Perturbative search for dead end CFTs

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Abstract

To explore the possibility of self-organized criticality, we look for CFTs without any relevant scalar deformations (a.k.a dead end CFTs) within powercounting renormalizable quantum field theories with weakly coupled Lagrangian description. In $d = 3$ dimensions, the only candidates are the pure (Abelian) gauge theories, which may be further deformed by Chern-Simons terms. In $d = 4$ dimensions, we show that there are infinitely many non-trivial candidates based on chiral gauge theories. From the three-loop beta functions, we compute the gap of scaling dimensions above the marginal value, and it may be as small as $\mathcal{O}(10^{-5})$ and robust under the perturbative expansions. These classes of candidates are very weakly coupled and our perturbative conclusion seems difficult to refute. Thus, the hypothesis that the non-trivial dead end CFTs do not exist is likely to be false in $d = 4$ dimensions.

1 Introduction

The rule of the game is as follows:

- We look for conformal field theories (CFTs) without any relevant scalar deformations. We name them dead end CFTs.

- We do not ask what will happen after introducing the relevant deformations (if any).

- We do not impose any continuous global symmetries nor discrete global/gauge symmetries.[1]

- We assume dead end CFTs are unitary, causal, and have finite energy-momentum tensor.[2]

- Deformations must be physical. In gauge theories, they must be BRST invariant.[3]

- (Optional) In our paper, we assume the gravitational anomaly does not exist.

- (Optional) In our paper, we only discuss powercounting renormalzable weakly coupled Lagrangian field theories.[4]

Let's play!

1.1 Physical background of the game

This game is designed to understand a possibility of self-organized criticality [1] (see e.g. [2] for a review) in quantum field theories. In many statistical systems, it is typically the case that in order to obtain the criticality, we have to tune at least one parameter of the system (e.g. temperature). It is interesting to see if we can construct a self-tuning model so that the criticality is automatically attained by just making the size of the system larger without tuning anything else. A naive guess is unless we use some symmetry principles

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[1]Otherwise, we have scalars with a shift symmetry or fermions with a (discrete) chiral symmetry as trivial examples.

[2]Otherwise, generalized free CFTs are trivial examples.

[3]Otherwise, the ghost mass terms or gauge non-invariant mass terms give unphysical relevant defromations.

[4]To the author's knowledge, there is no known example beyond perturbation theory arguably except for AdS/CFT inspired ones. See section 4.

(e.g. Nambu-Goldstone mechanism or anomaly cancellation mechanism) generic gapless systems are unstable, and the self-organized criticality is difficult to achieve.

However, we know one example in our nature. The theory of photon. The theory of photon is always at its criticality and we cannot detune the theory to make it gapped unless we introduce extra light matter fields (like Higgs mechanism). The fact that it is always at its criticality led Einstein to the discovery of the special relativity. The speed of light is absolute. It is hard to imagine if he could have come up with various gedanken experiments if the photon is massive and the propagation of light is not critical. The criticality of the photon is not protected by any global symmetry. It is the intrinsic nature of the Maxwell theory that does not allow any relevant deformations.[5] It is an example of dead end CFTs.

Is this just a peculiar coincidence or deep feature of our particle physics in the particular space-time dimensions of four?

Putting the philosophical questions aside, one technical reason we are interested in the (non-)existence of the dead end CFT is whether we may regularize various infrared singularities in "S-matrix" of the CFTs. Strictly speaking, the S-matrix does not exist in CFTs, but once they are deformed to a massive/gapped/topological phase, the concept makes sense. Indeed, the clever use of the (regularized) S-matrix and its analyticity properties has led to many important results in quantum field theories such as the proof of a-theorem in $d = 4$ dimensions [3], enhancement of conformal invariance from scale invariance [4][5], the convexity properties of large twist operators in general CFTs [6], and so on.

For example, one crucial point in the argument of enhancement of conformal invariance from scale invariance is as follows. If the theory were scale invariant but not conformal invariant, the argument in [4][5] suggests that the "c"-function (or "a"-function in $d = 4$ dimensions) would be decreasing forever along the RG flow. However, *if* the theory can be deformed to the massive/gapped/topological phase, the central charge is bounded $c \geq 0$ (or $a \geq 0$), and hence it is in contradiction. The argument does not apply if the theory under consideration is a dead end CFT or a dead end scale invariant field theory [7].

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[5]In the BRST quantization, one may regard photon as a Nambu-Goldstone boson for the residual gauge symmetry $\delta A_\mu = a_\mu$. Since there is no way to break this symmetry in a physical manner, this fact is not important for our discussions. See also footnote 3.

In this paper, we look for candidates of dead end CFTs within powercounting renormalizable quantum field theories with weakly coupled Lagrangian description. Of course, it is desirable to give a non-perturbative argument that does not rely on perturbation theory or Lagrangian descriptions. It is, however, sufficient to give a perturbative example if we would like to disprove the claim that the dead end CFTs do not exist. We will give some further thoughts on the non-perturbative aspects of the game in section 4.

2   No non-trivial candidates in $d = 3$

We begin with the matter content of the renormalizable quantum field theories with weakly coupled Lagrangian description in $d = 3$ dimensions. It consists of a certain number of bosonic spin zero scalar fields and fermionic spin half spinor fields charged under gauge groups that have suitable kinetic terms. In search for dead end CFTs, we would like to first ask under which conditions, these Lagrangian theories may or may not admit the Lagrangian mass terms.

We can always take the real basis for the scalar fields ϕ^I, and they transform as real (linear) representations under the gauge groups. The existence of the kinetic term means that there exists a positive bilinear form g_{IJ} so that the kinetic term $g_{IJ}D_\mu\phi^I D^\mu\phi^J$ is gauge invariant and non-degenerate. One can use the same bilinear form to construct the gauge invariant mass term for the scalars as $g_{IJ}\phi^I\phi^J$. These mass terms are relevant deformations with the powercounting scaling dimension $\Delta = 1$.

In a similar way, we can always take the Majorana basis for the fermionic spinor fields ψ^a in $d = 3$ dimensions, and they transform as real representations under the gauge groups. Again, the existence of the kinetic term means that there exists a positive bilinear form g_{ab} so that the kinetic term $g_{ab}\bar{\psi}^a\gamma^\mu D_\mu\psi^b$ is gauge invariant and non-degenerate. As is the scalar case, one can use the same bilinear form to construct the gauge invariant Majorana mass term (or real mass term) for the fermions as $g_{ab}\bar{\psi}^a\psi^b$. These are relevant deformations with the powercounting scaling dimension $\Delta = 2$.

Therefore, in $d = 3$ dimensions, gauge theories with any matter may admit relevant deformations as mass terms irrespective of their representations under the gauge groups as long as the kinetic terms exist. Within the weakly coupled Lagrangian description, they cannot be candidates for dead end CFTs because the mass terms are always (per-

turbatively) relevant.

The only remaining possibilities are pure gauge theories. There are two possible choices of the kinetic terms, i.e. Yang-Mills kinetic terms or Chern-Simons terms. The latter is a little subtle in our discussions because the action density is not gauge invariant. In any way, the introduction of the latter makes the theory topological in the infrared so they are not candidates of the dead end CFTs. As for the Yang-Mills kinetic terms, in $d = 3$ dimensions, we believe that gauge theories with non-Abelian gauge groups will confine with a mass gap (although we do not know the rigorous proof), so the infrared theory is massive and they are not candidates of the dead end CFTs. Therefore, the only candidates we have is the pure Abelian gauge theories with the Maxwell type action. It is scale invariant but not manifestly conformal invariant (see e.g. [8][9]), so it may be better to call them dead end scale invariant field theories. Since the change of the gauge coupling constant is a marginal deformation, they do not possess any relevant scalar deformations as an integral of gauge invariant local operators, but we may add the Chern-Simons terms so that they become topological in the infrared.

To avoid misunderstanding, we would like to comment on the non-perturbative fixed point which was claimed to be an example of self-organized criticality in certain spin liquid systems in $d = 1 + 2$ dimensions (see e.g. [10][11] and references therein) with emergent Lorentz invariance. The effective field theories describing such spin liquids are given by (emergent) $U(1)$ gauge theories coupled with N_f Dirac fermions (in the above Majorana basis we have used, a real vector representation of $O(2)$ gauge symmetry). In the large N_f limit, the theory is supposed to be conformal invariant in the infrared. The crucial claim here is that all the relevant deformations such as fermion mass terms are forbidden by global symmetries such as N_f flavor symmetries, parity, and time-reversal. While physically relevant, we do not consider them as an example of our dead end CFTs because it violates the third rule of the game.[6]

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[6]Indeed, we can see that the introduction of the symmetry principle makes the better-known Banks-Zaks fixed point in $d = 4$ dimensions as an example of self-organized criticality. Given so many examples, there is less interest in pursuing such possibilities from the purely theoretical (zoological) viewpoint.

3   Non-trivial candidates in $d = 4$

We have seen in $d = 3$ dimensions, there are no non-trivial candidates of the dead end CFTs within the weakly coupled Lagrangian description. The situation is drastically different in $d = 4$ dimensions because mass terms of the fermions can be forbidden without using any global symmetries.

We start with the field contents. In renormalizable field theories in $d = 4$ dimensions with weakly coupled Lagrangian description, we have bosonic spin zero scalar fields and fermionic spin half spinor fields charged under the gauge groups with finite kinetic terms. The argument for the scalars is the same as in $d = 3$ dimensions. We can always take the real basis for the scalar fields ϕ^I, and they transform as real (linear) representations of the gauge groups. The existence of the kinetic term means that there exists a positive bilinear form g_{IJ} so that the kinetic term $g_{IJ} D_\mu \phi^I D^\mu \phi^J$ is gauge invariant and non-degenerate. One can use the same bilinear form to construct the gauge invariant mass term for the scalars as $g_{IJ} \phi^I \phi^J$. These are relevant deformations with the powercounting scaling dimension $\Delta = 2$.

However, the situation is different in spinors. We can take the Weyl basis of the fermions ψ^a so that the representations of the gauge group are complex in general. The complex conjugate $\bar{\psi}_a$ (with the opposite chirality) transforms under the complex conjugate representations of ψ_a. The existence of the Weyl kinetic term means that there exists a Hermitian bilinear form $\delta^a_{\ b}$ so that the kinetic term $\delta^a_{\ b} \bar{\psi}_a \sigma^\mu D_\mu \psi^b$ is gauge invariant and non-degenerate. The crucial difference here is that unlike in $d = 3$ dimensions, we cannot use the bilinear form $\delta^a_{\ b}$ to construct the Lorentz invariant mass term because $\bar{\psi}_a$ and ψ^a have different chiralities. The gauge theories with Weyl fermions in non-real representations are called chiral gauge theories and since they do not (always) possess the mass deformations, they are good candidates for the dead end CFTs.

Not every chiral gauge theories are consistent. They may suffer gauge anomaly. The anomaly cancellation conditions are well-known. For each gauge group, we require

$$\sum_F \text{Tr}(R_F^a \{R_F^b, R_F^c\}) = 0 \ , \tag{1}$$

where R_F is the representation matrix and the sum is taken over all the Weyl fermions. Note that the condition is linear in the matter representation, so we can add the anomaly free matter combinations and still it is anomaly free. We only focus on the anomaly free

gauge theories.

Extreme examples are pure gauge theories. We do not have any matter at all, and we cannot add any mass terms to the gauge bosons by hand. However, we believe that the non-Abelian gauge theories in $d = 4$ dimensions will confine and show the mass gap. Therefore they are not candidates of the dead end CFTs. On the other hand, the pure Abelian gauge theories are perfectly good examples of dead end CFTs. The gauge coupling constant is a marginal deformation and they do not possess any relevant deformations at all. Indeed, we know that our standard model ends up with the free Maxwell field theory in the far infrared, and it is a dead end CFT! Are there any other non-trivial examples? This is what we want to persue in the rest of this section.

Given the above discussions, the non-trivial candidates we have in mind are anomaly free chiral gauge theories without any scalar fields. Classically these candidates are all conformal invariant and the gauge coupling constants are marginal. The renormalization makes the gauge coupling constants run, and the question is whether we may find the non-trivial zero of the beta functions of the gauge coupling constants. The answer depends on the details of the gauge groups and representations of fermions. If these fixed points are infrared stable, all the gauge coupling constants are irrelevant, and we do not have any relevant deformations at the fixed point. These fixed points are dead end CFTs.[7] This leads to the question of conformal windows in chiral gauge theories. We will not try to determine the boundary of the conformal windows, but our strategy is to find the infinitely many examples of non-trivial zeros of the beta functions in which the purturbative computation of the beta functions (up to three loop order in this paper) is reliable.

One comment on the renormalizability is in order. One may ask if our chiral gauge theories we will discuss are really renormalizbale. At least within the power-counting renormalization, they are proved to be renormalizble, and certainly we are able to compute the physical observables in these CFTs at the three loop order we study. After all, our examples will turn out to be no more exotic than the standard model as chiral gauge theories, and if we doubt their renormalizabilities (or realizabilities in nature), we should ask the same question to the standard model. See e.g. [12] and references therein for

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[7]Within perturbation theory, there is no candidates for the Virial current, so the fixed point is conformal invariant rather than merely scale invariant (see e.g. [7] and reference therein for more details).

further discussions on the non-perturbative renormalizabilities.

3.1 Simple quiver type chiral gauge theories

The easiest way to solve the anomaly free condition is to study the quiver-type gauge theories of $SU(N_c)^K$. The matter Weyl fermions are in bifundamental representations of adjacent gauge groups and represented by arrows. When the number of incoming arrows and outgoing arrows are the same at each node that represents a simple gauge group, the theory is anomaly free. In order to forbid the fermion mass term, it is sufficient to make the directions of the arrows only one way between any pair of nodes.

For simplicity, we focus on the circular quiver gauge theories of $SU(N_c)^K$ with N_f generations of bifundamental Weyl fermions[8]:

$$\cdots \xrightarrow{\times N_f} SU(N_c)_1 \xrightarrow{\times N_f} SU(N_c)_2 \xrightarrow{\times N_f} \cdots \xrightarrow{\times N_f} SU(N_c)_K \xrightarrow{\times N_f} SU(N_c)_1 \xrightarrow{\times N_f} \cdots \tag{2}$$

The beta functions of the system can be computed up to three loops by using the recent results reviewed in Appendix. The three-loop beta functions in the Modified Minimal Subtraction scheme are given by

$$
\begin{aligned}
\beta_i =\ & \frac{g_i^3}{(4\pi)^2}\left[-\frac{11}{3}N_c + \frac{2}{3}N_f N_c\right] \\
& + \frac{g_i^3}{(4\pi)^2}\left[\frac{g_i^2}{(4\pi)^2}\left\{-\frac{34}{3}N_c^2 + N_c N_f\left(\frac{10}{3}N_c + \frac{N_c^2-1}{N_c}\right)\right\}\right. \\
& \left. + \left(\frac{g_{i-1}^2}{(4\pi)^2} + \frac{g_{i+1}^2}{(4\pi)^2}\right)N_c N_f\frac{N_c^2-1}{2N_c}\right] \\
& + \frac{g_i^5}{(4\pi)^4}\left[\frac{g_i^2}{(4\pi)^2}\left\{-\frac{2857}{54}N_c^3 + N_c N_f\left(\frac{1415}{54}N_c^2 + \frac{205}{18}N_c\frac{N_c^2-1}{2N_c} - \left(\frac{N_c^2-1}{2N_c}\right)^2\right)\right.\right. \\
& \left. - N_c^2 N_f^2\left(\frac{79}{54}N_c + \frac{11}{9}\frac{N_c^2-1}{2N_c}\right)\right\} \\
& \left. + \left(\frac{g_{i-1}^2}{(4\pi)^2} + \frac{g_{i+1}^2}{(4\pi)^2}\right)N_f 2\left(2N_c - \frac{N_c^2-1}{2N_c}\right)\frac{N_c^2-1}{2N_c}\right] \\
& + \frac{g_i^3}{(4\pi)^2}\left[\left(\frac{g_{i-1}^4}{(4\pi)^4} + \frac{g_{i-1}^4}{(4\pi)^4}\right)\left\{N_f\left(\frac{133}{18}N_c - \frac{N_c^2-1}{2N_c}\right)\frac{N_c^2-1}{2N_c}\frac{1}{2}\right.\right. \\
& \left.\left. -2N_f^2\frac{11}{9}\frac{N_c^2-1}{2N_c}\frac{1}{2}\frac{1}{2}N_c\right\}\right].
\end{aligned}
\tag{3}
$$

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[8]The conformal window of the model was also discussed in [13].

for each gauge coupling constant g_i ($i = 1, 2 \cdots, K$). The asymptotic freedom requires $N_f < 5.5$.[9] In order to obtain the weakly coupled fixed point, it is desirable that N_f is close to the upper boundary of the asymptotic freedom limit, so our main focus will be $N_f = 5$.

We look for the zeros of the beta functions. When $N_f^* < N_f < 5.5$ with a certain critical number N_f^*, the zeros of the beta functions correspond to infrared stable fixed points, and we find good candidates of dead end CFTs. Once we find the zero of the beta functions, one may compute the anomalous dimensions of the field strength operators $\mathrm{Tr}_i(F_{\mu\nu}F^{\mu\nu})$ from the Hessian matrix $\partial_i \beta^j|_{g_i = g_i^*}$. Up to three-loop orders, the beta functions of the gauge coupling constants actually do not depend on the number of nodes K in the quiver. This is because we need at least K numbers of fermion loops to obtain the non-trivial K dependence in the beta functions. On the other hand, the anomalous dimensions of the field strength do depend on K because we have to diagonalize the $K \times K$ Hessian matrix.

In principle, we also need to study the CP odd operators $\mathrm{Tr}_i(\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma})$ with their coupling constants θ_i as theta terms beyond the perturbation theory. Actually, $K - 1$ out of K theta terms are redundant operators in this theory because they can get removed by the (anomalous) phase rotations of Weyl fermions. The overall theta term, however, may be non-trivial. In perturbation theory nothing depends on its value. We do not know if the theta term is non-perturbatively renormalized or it will affect the beta functions. In any way, if we have an infrared fixed point, the anomalous dimension must be positive and our discussions are still valid. In the other examples that we discuss in later subsections, all the theta terms are redundant operators in the action.

In our perturbative search, we may set $g_1 = g_2 = \cdots = g_K$. We find that the other fixed points make some of the gauge coupling constants vanish, so we end up with effectively decomposed non-circular quivers. We have listed the two-loop and three-loop anomalous dimensions of the permutation symmetric field strength $\sum_i \mathrm{Tr}_i F_{\mu\nu}F^{\mu\nu}$ for smaller values of N_c in table 1. They do not depend on the number of nodes K. The anomalous dimensions of permutation non-symmetric field strength do depend on K, and for example, $N_c = 3$, $N_f = 5$, $K = 3$, we have the eigenvalues

$$(0.0155313, 0.00919003, 0.00919003) \tag{4}$$

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[9]We may relax the condition of the asymptotic freedom, but in practice, we cannot find any additional weakly coupled fixed point by relaxing the condition.

at the three-loop order. For $N_c = 3$, $N_f = 5$, $K = 4$, we have the eigenvalues

$$(0.0155313, 0.0113038, 0.0113038, 0.00707626) \,. \tag{5}$$

For $N_c = 3$, $N_f = 5$, $K = 5$, we have the eigenvalues

$$(0.0155313, 0.0126102, 0.0126102, 0.00788365, 0.00788365) \,, \tag{6}$$

and so on. In every cases, all the eigenvalues are positive, meaning that the fixed points are infrared stable.

N_c	N_f	2-loop	3-loop
3	5	0.01563	0.0155313
5	5	0.01488	0.0148063
3	4	0.220	0.203393
5	4	0.207	0.193566
3	3	1.39	0.978207
5	3	1.26	0.930279

Table 1: The anomalous dimension of the permutation symmetric field strength of the $SU(N_c)$ chiral quiver gauge theories with N_f generations of bifundamental Weyl fermions. Each entry has an additional integer label $K \geq 3$.

Although the beta functions are renormalization group scheme dependent, the anomalous dimensions at the fixed point are physical quantities and they do not depend on the choice of the renomalization scheme. Also note that the smallness of the coupling constant g_i at the fixed point itself is not that important because the physical expansion parameters may be different (e.g. t' Hooft coupling $g_i^2 N_c$ may be more relevant). The ratio between the two-loop predictions and the three-loop predictions may be regarded as a good barometer how the perturbation theory is reliable or not (assuming there is no accidental cancellation).

It turns out that in all the examples we have studied, the three-loop predictions actually make anomalous dimensions smaller than the two-loop predictions. We find that the loop expansion is not terribly bad for the anomalous dimensions of permutation symmetric field strength for $N_f = 5$, which is at the percent order. For comparison, we

show that the two-loop and three-loop predictions of the Banks-Zaks fixed point [14][15] of $SU(N_c)$ gauge theory with n_f Dirac fermions in fundamental representation in table 2.

In comparison with the Banks-Zaks theories, we realize that the structure of the beta functions of our chiral quiver with N_f generations of Weyl fermions in bifundamental representation is more or less similar to that of the Banks-Zaks theory with $n_f = N_f N_c$ Dirac fermions in fundamental representation. The only difference at the two-loop level is that we have twice more contributions to the wave-function renormalization factors of fermions, which makes the fixed point coupling smaller in our chiral quiver gauge theories than in the Banks-Zaks fixed point. It is generically believed that the Banks-Zaks theory with $n_f = 5N_c$ Dirac fermions in fundamental representations are well in the conformal window, so it makes it plausible (although it does not prove) that our chiral quiver gauge theories with $N_f = 5$ for any N_c are in the conformal window as well. If this is the case, we would have infinitely many classes of dead end CFTs labelled by N_c and K. The K dependence on the anomalous dimensions are very small, but we recall that the number of these slightly irrelevant deformations (gauge kinetic terms) are given by K and the operator contents are different.

N_c	n_f	2-loop	3-loop
3	16	0.0022075	0.00220301
3	15	0.02272	0.022307
3	12	0.36	0.296
5	27	0.0007501	0.000749578
5	25	0.02192	0.021558
5	20	0.34	0.285

Table 2: The anomalous dimension of the field strength in Banks-Zaks fixed point with n_f Dirac fermions in fundamental representation.

The lower values of N_f may admit more strongly coupled dead end CFTs. For example, $N_f = 4$ generations of $SU(3)$ chiral quiver gauge theories may be compared with $SU(3)$ Banks-Zaks theory with $n_f = 12$ Dirac fermions in fundamental representation. The recent lattice simulations seem to more or less agree that the latter is indeed in the conformal window (see e.g. [16][17][18][19][20] and reference therein), and it may suggest

that the chiral quiver gauge theory with $N_f = 4$ is also in the conformal window. Even $N_f = 3$ generations of $SU(3)$ chiral quiver gauge theory can be compared with the $SU(3)$ Banks-Zaks theory with $n_f = 9$ Dirac fermions in fundamental representation and it may possess the fixed point (with some controversies in the lattice simulations). The analysis based on the existence of topological excitations in [13], however, predicts (but does not prove) that the chiral quiver gauge theories have a smaller conformal window than in the vector like models of Banks-Zaks theories, and $N_f = 4$ might have been already excluded. It would be interesting to settle the conformal window, but this is not the main scope of our paper. We only attempt to offer the existence proof of dead-end CFTs so we are more interested in the weakly coupled fixed points. With this respect, we have no (known) arguments against that $N_f = 5$ in chiral quiver gauge theories do not possess the infrared fixed point.

3.2   Anomaly free chiral matters

A more non-trivial way to obtain the anomaly free chiral gauge theories is to use the cancellation among various matter representations of Weyl fermions in gauge/gravitational anomalies. A particularly well-known matter combinations that cancel the anomaly in $SU(N_c)$ gauge group is generalized Georgi-Glashow model of one anti-symmetric representation and $N_c - 4$ anti-fundamental representation [21], and the generalized Bars-Yankielowicz model of one symmetric representation and $N_c + 4$ anti-fundamental representations [22].

We may generically consider N_a generations of generalized Georgi-Glashow model and N_s generations of generalized Bars-Yankielowicz model. In this subsection, we focus on the single gauge group and we will discuss the quiver generalization in the next section. We remark here that $N_s = 3$ model is the $SU(5)$ grand unified model of our standard model (without Higgs fields). In fact, all these chiral gauge theories are introduced in the model of our particle physics.

From the formula in Appendix, the three-loop beta functions are computed as

$$\beta = \frac{g^3}{(4\pi)^2}\left[-\frac{11}{3}N_c + \frac{2}{3}N_a(N_c-3) + \frac{2}{3}N_s(N_c+3)\right]$$

$$+ \frac{g^5}{(4\pi)^4}\left[-\frac{34}{3}N_c^2 + \right.$$

$$+ N_a\left\{\left(\frac{10}{3}N_c + 2\frac{(N_c+1)(N_c-2)}{N_c}\right)\frac{N_c-2}{2} + (N_c-4)\left(\frac{10}{3}N_c + 2\frac{N_c^2-1}{2N_c}\right)\frac{1}{2}\right\}$$

$$+ N_s\left\{\left(\frac{10}{3}N_c + 2\frac{(N_c-1)(N_c+2)}{N_c}\right)\frac{N_c+2}{2} + (N_c+4)\left(\frac{10}{3}N_c + 2\frac{N_c^2-1}{2N_c}\right)\frac{1}{2}\right\}\right]$$

$$+ \frac{g^7}{(4\pi)^6}\left[-\frac{2857}{54}N_c^3\right.$$

$$+ N_a\left\{\frac{1415}{54}N_c^2 + \frac{205}{18}N_c\frac{(N_c+1)(N_c-2)}{N_c} - \left(\frac{(N_c+1)(N_c-2)}{N_c}\right)^2\right\}\frac{N_c-2}{2}$$

$$+ N_s\left\{\frac{1415}{54}N_c^2 + \frac{205}{18}N_c\frac{(N_c-1)(N_c+2)}{N_c} - \left(\frac{(N_c-1)(N_c+2)}{N_c}\right)^2\right\}\frac{N_c+2}{2}$$

$$+ (N_a(N_c-4) + N_s(N_c+4))\left(\frac{1415}{54}N_c^2 + \frac{205}{18}N_c\frac{N_c^2-1}{2N_c} - \left(\frac{N_c^2-1}{2N_c}\right)^2\right)\frac{1}{2}$$

$$- N_a^2\left(\frac{79}{54}N_c + \frac{11}{9}\frac{(N_c+1)(N_c-2)}{N_c}\right)\left(\frac{N_c-2}{2}\right)^2$$

$$- N_aN_s\left(\frac{79}{54}N_c + \frac{11}{9}\frac{(N_c+1)(N_c-2)}{N_c}\right)\left(\frac{(N_c-2)}{2}\frac{(N_c+2)}{2}\right)$$

$$- N_a(N_a(N_c-4) + N_s(N_c+4))\left(\frac{79}{54}N_c + \frac{11}{9}\frac{(N_c+1)(N_c-2)}{N_c}\right)\left(\frac{N_c-2}{2}\frac{1}{2}\right)$$

$$- N_sN_a\left(\frac{79}{54}N_c + \frac{11}{9}\frac{(N_c-1)(N_c+2)}{N_c}\right)\left(\frac{(N_c-2)}{2}\frac{(N_c+2)}{2}\right)$$

$$- N_s^2\left(\frac{79}{54}N_c + \frac{11}{9}\frac{(N_c-1)(N_c+2)}{N_c}\right)\left(\frac{N_c+2}{2}\right)^2$$

$$- N_s(N_a(N_c-4) + N_s(N_c+4))\left(\frac{79}{54}N_c + \frac{11}{9}\frac{(N_c-1)(N_c+2)}{N_c}\right)\left(\frac{N_c+2}{2}\frac{1}{2}\right)$$

$$- (N_a(N_c-4) + N_s(N_c+4))N_a\left(\frac{79}{54}N_c + \frac{11}{9}\frac{N_c^2-1}{2N_c}\right)\left(\frac{N_c-2}{2}\frac{1}{2}\right)$$

$$- (N_a(N_c-4) + N_s(N_c+4))N_s\left(\frac{79}{54}N_c + \frac{11}{9}\frac{N_c^2-1}{2N_c}\right)\left(\frac{N_c+2}{2}\frac{1}{2}\right)$$

$$\left.- (N_a(N_c-4) + N_s(N_c+4))^2\left(\frac{79}{54}N_c + \frac{11}{9}\frac{N_c^2-1}{2N_c}\right)\left(\frac{1}{2}\frac{1}{2}\right)\right] . \tag{7}$$

As mentioned, the theta term in these models is redundant, so we only have to consider the non-trivial zero of the gauge coupling constant.

We can now play the game of finding very weakly coupled fixed points by changing N_c, N_a and N_s.[10] For example, when $N_c = 5$, we present the most weakly coupled fixed point given a fixed value of N_s together with the anomalous dimension of field strength in table 3. We see that they are more weakly coupled than the $SU(3)$ Banks-Zaks fixed point with $n_f = 15$ Dirac fermions in fundamental representation (see table 2). We also see that the difference between the two-loop prediction and the three-loop prediction is order of percent and the perturbation theory seems fairly reliable.

N_c	N_s	N_a	2-loop	3-loop
5	0	13	0.00622	0.006194
5	1	9	0.00607	0.006046
5	2	5	0.00592	0.005904
5	3	1	0.00579	0.0057688

Table 3: The anomalous dimension of the field strength in weakly coupled chiral $SU(5)$ gauge theories with N_s generations of Bars-Yankielowicz model and N_a generations of Georgi-Glashow model.

We may further investigate much more weakly coupled fixed points. In table 4, we show the available very weakly coupled fixed points under the criterion that the anomalous dimension of the field strength is smaller than that of $SU(3)$ Banks-Zaks fixed point with $n_f = 16$ Dirac fermions in fundamental representation. For another reference, we also note that the anomalous dimension of the field strength of photons in QED at the scale of electron mass is $\partial_\alpha \beta_\alpha|_{\alpha=\frac{1}{137}} = \frac{4\alpha}{3\pi}|_{\alpha=\frac{1}{137}} + O(\alpha^2) \sim 0.003$ and comparable.

We can see that some of these examples such as $SU(35)$ with $N_s = 0$, $N_f = 6$ are extremely weakly coupled. Their anomalous dimensions are 10^{-2} times smaller than that of QED and so are their loop corrections. It is hard to imagine that the conclusion that these models have non-trivial conformal fixed points will be refuted by any other methods. Since the loop suppression is very large, we do not have to worry about the scheme dependence of the beta function at the higher loop order, either.

---

[10]We can find the two-loop discussions with $N_s N_a = 0$ case in [23]. When $N_s N_a = 0$, [13] also gives the estimate of the conformal window from the topological excitations. The latter claims that theirs is the first estimate of the conformal window of these models. Apparently, the existence of non-trivial conformal fixed points in chiral gauge theories have been much less studied in the literature.

N_c	N_s	N_a	2-loop	3-loop
9	0	8	0.001762	0.00175977
9	1	6	0.0017432	0.0017414
13	0	7	0.0008206	0.00082021
11	2	4	0.000123063	0.000123055
23	1	5	2.79044×10^{-5}	2.79039×10^{-5}
35	0	6	1.20187×10^{-5}	1.20186×10^{-5}
7	3	2	0.00030422	0.000304164
19	4	1	4.05191×10^{-5}	4.05182×10^{-5}
31	5	0	1.51642×10^{-5}	1.51641×10^{-5}
9	4	0	0.0016901	0.00168861

Table 4: Examples of extremely small anomalous dimension of the field strength in $SU(N_c)$ chiral gauge theories with N_s generations of Bars-Yankielowicz model and N_a generations of Georgi-Glashow model.

3.3 Quiver with external matter

One may wonder if the extremely weakly coupled examples presented in subsection 3.2 are isolated exotic examples. Unlike the Banks-Zaks fixed point with Dirac fermions in fundamental representation, there is no Veneziano limit [24] that produces infinitely many arbitrarily weakly coupled fixed points in a controllable manner. Nevertheless, we would like to show that there exist infinitely many such examples of (numerically) very weakly coupled dead end CFTs from combining the quiver constructions in section 3.1 and the non-trivial chiral matter in section 3.2.

We study $SU(N_c)^K$ quiver gauge theories with N_f generations of Weyl fermions (arrows between nodes) in bifundamental representations. Again, for simplicity, we consider the circular quiver. In addition, at each nodes we add N_a copies of generalized Georgi-

Glashow model and N_s copies of generalized Bars-Yankielowicz model.

$$
\begin{array}{ccccc}
N_a\text{GG} & & N_a\text{GG} & & N_a\text{GG} \\
\uparrow & & \uparrow & & \uparrow \\
\cdots \xrightarrow{\times N_f} SU(N_c)_K \xrightarrow{\times N_f} SU(N_c)_1 \xrightarrow{\times N_f} SU(N_c)_2 \xrightarrow{\times N_f} \cdots \\
\downarrow & & \downarrow & & \downarrow \\
N_s\text{BY} & & N_s\text{BY} & & N_s\text{BY}
\end{array}
\tag{8}
$$

The model is chiral and dose not admit any mass term.

The two-loop beta functions at each node is given by

$$
\begin{aligned}
\beta_i = {} & \frac{g_i^3}{(4\pi)^2}\left[-\frac{11}{3}N_c + \frac{2N_a}{3}(N_c - 3) + \frac{2N_s}{3}(N_c + 3) + \frac{2}{3}N_f N_c\right] \\
& + \frac{g_i^3}{(4\pi)^2}\left[\frac{g_i^2}{(4\pi)^2}\left\{-\frac{34}{3}N_c^2 + N_c N_f\left(\frac{10}{3}N_c + \frac{N_c^2 - 1}{N_c}\right)\right.\right. \\
& + N_a\left\{\left(\frac{10}{3}N_c + 2\frac{(N_c+1)(N_c-2)}{N_c}\right)\frac{N_c-2}{2} + (N_c-4)\left(\frac{10}{3}N_c + 2\frac{N_c^2-1}{2N_c}\right)\frac{1}{2}\right\} \\
& + N_s\left\{\left(\frac{10}{3}N_c + 2\frac{(N_c-1)(N_c+2)}{N_c}\right)\frac{N_c+2}{2} + (N_c+4)\left(\frac{10}{3}N_c + 2\frac{N_c^2-1}{2N_c}\right)\frac{1}{2}\right\}\right\} \\
& + \left.\left(\frac{g_{i-1}^2}{(4\pi)^2} + \frac{g_{i+1}^2}{(4\pi)^2}\right)N_c N_f \frac{N_c^2-1}{2N_c}\right].
\end{aligned}
\tag{9}
$$

We do not write down the three-loop terms here, which would not fit into one page length. One may derive them from the general formula in Appendix. As in section 3.1, there is no K dependence in the beta functions at the two (or three) loop level.

We look for non-trivial zeros of the beta functions by varying N_c, N_f, N_a and N_s. We present some examples of extremely weakly coupled fixed points together with the anomalous dimension of the permutation symmetric field strength in table 5. All these examples are good candidates of dead end CFTs. In particular, for each values of N_c, N_f, N_a and N_s listed there, we can choose the number of nodes K in the quiver arbitrarily, so each on the list will give us infinitely many examples of extremely weakly coupled dead end CFTs. It seems difficult to refute the existence of these fixed points by any other methods.

4   Discussions

In this paper, we have looked for dead end CFTs in the perturbative regime. There is no such candidate in $d = 3$ dimensions, but there are infinitely many candidates in $d = 4$

N_c	N_f	N_s	N_a	2-loop	3-loop
29	1	0	5	1.69639×10^{-5}	1.69638×10^{-5}
25	1	4	0	2.26103×10^{-5}	2.26100×10^{-5}
17	1	1	4	4.95841×10^{-5}	4.95828×10^{-5}
19	2	3	0	3.80362×10^{-5}	3.80356×10^{-5}
23	2	0	4	2.61755×10^{-5}	2.61752×10^{-5}
17	3	0	3	4.66097×10^{-5}	4.66088×10^{-5}

Table 5: Examples of infinite series of very small anomalous dimension of the permutation symmetric field strength based on $SU(N_c)^K$ chiral quiver gauge theories with N_s generations of Bars-Yankielowicz model and N_a generations of Georgi-Glashow model. Each entry has an additional integer label $K \geq 3$.

dimensions. It would be interesting to see how far we can reach beyond perturbation theory. In $d = 2$ dimensions, there is an intriguing non-perturbative result reported in [25] based on the modular invariance. It was proved that in order to obtain the dead end CFTs, the total central charge $c + \bar{c}$ must be greater than 18.270. A further refinement of the argument and the lower bound may be found in [26][27]. In $d = 2$ dimensions, extremal CFTs (see e.g. [28] and reference therein), if they exist, give examples of dead end CFTs.

Such a bound from the central charge seems interesting in higher dimensions if any. We have found infinitely many candidates of dead end CFTs in $d = 4$ dimensions, but certainly, the construction based on chiral gauge theories required a large number of fields, and the infinite series we have found require more and more matter. We may conjecture that there is a lower bound on the central charge (say "a" that couples to the Euler number in trace anomaly) that is needed to construct dead end CFTs.[11]

We cannot resort to the modular invariance in higher dimensions, but the recent developments in conformal bootstrap may shed some light. In particular, the study of

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[11]Without extra conditions, the author believes that the lowest bound for the dead end CFT comes from the free $U(1)$ gauge theory. Unfortunately, we even do not know examples of non-free CFTs whose central charge is less than that of the free $U(1)$ gauge theory in $d = 4$ dimensions. To the author's knowledge, the only non-trivial candidate is the hypothetical CFT sitting at a kink of $\mathcal{N} = 1$ superconformal bootstrap discussed in [29]. It, however, possesses a relevant deformation.

the energy-momentum tensor correlation functions may help. We stress again that at least in $d = 4$ dimensions, we do have candidates of the dead end CFTs, so the naive search without any further assumptions should go nowhere. Of course, we may add the constraint on the central charges, and the game will become non-trivial. On the other hand, it is interesting to see what the conformal bootstrap tells us in $d = 3$ dimensions.[12]

Despite the failure of our perturbative search in $d = 3$ dimensions, the author believes that dead end CFTs will exist in $d = 3$ dimensions, at least in the "large central charge limit". This conviction comes from the AdS/CFT. It seems that there is nothing wrong with having classical gravity in the large AdS space-time without any massless or "tachyonic" matter in $d = 1 + 3$ dimensions. Indeed, if our universe had a tiny *negative* cosmological constant, the AdS/CFT dual of our universe would be a dead end CFT because all the scalar masses are much larger than the AdS scale. The conformal bootstrap should be satisfied in this regime, and we would not be able to exclude it from the conformal bootstrap analysis. Recent attempts to obtain string constructions with a large gap in spectrum may be found in [30][31].

We would like to end this paper with some variations of the game. Does an $\mathcal{N} = 4$ supersymmetric dead end CFT in $d = 4$ dimensions exist? The answer is no. The energy-momentum tensor multiplet always contain the dimension two scalar. Does an $\mathcal{N} = 2$ supersymmetric dead end CFT in $d = 4$ dimensions exist? The answer is no. The energy-momentum tensor multiplet always contain the dimension two scalar. How about $\mathcal{N} = 1$? At this point, the energy-momentum tensor multiplet does not contain a relevant scalar operator, so there is a chance that a dead end SCFT may exist. As a bonus such a theory does not possess any continuous global symmetries (except R-symmetry) because the current multiplet contains the dimension two scalar. However, in the Lagrangian description, one can always construct the relevant deformations such as gaugino mass for vector multiplets or scalar mass for chiral multiplets (see e.g. [32] for a related remark) so the construction should be non-perturbative. On the other hand, the pure (gauged) supergravity in $d = 1 + 4$ dimensions may couple to only heavy matter. Maybe the $\mathcal{N} = 1$

---

[12]The conformal bootstrap that we employ today cannot tell the difference between a free scalar and free Maxwell theory in $d = 3$ dimensions, so the Maxwell theory will be counted as a non dead end CFT. Thus we have no candidates at all for this purpose. In relation, we should note that it is hard to exclude the possibility that the global symmetry forbids the relevant deformations from the conformal bootstrap approach.

variation of the game is as interesting as the one we discussed in this paper.

Acknowledgements

A   Three-loop beta functions of general multiple gauge theories

In this appendix, we review the recent results of the three-loop beta functions for gauge coupling constants for general multiple gauge theories [33] (see also [34] for the single gauge group). We consider the direct product of simple gauge groups G_i with the gauge coupling constants g_i $(i = 1, \cdots, n)$. For a field transforming under the representation R of the gauge group G_i with the generators R^a in the matrix notation satisfying

$$[R^a, R^b] = i f^{abc} R^c \ , \tag{10}$$

we define Casimir invariants as

$$\mathrm{Tr}(R^a R^b) = \delta^{ab} T(R) \ ,$$
$$R^a R^a = \mathbf{1}_{d(R)} C(R),$$
$$f^{acd} f^{bcd} = \delta^{ab} C(G)$$
$$\delta^{aa} = d(G) \ . \tag{11}$$

The following identity holds

$$C(R) d(R) = T(R) d(G) \ , \tag{12}$$

18

where $d(R)$ is the dimension of the representation R and $d(G)$ is the dimension of the group.

Explicitly for $SU(N_c)$ group, we have

$$d(G) = N_c^2 - 1$$

$$C(G) = N_c$$

$$d(R) = \left(N_c, \frac{N_c(N_c+1)}{2}, \frac{N_c(N_c-1)}{2} \right) \quad \text{for} \quad R = (\bar{F}, S, AS) \ ,$$

$$T(R) = \left(\frac{1}{2}, \frac{N_c+2}{2}, \frac{N_c-2}{2} \right)$$

$$C(R) = \left(\frac{N_c^2-1}{2N_c}, \frac{(N_c-1)(N_c+2)}{N_c}, \frac{(N_c+1)(N_c-2)}{N_c} \right) \ . \tag{13}$$

In general, the matter Weyl fermion F is charged under multiple gauge groups. Following [33], we use the notation $d(F_i)$ to specify the dimensions of the representation R with respect to the gauge group G_i. Furthermore, we also define the multiplicity of a representation with respect to a subset of the original direct product of simple gauge group as

$$D(F_i) = \prod_{\substack{j \neq i \\ j=1}}^{n} d(F_j) \ , \quad D(F_{ij}) = \prod_{\substack{k \neq i,j \\ k=1}}^{n} d(F_k) \ , \quad D(F_{ijk}) = \prod_{\substack{l \neq i,j,k \\ l=1}}^{n} d(F_l) \ . \tag{14}$$

For generic multiple gauge theories with arbitrary representations of Weyl fermions, the three-loop beta function of the coupling constant g_i in the Modified Minimal Subtrac-

tion scheme is given by

$$
\begin{aligned}
\beta_i ={}& \frac{g_i^3}{(4\pi)^2}\left[-\frac{11}{3}C(G_i)+\sum_F \frac{2}{3}T(F_i)D(F_i)\right] \\
&+\frac{g_i^3}{(4\pi)^2}\left[\frac{g_i^2}{(4\pi)^2}\left\{-\frac{34}{3}C(G_i)^2+\sum_F\left(\frac{10}{3}C(G_i)+2C(F_i)\right)T(F_i)D(F_i)\right\}\right. \\
&\left.+\sum_{j\neq i}\frac{g_j^2}{(4\pi)^2}\sum_F 2C(F_j)d(F_j)T(F_i)D(F_{ij})\right] \\
&+\frac{g_i^5}{(4\pi)^4}\left[\frac{g_i^2}{(4\pi)^2}\left\{-\frac{2857}{54}C(G_i)^3+\sum_F\left(\frac{1415}{54}C(G_i)^2+\frac{205}{18}C(G_i)C(F_i)-C(F_i)^2\right)T(F_i)D(F_i)\right.\right. \\
&\left.-\sum_{F_m,F_n}\left(\frac{79}{54}C(G_i)+\frac{11}{9}C(F_{m,i})\right)T(F_{m,i})T(F_{n,i})D(F_{m,i})D(F_{n,i})\right\} \\
&\left.+\sum_{j\neq i}\frac{g_j^2}{(4\pi)^2}\sum_F 2\left(2(C(G_i)-C(F_i))T(F_i)C(F_j)D(F_{ij})\right)\right] \\
&+\frac{g_i^3}{(4\pi)^2}\left[\sum_{j\neq i}\frac{g_j^4}{(4\pi)^4}\left\{\sum_F\left(\frac{133}{18}C(G_j)-C(F_j)\right)C(F_j)T(F_i)D(F_{ij})\right.\right. \\
&\left.-\sum_{F_m,F_n}\frac{11}{9}C(F_{m,j})T(F_{n,j})T(F_{m,i})D(F_{m,ij})D(F_{n,j})\right\} \\
&\left.+\sum_{j\neq k\neq i}\frac{g_j^2}{(4\pi)^2}\frac{g_k^2}{(4\pi)^2}\left(-\sum_F C(F_j)C(F_k)T(F_i)D(F_{ijk})\right)\right]\,. \qquad (15)
\end{aligned}
$$

In our applications, there is no matter Weyl fermions that is charged under three different gauge groups, so the last line in (15) will be dropped. In [33], one may also find the additional contributions from scalars that we do not use in this paper.

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