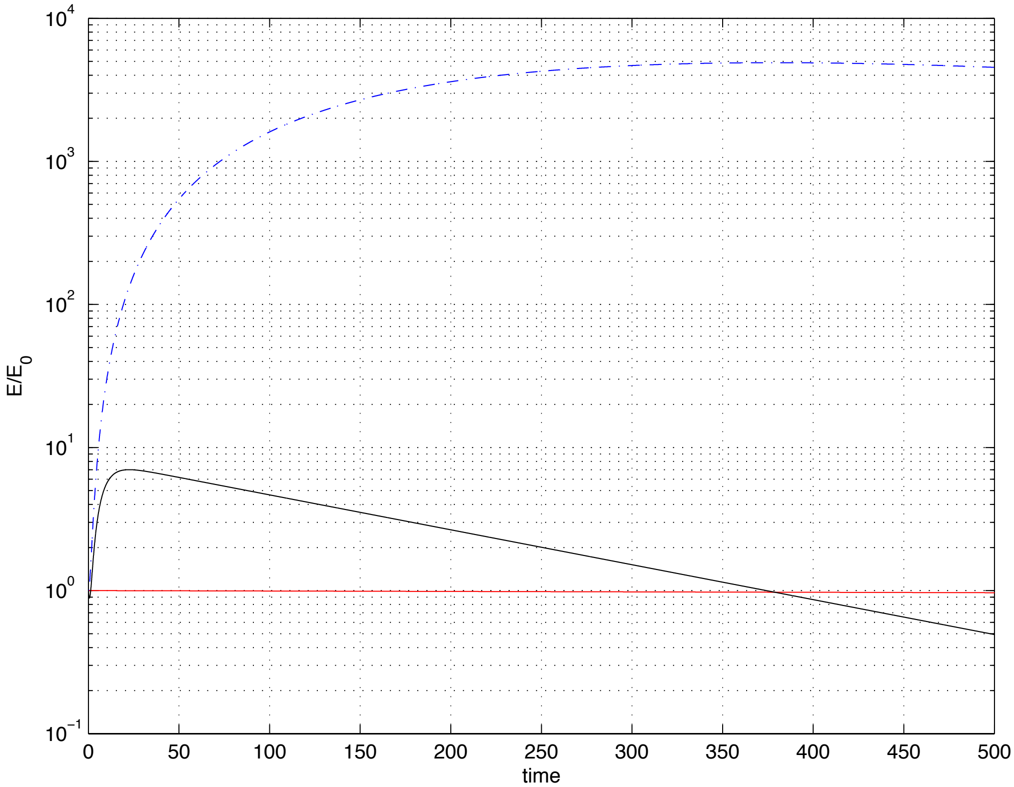

Figure 3. Energy growth of optimal perturbation for $Re = 5000$, corresponding to $k_x = 0,\ k_z = 2.044$. The upper line corresponds to the uncontrolled case, the middle to the high-penalty controlled case, and the lowest to the low-penalty controlled case. The uncontrolled case illustrates a perturbation energy growth of 4897. The high-penalty controller effectively limits the growth of the initial perturbation energy. The low-penalty controller completely eliminates the transient growth.

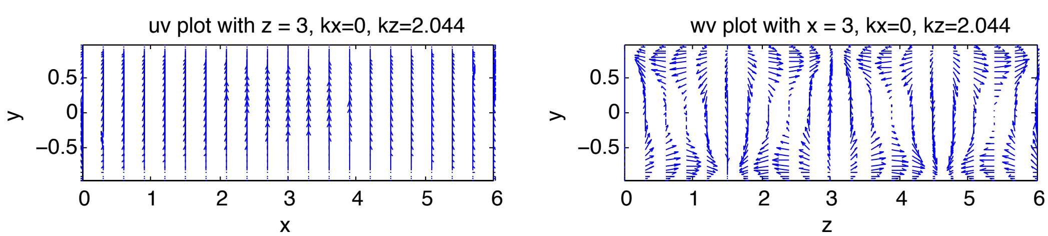

Figure 4. The flowfield at $t = 0$ (E=1), magnitudes have been normalised.

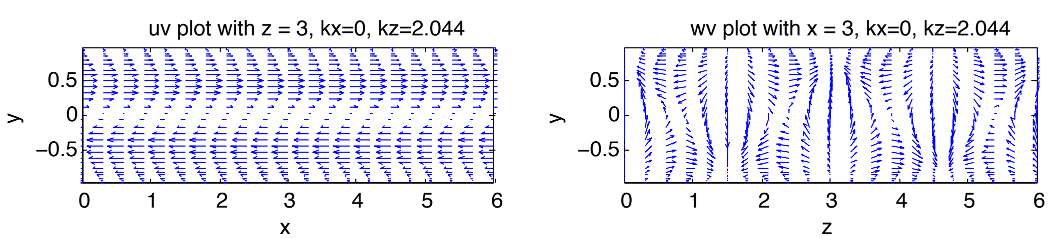

Figure 5. The uncontrolled flowfield at $t = 379$ (peak growth, E=4897)

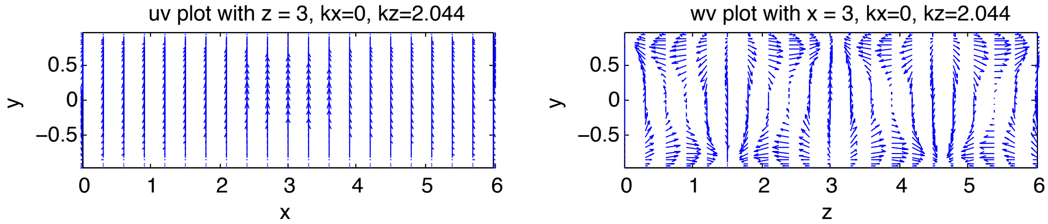

Figure 6. The low-penalty controlled flowfield at $t = 379$ (E=0.97)

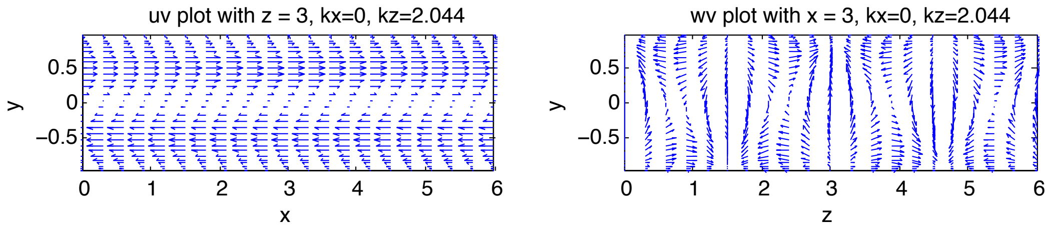

Figure 7. The high-penalty controlled flowfield at $t = 379$ (E=0.97)

control signal) an optimal reduction in energy of the transient mode, ultimately increasing the threshold on perturbation amplitude leading to bypass transition.

The full-field wall-normal velocity volume forcing and sensing case provides a benchmark for extension of the procedure to more physically realisable forcing and sensing scenarios. The method permits investigation of control effectiveness under different forcing methods and given limited sensing and actuation capabilities, which will form part of our future work in this area. Application to a full channel simulation is under way, allowing investigation of the effect of sensor or actuator outage and design of component redundancy for optimal efficiency. An experimental study is also planned.

Acknowledgments

A. Sharma, J. Morrison and D. Limebeer thank BAE systems and the UK EPSRC for their support under the FLAVIIR project. B. McKeon thanks the Royal Society for their support under a Dorothy Hodgkin Fellowship.

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