Computing Global Combine Operations in the Multiport Postal Model

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Abstract—Consider a message-passing system of n processors, in which each processor holds one piece of data initially. The goal is to compute an associative and commutative reduction function on the n pieces of data and to make the result known to all the n processors. This operation is frequently used in many message-passing systems and is typically referred to as *global combine*, *census computation*, or *gossiping*. This paper explores the problem of global combine in the *multiport postal model*. This model is characterized by three parameters: n —the number of processors, k —the number of ports per processor, and λ —the communication latency. In this model, in every round r , each processor can send k distinct messages to k other processors, and it can receive k messages that were sent from k other processors $\lambda - 1$ rounds earlier. This paper provides an optimal algorithm for the global combine problem that requires the least number of communication rounds and minimizes the time spent by any processor in sending and receiving messages.

Index Terms—Census computation, distributed systems, global combine, gossiping, message-passing systems, multiple ports, parallel computers, postal model.

I. INTRODUCTION

This paper explores the *global combine* operation in the *multiport postal model* for message-passing systems. In this model, the system consists of n processors, each processor has k communication ports, and the communication latency is λ . We denote such a message-passing system by $MPS(n, k, \lambda)$, and we assume that the n processors are labeled from 0 through $n - 1$. In the global combine operation, each processor i initially holds a data item d_i . Given a commutative and associative combining operator " $\oplus$," the goal is to compute $D = d_0 \oplus d_1 \oplus \dots \oplus d_{n-1}$ and to place the result D in all the n processors. Examples of such combining operators are MAX, MIN, (logical or bitwise) AND, OR, XOR, integer addition, and integer multiplication.

The global combine operation is frequently used in many applications for message-passing systems (see [20]). Several collective communication libraries, such as Express [19] by Parasoftware and the External User Interface (EUI) [2], [3] of the Scalable POWER-parallel System (SP-1) by IBM, provide primitives for computing a global combine operation. This operation has also been included as part of the collective communication routines in the Message-Passing Interface (MPI) standard proposal [18]. (This operation is called `mpi_allreduce` in MPI.) In the literature, this operation is also referred to as *census computation* (e.g., [1], [8]) or *gossiping* [21].

In the multiport postal model, two integer parameters, $k \geq 1$ and $\lambda \geq 1$, are used to model the communication characteristics of a message-passing system $MPS(n, k, \lambda)$. We assume that each of the

n processors has "effectively" k distinct input ports and k distinct output ports, and we assume that communicating a data item between any pair of processors exhibits a latency of λ . In other words, we assume that in each communication round r , every processor can send k distinct messages to k other processors, and in the same round r each processor can receive k distinct messages that were sent out from k other processors $\lambda - 1$ rounds earlier (i.e., in round $r - \lambda + 1$, provided that $r - \lambda + 1 \geq 1$). It is assumed that all processors know the values of n , k , and λ .

The multiport postal model generalizes existing models in two ways:

- 1) by addressing multiport capabilities of several systems ($k > 1$), and
- 2) by addressing communication latencies of systems ($\lambda > 1$). Although we do not know any existing machines that exhibit the multiport postal model with $k > 1$ and $\lambda > 1$, there have been many machines exhibit either the multiport model or the postal model. This is the first attempt to unify the algorithm design and implementation for both models, by considering algorithm for the more general model. Although we consider the global combine operation only in this paper, the same techniques can be easily applied to algorithms for all-to-all broadcast under the same model.

In the first way, the multiport model generalizes the one-port model that has been widely investigated. There are two prevailing variants of the one-port model: unidirectional communication and bidirectional communication. In unidirectional communication, a processor can either send or receive a message in any given round (see [21]). In bidirectional communication, distinct pairs of processors can exchange messages in any given round (see [21], [22], [26]). Recently, a third variant was explored, in which in any given round, each processor can send a message to one processor and receive a message from another processor (see [7], [8], [11], [12]). However, there are examples of parallel systems with k -port capabilities for $k > 1$, such as the Ncube/2 [25] and the CM-2 (where k is the dimension of the hypercube in both machines).

In the second way, the postal model, which was recently introduced by Bar-Noy and Kipnis [5], [6] addresses emerging trends in modern distributed-memory parallel computers and high-speed communication networks. (A related model, the LogP model [17], was also proposed recently to address similar goals.) Systems that are well modeled by the postal model include parallel computers like TMC's CM-5 [24] and IBM's SP-2 [28], and communication networks such as PARIS [14] and AURORA [15]. Bruck et al. has confirmed experimentally in [10] that a properly tuned broadcast based on the postal model on the Delta system can improve the best known implementation by more than 20%. Culler et al. also confirmed experimentally in [16] that LogP model is helpful in the development of the fast parallel sorting algorithms on the CM-5.

Two complexity measures are frequently used in evaluating communication algorithms. The first measure, denoted by C_1 , is the number of communication rounds required by an algorithm. Notice that in each round, the messages sent and received by each processor can be of arbitrary sizes. Let s_i be the size of the largest message sent or received by any processor in round i . The second measure, denoted by C_2 , is the sum of the s_i s over all the rounds. Note that C_2 gives an upper bound on the number of data items sent or received *per processor*. Hence, the communication time complexity of an algorithm can be modeled as $C_1 t_s + C_2 t_c$, where t_s is the communication start-up time and t_c is the per-data-item communication time.

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Many algorithms for the global combine operation have appeared in the literature. Most of these algorithms are based on the one-port no-latency model ($k = \lambda = 1$) and are restricted to certain topologies, such as meshes [4] or hypercubes [20], [27]. A naive way to compute the global combining operation is to solve the concatenation problem (see [12] for an optimal solution). However, since in this solution C_2 is very large (order of n), we rule out such an approach.

In the general $MPS(n, k, \lambda)$ model, optimal solutions for the global combine problem were known only for specific values of the parameters, n , k , and λ . For example, for the case of $k = \lambda = 1$, a known solution first reduces the number of data pieces to be a power of 2, then applies a butterfly combining algorithm on the reduced data, and finally sends the reduction result to the remaining processors. This solution is optimal in both C_1 and C_2 only when n is a power of 2. For the same parameters, an optimal solution for any value of n was recently developed in [8]. This solution is based on the binary representation of n . Lately, an optimal solution for $k = 1$, for any integer $\lambda \geq 1$, but only for values of n of certain form, was presented in [23]. It should be noted that the solution of [23] can be extended to any value of n (still with $k = 1$). This extension results in an algorithm that is within an additive term of $2\lambda - 1$ of optimal. More recently, two new optimal solutions for the global combine problem in restricted models have appeared: 1) a solution for $k = 1$ and for any integers $\lambda \geq 1$ and $n \geq 1$ was presented in [9], and 2) a solution for $\lambda = 1$ and for any integers $k \geq 1$ and $n \geq 1$ appeared in [13]. Finally, it should be noted that if the combining operator " $\oplus$ " satisfies $a \oplus a = a$ for all possible a (such as MAX and MIN), then solutions for "nice" numbers can be extended to an optimal solution for any value of n [8]. However, many operators do not have this property (e.g., Exclusive Or, integer addition, and integer multiplication).

In this paper, we provide for the first time an optimal algorithm (with respect to both C_1 and C_2) for the global combine problem for any commutative and associative operator and for any integers $n \geq 1$, $k \geq 1$ and $\lambda \geq 1$.

II. PRELIMINARIES

In this section, we provide definitions that are used by the general algorithm in Section III. We also provide the lower bound for the global combine problem.

A. Sequence Representation of Integers

We first define a sequence of *generalized Fibonacci numbers* that is indexed by k and λ . Given integers $k \geq 1$ and $\lambda \geq 1$, the function $F(\cdot)$ is defined from the nonnegative integers into the positive integers as follows:

$$F(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \lambda - 1, \\ F(r-1) + kF(r-\lambda) & \text{if } \lambda \leq r. \end{cases}$$

For clarity, the two parameters k and λ , which are fixed throughout the paper, are omitted from the function F . For example, for $\lambda = 1$, we have $F(r) = (k+1)^r$, and for $k = 1$ and $\lambda = 2$ the value of $F(r)$ is the Fibonacci number whose index is r .

In Section III, we show that the variable r in $F(r)$ corresponds to the round number in the algorithm. Thus, if the number of processors $n > 1$ is of the form $n = F(m)$, for some integer m , then the global combine algorithm finishes optimally in m rounds. However, it is more complicated to give an optimal algorithm that takes m rounds for $n > 1$ when $F(m-1) < n < F(m)$. In order to do so, we need to introduce a sequence of *deficiency parameters* $\epsilon = (\epsilon_0, \epsilon_1, \dots, \epsilon_{m-\lambda})$, where the ϵ_i s are integers.

Formally, let $\epsilon = (\epsilon_0, \epsilon_1, \dots, \epsilon_{m-\lambda})$ be a sequence in which $0 \leq \epsilon_j \leq k$, for all $0 \leq j \leq m - \lambda$. Define the following recurrence:

$$F_\epsilon(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \lambda - 1, \\ F_\epsilon(r-1) + kF_\epsilon(r-\lambda) - \epsilon_{r-\lambda} & \text{if } \lambda \leq r \leq m. \end{cases}$$

Note that the recurrence $F(\cdot)$ is identical to $F_\epsilon(\cdot)$, for $\epsilon = (0, \dots, 0)$ and $r \leq m$. We denote this recurrence by $F_0(\cdot)$.

Assume, throughout the paper, that $F_0(m-1) < n \leq F_0(m)$. In the following, we show that there exists a sequence $\epsilon = (\epsilon_0, \epsilon_1, \dots, \epsilon_{m-\lambda})$, in which $0 \leq \epsilon_j \leq k$, for all $0 \leq j \leq m - \lambda$, such that $n = F_\epsilon(m)$. First, we need two lemmas.

LEMMA 1. For every $\epsilon = (\epsilon_0, \epsilon_1, \dots, \epsilon_{m-\lambda})$, where $0 \leq \epsilon_j \leq k$, for all $0 \leq j \leq m - \lambda$, we have

$$F_\epsilon(m) = F_0(m) - \sum_{j=0}^{m-\lambda} \epsilon_j F_0(m - \lambda - j).$$

PROOF. The proof is by induction on $m \geq 0$. To obtain the induction basis, let $0 \leq m \leq \lambda - 1$. By our definition $F_\epsilon(m) = F_0(m) = 1$, and the claim is trivially true since there are no summands in the summation. Assume now that the claim is true for all $0 \leq j \leq m - 1$. By the induction hypothesis we get the following two equations:

$$\begin{aligned} F_\epsilon(m - \lambda) &= F_0(m - \lambda) - \sum_{j=0}^{m-\lambda-\lambda} \epsilon_j F_0(m - \lambda - \lambda - j) \\ F_\epsilon(m - 1) &= F_0(m - 1) - \sum_{j=0}^{m-1-\lambda} \epsilon_j F_0(m - 1 - \lambda - j) \\ &= F_0(m - 1) - \sum_{j=0}^{m-\lambda-1} \epsilon_j F_0(m - 1 - \lambda - j) \\ &\quad - \sum_{j=0}^{\lambda-2} \epsilon_{m-1-\lambda-j} F_0(j). \end{aligned}$$

Notice that we partitioned the sum in the second equation into two, and changed the running index of the second sum. By our definition

$$\begin{aligned} F_\epsilon(m) &= F_\epsilon(m-1) + kF_\epsilon(m-\lambda) - \epsilon_{m-\lambda} \\ &= F_0(m-1) + kF_0(m-\lambda) - \sum_{j=0}^{m-2\lambda} \epsilon_j F_0(m-1-\lambda-j) \\ &\quad - \sum_{j=0}^{m-2\lambda} k\epsilon_j F_0(m-\lambda-\lambda-j) - \sum_{j=0}^{\lambda-2} \epsilon_{m-1-\lambda-j} F_0(j) - \epsilon_{m-\lambda} \\ &= F_0(m) - \sum_{j=0}^{m-2\lambda} \epsilon_j F_0(m-\lambda-j) - \sum_{j=0}^{\lambda-2} \epsilon_{m-1-\lambda-j} F_0(j) - \epsilon_{m-\lambda}. \end{aligned}$$

The first equality follows from the definition of $F_\epsilon(\cdot)$. The second equality follows from the induction hypothesis and by rearranging operands. The third equality follows from the definition of $F_0(\cdot)$. Now, observe that $F_0(j) = F_0(j+1) = 1$, for $0 \leq j \leq \lambda - 2$, and thus

$$\begin{aligned} F_\epsilon(m) &= F_0(m) - \sum_{j=0}^{m-2\lambda} \epsilon_j F_0(m-\lambda-j) \\ &\quad - \sum_{j=0}^{\lambda-2} \epsilon_{m-1-\lambda-j} F_0(j+1) - \epsilon_{m-\lambda} F_0(0). \end{aligned}$$

Finally, by changing the running index in the second summation and by combining all the terms under one summation, we get that $F_\varepsilon(m) = F_0(m) - \sum_{j=0}^{m-\lambda} \varepsilon_j F_0(m-\lambda-j)$. $\square$

Recall that $F_0(m-1) < n \leq F_0(m)$, and let $\alpha = F_0(m) - n$.

LEMMA 2. *There exists $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{m-\lambda})$, in which $0 \leq \varepsilon_j \leq k$, for all $0 \leq j \leq m - \lambda$, such that $\alpha = \sum_{j=0}^{m-\lambda} \varepsilon_j F_0(m-\lambda-j)$.*

PROOF. If $\alpha = 0$, then by letting $\varepsilon_j = 0$ for all j we are done. Suppose that $\alpha > 0$. By definition we get that $\alpha < F_0(m) - F_0(m-1) = kF_0(m-\lambda)$. Therefore, there exists $0 \leq \varepsilon_0 < k$ and $\alpha' < F_0(m-\lambda)$ such that $\alpha = \varepsilon_0 F_0(m-\lambda) + \alpha'$. If $\alpha' = 0$, then we are done. Otherwise, there exists some x , $0 < x \leq m - 2\lambda + 1$, such that $F_0(m-\lambda-x) \leq \alpha' < F_0(m-\lambda+1-x)$. Set

$$\varepsilon_x = \left\lfloor \frac{\alpha'}{F_0(m-\lambda-x)} \right\rfloor,$$

and let $\alpha'' = \alpha' - \varepsilon_x F_0(m-\lambda-x)$. It follows that $\alpha'' < F_0(m-\lambda-x)$. We repeat the above process for α'' as was done for α' . Since $0 \leq \alpha'' < \alpha'$, it follows that eventually the remainder will be zero and the proof is completed. $\square$

Note that for $\lambda > 1$ it is enough to set $\varepsilon_x = 1$ instead of $\varepsilon_x = \left\lfloor \frac{\alpha'}{F_0(m-\lambda-x)} \right\rfloor$. This is true since in this case $\alpha'' < F_0(m-\lambda+1-x) - F_0(m-\lambda-x) = kF_0(m-\lambda+1-x-\lambda) < kF_0(m-\lambda-x)$.

The above lemmas imply the following theorem. Note that for a given n , the sequence ε for which $n = F_\varepsilon(m)$ is not necessarily unique.

THEOREM 3. *For any n , such that $F_0(m-1) < n \leq F_0(m)$, there exists a sequence $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{m-\lambda})$, in which $0 \leq \varepsilon_j \leq k$, for all $0 \leq j \leq m - \lambda$, such that $n = F_\varepsilon(m)$.*

B. The Lower Bound

Given the implicit integer parameters $k \geq 1$ and $\lambda \geq 1$, we define $f(n) = \min_r \{F(r) \geq n\}$. Notice that $F(f(n)-1) < n \leq F(f(n))$. We now show that for any $n > 1$, any algorithm for the global combine problem in $\mathcal{MPS}(n, k, \lambda)$ requires at least $C_1 = f(n)$ rounds and communicates at least $C_2 = f(n)$ data items per processor.

These bounds are derived by reducing the broadcasting problem to the problem of computing a global combine operation. This can be done by assuming that the broadcast source has a data item while all other processors have identity element I as their data items. Thus, the lower bound for the broadcast problem in $\mathcal{MPS}(n, k, \lambda)$ implies the lower bound for the global combine operation.

Now notice that the recursive definition of $F(r)$ is identical to the recursive definition describing the maximum number of processors that can receive the data item of the source after r rounds in $\mathcal{MPS}(n, k, \lambda)$. Thus, the lower bound for broadcast (and also for global combine) in $\mathcal{MPS}(n, k, \lambda)$ is $C_1 = C_2 = f(n)$.

III. THE ALGORITHM

In this section, we present an optimal algorithm for the global combine operation in $\mathcal{MPS}(n, k, \lambda)$. The basic idea of the algorithm is to treat n as $F_\varepsilon(m)$, where $F_\varepsilon(m)$ is recursively defined in Section II.

Using this representation, one can start with the value of 1 and get to the value of n in m rounds. During each round r , where $\lambda \leq r \leq m$, a processor receives k values from k other processors, each such value being a partial reduction result of either $F_\varepsilon(r-\lambda)$ items or $F_\varepsilon(r-\lambda)-1$ items, that were sent out during round $r-\lambda+1$.

Let I be the identity data item with respect to $\oplus$. (Note that the existence of the identity data item is assumed only for the purpose of describing the algorithm. In practice, this is not needed.) In the following, the arithmetic for all the processor IDs and the subscripts for the data items is done in modulo n . Each processor i maintains the sequence defined by n, k , and λ , another sequence of $m+1$ elements: $c_i^0, c_i^1, \dots, c_i^m$, and two local variables: S_i and T_i . For the sake of clarity, we denote the values of S_i and T_i after round r by S_i^r and T_i^r , respectively. The contents of these variables are as follows.

- c_i^r : denotes the number of data items on which the reduction function was already performed after round r . (The partial reduction result will be stored in T_i^r) Initially, $c_i^0 = 1$.
- S_i^r : holds the reduction result of $c_i^r - 1$ consecutive data items (cyclically) starting with d_{i+1} , i.e.,

$$S_i^r = d_{i+1} \oplus d_{i+2} \oplus \dots \oplus d_{i+c_i^r-1}.$$

Initially, $S_i^0 = I$.

- T_i^r : holds the reduction result of c_i^r consecutive data items (cyclically) starting with d_i , i.e.,

$$T_i^r = d_i \oplus d_{i+1} \oplus \dots \oplus d_{i+c_i^r-1}.$$

Initially, $T_i^0 = d_i$.

Before the algorithm starts, each processor computes the sequence $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{m-\lambda})$ in which $0 \leq \varepsilon_j \leq k$ for all $0 \leq j \leq m - \lambda$ and such that $n = F_\varepsilon(m)$. (This computation can be done following the proof of Lemma 2.) Next, each processor i sets $c_i^j = F_\varepsilon(j)$ for all $0 \leq j \leq m$.

Algorithm. The algorithm consists of m rounds. The following is the description of the algorithm performed by processor i in round r , for $1 \leq r \leq m$.

- 1) Perform the following k sends and k receives concurrently.
 - If $r \leq m - \lambda + 1$, send T_i^{r-1} to the following set of $k - \varepsilon_{r-1}$ processors: $\{i - c_i^{r+\lambda-2} - (x-1)c_i^{r-1} : 1 \leq x \leq k - \varepsilon_{r-1}\}$.
 - If $r \leq m - \lambda + 1$, send S_i^{r-1} to the following set of ε_{r-1} processors:

$$\{i - c_i^{r+\lambda-2} - (x-1)c_i^{r-1} + x - (k - \varepsilon_{r-1}) : \text{for } k - \varepsilon_{r-1} < x \leq k\}.$$
 - If $r \geq \lambda$, receive the k data items that were sent out to processor i during round $r - \lambda + 1$. Denote these k data items by $V_1, V_2, \dots, V_k$.
- 2) If $1 \leq r \leq \lambda - 1$, then $S_i^r = S_i^{r-1}$ and $T_i^r = T_i^{r-1}$.
- 3) If $\lambda \leq r \leq m$, compute the following:
 - $S_i^r = S_i^{r-1} \oplus V_1 \oplus V_2 \oplus \dots \oplus V_k$;
 - $T_i^r = T_i^{r-1} \oplus V_1 \oplus V_2 \oplus \dots \oplus V_k$.

The final result of the global combine operation over all the n data items is obtained for processor i in the variable T_i after round m (that is, in T_i^m).

We now prove the correctness of the algorithm.

THEOREM 4. *In the above algorithm $T_i^m = d_i \oplus d_{i+1} \oplus \dots \oplus d_{i-1}$.*

PROOF. The proof is by induction on round r , for $1 \leq r \leq m$. The induction hypothesis is that after round r , where $1 \leq r \leq m$, the following two properties hold for every $0 \leq i \leq n-1$.

- 1) $S_i^r = d_{i+1} \oplus d_{i+2} \oplus \dots \oplus d_{i+c_i^r-1}$.
- 2) $T_i^r = d_i \oplus S_i^r$.

For the basis, consider $1 \leq r \leq \lambda - 1$. We have $S_i^r = I$ and $T_i^r = d_i$, for all $0 \leq i \leq \lambda - 1$. Thus, the two properties hold.

Assume, for the sake of the induction hypothesis, that the two properties hold right before round r , for some $\lambda \leq r \leq m$. We now wish to show that they still hold after round r . Consider Property 1. Assume without loss of generality that $V_1, V_2, \dots, V_k$ are ordered according to the ascending order (modulo n) of the sending processor IDs starting from i . For convenience, let $y = k - \varepsilon_{r-\lambda}$. From the algorithm, for $1 \leq x \leq y$, we have that $V_x = T_{j(x)}^{r-\lambda}$ where $j(x) = i + c_i^{r-1} + (x-1)c_i^{r-\lambda}$ and for $y < x \leq k$, we have that $V_x = S_{j(x)}^{r-\lambda}$ where $j(x) = i + c_i^{r-1} + (x-1)c_i^{r-\lambda} - (x-y)$. Note that a value received from processor $j(x)$, where $1 \leq x \leq y$, during round r was the partial reduction result of $c_{j(x)}^{r-\lambda} = c_i^{r-\lambda}$ consecutive elements starting from processor $j(x)$. Also, a value received from processor $j(x)$, where $y < x \leq k$, during round r was the partial reduction result of $c_{j(x)}^{r-\lambda} - 1 = c_i^{r-\lambda} - 1$ consecutive elements starting from processor $j(x) + 1$. Thus, all we need to prove are two things: 1) the "last" element covered by S_i^{r-1} (and thus also by T_i^{r-1}) is followed by the "first" element covered by V_1 ; 2) the "last" element covered by V_x is followed by the "first" element covered by V_{x+1} , for $1 \leq x < k$.

To prove 1), it follows from the definitions that $(j(1) - i) \bmod n = c_i^{r-1}$ if $y > 0$, and is $c_i^{r-1} - 1$ if $y = 0$. To prove 2), one verifies that $(j(x+1) - j(x)) \bmod n = c_i^{r-\lambda}$ for $1 \leq x < y$ and $(j(x+1) - j(x)) \bmod n = c_i^{r-\lambda} - 1$ for $y \leq x \leq k-1$. Thus, Property 1 holds.

Similarly, it can be seen that Property 2 also holds from the induction. Thus, T_i^m is the reduction result of $c_i^m = F_\varepsilon(m) = n$ consecutive elements starting from item d_i . $\square$

The next complexity statement follows from the description of the algorithm.

THEOREM 5. *The algorithm takes $C_1 = m = f(n)$ rounds and communicates $C_2 = m = f(n)$ data items.*

IV. AN EXAMPLE

In this section, we elaborate on a concrete example for the integer addition operator. Let $n = 9$, $k = 2$, and $\lambda = 3$. From the definition of the function $F(\cdot)$ it follows that $F(0) = F(1) = F(2) = 1$, $F(3) = 3$, $F(4) = 5$, $F(5) = 7$, and $F(6) = 13$. Therefore, the number of rounds in the algorithm is $m = f(n) = 6$. Since $F(6) = 13$, it follows that $\alpha = F(m) - n = 4$. The constructive proof of Lemma 2 implies that $\varepsilon = (1, 1, 0, 0)$. However, for this example we choose another valid value, $\varepsilon = (1, 0, 0, 1)$, i.e., $\varepsilon_0 = 1$, $\varepsilon_1 = 0$, $\varepsilon_2 = 0$, and $\varepsilon_3 = 1$. From the definition of $F_\varepsilon(\cdot)$ it follows that $c_i^0 = F_\varepsilon(0) = 1$, $c_i^1 = F_\varepsilon(1)$

$= 1$, $c_i^2 = F_\varepsilon(2) = 1$, $c_i^3 = F_\varepsilon(3) = 2$, $c_i^4 = F_\varepsilon(4) = 4$, $c_i^5 = F_\varepsilon(5) = 6$, and $c_i^6 = F_\varepsilon(6) = 9$.

Suppose now that processor i 's input is 2^i , for $0 \leq i \leq 8$. Let $T^j = (T_0^j, T_1^j, \dots, T_8^j)$ and $S^j = (S_0^j, S_1^j, \dots, S_8^j)$. Initially,

$$T^0 = (1, 2, 4, 8, 16, 32, 64, 128, 256)$$

$$S^0 = (0, 0, 0, 0, 0, 0, 0, 0, 0).$$

We now follow the run of the algorithm round after round.

ROUND 1. Processor i sends T_i^0 to processor $i-1$ and S_i^0 to processor $i-2$. Processor i receives no messages. Therefore,

$$T^1 = (1, 2, 4, 8, 16, 32, 64, 128, 256)$$

$$S^1 = (0, 0, 0, 0, 0, 0, 0, 0, 0).$$

ROUND 2. Processor i sends T_i^1 to processors $i-2$ and $i-3$. Processor i receives no messages. Therefore,

$$T^2 = (1, 2, 4, 8, 16, 32, 64, 128, 256)$$

$$S^2 = (0, 0, 0, 0, 0, 0, 0, 0, 0).$$

ROUND 3. Processor i sends T_i^2 to processors $i-4$ and $i-5$. Processor i receives T_{i+1}^0 and S_{i+2}^0 . Therefore,

$$T^3 = (3, 6, 12, 24, 48, 96, 192, 384, 257)$$

$$S^3 = (2, 4, 8, 16, 32, 64, 128, 256, 1).$$

ROUND 4. Processor i sends T_i^3 to processors $i-6$ and S_i^3 to processor $i-7$. Processor i receives T_{i+2}^1 and T_{i+3}^1 . Therefore,

$$T^4 = (15, 30, 60, 120, 240, 480, 449, 387, 263)$$

$$S^4 = (14, 28, 56, 112, 224, 448, 385, 259, 7).$$

ROUND 5. Processor i sends no messages. Processor i receives T_{i+4}^2 and T_{i+5}^2 . Therefore,

$$T^5 = (63, 126, 252, 504, 497, 483, 455, 399, 287)$$

$$S^5 = (62, 124, 248, 496, 483, 451, 391, 271, 31).$$

ROUND 6. Processor i sends no messages. Processor i receives T_{i+6}^3 and S_{i+7}^3 . Therefore,

$$T^6 = (511, 511, 511, 511, 511, 511, 511, 511, 511)$$

$$S^6 = (510, 509, 507, 503, 495, 479, 447, 383, 255).$$

Indeed, $D = d_0 + d_1 + \dots + d_8 = 1 + 2 + \dots + 256 = 511$.

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