Supplemental Material for: "Generating topological order: no speedup by dissipation"

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Encoding into topological codes

Here we consider the problem of transferring information from a few qubits into the code space $\mathcal{C} \subset (\mathbb{C}^2)^{\otimes n}$ of a topological code (i.e., the degenerate ground space of a Hamiltonian of the form (1) in the main text, defined on a qubit lattice with local stabilizers Π_a having support with diameter bounded by an 'interaction length' ξ and a code distance growing with L – see below). An encoder is a CPTPM $\mathcal{E}$ which converts 'simple' states into code states, i.e., states supported on $\mathcal{C}$. We will use $A = A_1 \cdots A_k \subset \Lambda$ to refer to the k-qubits that are being encoded (assuming that k is the number of logical qubits). Since the encoder $\mathcal{E}$ is supposed to take locally encoded information into the code space, we assume that the qubits in A are nearest neighbors (i.e., form a simply connected subset of the lattice). Independently of the logical information that is being encoded, we assume that the remaining $n-k$ qubits $A^c = \Lambda \backslash A$ are initially in a fixed product state $|\Phi\rangle = \bigotimes_{j=1}^{n-k} |\Phi_j\rangle \in (\mathbb{C}^2)^{\otimes(n-k)}$. This is intended to be a state which is easy to prepare (see [3]). In fact, neither the product form nor the fact that it is pure are essential for our lower bound on encoding time. We use this convention as it corresponds to a natural operational restriction.

Given this setup, the notion of a (perfect) encoder is particularly easy to define for unitary maps $U : (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^p)^{\otimes n}$. A unitary encoder U takes the subspace $\mathcal{C}_* := (\mathbb{C}^2)^{\otimes k} \otimes \mathbb{C}|\Phi\rangle \subset (\mathbb{C}^p)^{\otimes n}$ isomorphically to $\mathcal{C}$. For general physical maps, that is, completely positive trace-preserving maps (CPTPMs), we define (approximate) encoders similarly as follows.

Definition 1. *A CPTPM $\mathcal{E}$* encodes $\mathcal{C}_*$ into the code $\mathcal{C}$ with error ϵ *if $\|\mathcal{E}(\cdot) - U \cdot U^\dagger\|_{\mathcal{S}(\mathcal{C}_*)} \leq \epsilon$. Here U is an (arbitrary) unitary encoder for $\mathcal{C}$, and we use the norm $\|\mathcal{E}(\cdot)\|_{\mathcal{S}(\mathcal{C}_*)} := \max_{\rho \in \mathcal{B}_1^+(\mathcal{C}_*)} \|\mathcal{E}(\rho)\|_1$ obtained by maximizing over the set $\mathcal{B}_1^+(\mathcal{C}_*) = \{\rho : \rho \geq 0, \mathrm{tr}(\rho) = 1, \mathrm{supp}(\rho) \subset \mathcal{C}_*\}$ of normalized density matrices supported on $\mathcal{C}_*$.*

Note that using the given notion of distance in Definition 1 (instead of the diamond norm) strengthens our lower bound on the encoding time.

We provide a lower bound on the time required to encode information into a topological code $\mathcal{C}$. It is expressed in terms of the *distance* d of the code, i.e., the minimum weight of an Pauli operator O such that $POP \neq c_O P$ for any constant c_O, where P is the projection onto $\mathcal{C}$.

Theorem 2. *Let $\mathcal{C}$ be a topological code with distance d on a D-dimensional lattice. Assume that $\{\mathcal{E}^{(t)}\}_t$ is a family of CPTPMs satisfying a Lieb-Robinson bound. Assume further that for some $t > 0$, $\mathcal{E}^{(t)}$ encodes $\mathcal{C}_*$ into $\mathcal{C}$ with constant error $\epsilon \ll 1$. Then $t \geq \Omega(d^{1/(D-1)})$.*

Observe that this result only involves basic information-theoretic properties of the code, and confirms the qualitative intuition that encoding becomes harder with growing code distance.

Let us discuss the relationship between Theorem 2 and Theorem 1 in the main text. First observe that encoding is generally a more difficult task than the problem of state preparation: the goal is to be able to prepare arbitrary encoded states instead of (any) state supported on the code space. Furthermore, since the code distance of any topological code is bounded by [4] $d \leq O(L^{D-1})$, the bound $t \geq \Omega(d^{1/(D-1)})$ on the encoding time in Theorem 2 is generally less stringent than the $t \geq \Omega(L)$ bound of Theorem 1. Theorem 2 is therefore of main interest in cases where the assumptions of Theorem 1 are either not satisfied or not known to be.

Proof of Theorem 2. – The first step is to show that there exists a region B which supports a non-trivial logical operator and is distant from the region A containing the physical qubits to be encoded. To do so, we will use the notion of correctable regions introduced in [2] and corresponding 'cleaning' results of [1]. This proof also clarifies how the macroscopic code distance comes into play.

Recall that a subset of qubits $\Gamma \subset \Lambda$ is called correctable if the encoded information can be recovered even after losing all qubits in Γ, i.e., if there is a decoding CPTPM $\mathcal{D}$ such that $\mathcal{D} \circ \mathrm{tr}_\Gamma(\bar{\rho}) = \bar{\rho}$ for all encoded states $\bar{\rho} \in \mathcal{B}_1^+(\mathcal{C})$. A simple example is any set Γ containing fewer qubits than the code distance, that is, $|\Gamma| < d$. A much less trivial statement which

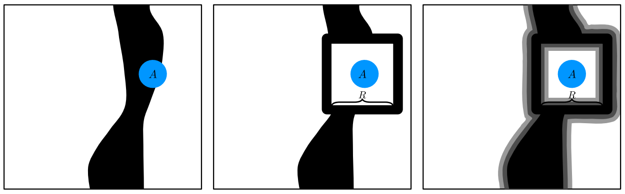

FIG. 1. This figure illustrates the proof of Theorem 2. Region A consists of the qubits carrying the logical information before the encoding. A logical operator may have support on this region, but can be cleaned out. Applying the Heisenberg evolution of the encoding map to the operator smears out its support. If the evolution is sufficiently local, the resulting operator still has no support in A and can therefore not distinguish unencoded states.

is shown and used in [1] is the following. Let us define the cube $\Gamma_R(v) \subset \Lambda$ for $v \in \Lambda$ as the rectangular block of size $R \times R \times \cdots \times R$, i.e., a D-dimensional cube of linear size R aligned with the coordinate axes and centered at a location v of the lattice (according to some convention).

Lemma 2 (see [1])**.** *Let $\mathcal{C}$ be a D-dimensional topological code with distance d and interaction length ξ. Then there is a constant $c = c(\xi) > 0$ such that all cubes $\Gamma_R(v)$, $v \in \Lambda$ with $R \leq cd^{\frac{1}{D-1}}$ are correctable.*

Correctable regions conveniently allow the application of the cleaning lemma.

Lemma 3 (*Cleaning* see [2])**.** *If $\Gamma \subset \Lambda$ is correctable, and $\bar{P}$ is a logical operator of the code, then there exists a logical operator $\bar{P}'$ with support outside Γ, such that the actions of $\bar{P}$ and $\bar{P}'$ on the code space $\mathcal{C}$ agree.*

In other words, the support of a logical operator $\bar{P}$ can be modified to exclude qubits in Γ (explaining the terminology 'cleaning') without affecting its action on encoded states. With it, we are equipped to give the full proof of Theorem 2.

Proof. Let $A \subset \Lambda$, $|A| = k$ be the set of qubits carrying the quantum information to be encoded. Our first goal is to show the existence of a suitable set B. Let c be the constant from Lemma 2, that is, any cube Γ_R of linear size $R = cd^{\frac{1}{D-1}}$ is correctable see Fig. 1. We choose a block Γ_R that contains the qubits A near its center. Then we set $B := \Gamma_R^c = \Lambda \backslash \Gamma_R$. Since $k \ll d$, this guarantees that

(i) $d(B, A) \geq \frac{c}{3} d^{\frac{1}{D-1}}$.

Since B is the complement of Γ_R, Lemma 2 and the cleaning lemma applied to the region Γ_R imply that for any logical operator, we can find an equivalent logical operator with support completely contained in B. This implies, in particular, that

(ii) there is a logical operator $\bar{P}_B$ with eigenvalues $\{+1, -1\}$ with support $\mathrm{supp}(\bar{P}_B) \subset B$.

Consider two orthogonal states $\bar{\rho}_0, \bar{\rho}_1 \in \mathcal{B}_1^+(\mathcal{C})$ with

$$\mathrm{tr}[\bar{P}_B(\bar{\rho}_0 - \bar{\rho}_1)] = 2 \,. \tag{1}$$

Defining ρ_b for $b \in \{0, 1\}$ by $(\rho_b)_A \otimes \Phi_{A^c} := U^\dagger \bar{\rho}_b U$ (where U is a unitary encoder)

$$\mathrm{tr}[\bar{P}_B(\bar{\rho}_0 - \bar{\rho}_1)] = \mathrm{tr}[\bar{P}_B U(\rho_0 \otimes \Phi - \rho_1 \otimes \Phi)U^\dagger]. \tag{2}$$

Assuming $\mathcal{E}^{(t)}$ encodes into $\mathcal{C}$ with error ϵ, this implies (with $\mathrm{tr}(MN) \leq \|M\|_\infty \cdot \|N\|_1$ and $\|\bar{P}_B\|_\infty = 1$) that

$$\left| \mathrm{tr}[\bar{P}_B(\bar{\rho}_0 - \bar{\rho}_1)] - \mathrm{tr}[\bar{P}_B \mathcal{E}^{(t)}(\rho_0 - \rho_1) \otimes \Phi] \right| \leq \epsilon. \tag{3}$$

Combining (1) and (3), we get for $P(t) := (\mathcal{E}^{(t)})^\dagger(\bar{P}_B)$

$$\mathrm{tr}\left[P(t)(\rho_0 - \rho_1) \otimes \Phi\right] \geq 2 - \epsilon. \tag{4}$$

Set $P_r(t) := (\mathcal{E}_{B_r}^{(t)})^\dagger(\bar{P}_B)$, where $\mathcal{E}_{B_r}^{(t)}$ is the localized evolution (according to the Lieb-Robinson property). Using $|B| \leq n$, $\|\bar{P}_B\|_\infty = 1$ and choosing $r = \frac{c}{3} d^{\frac{1}{D-1}}$, we have

$$\|P(t) - P_r(t)\|_\infty \leq Cn \exp(vt - \gamma \frac{c}{3} d^{\frac{1}{D-1}}) \,,$$

for some nonnegative constants C, v, γ. This, combined with (4) and $\|(\rho_0 - \rho_1) \otimes \Phi\|_1 \leq 2$ gives

$$\begin{aligned} &\mathrm{tr}\left[P_r(t)(\rho_0 - \rho_1) \otimes \Phi\right] \geq \\ &\qquad 2 - \epsilon - 2Cn \exp(vt - \gamma \frac{c}{3} d^{\frac{1}{D-1}}) \,. \end{aligned} \tag{5}$$

But $P_r(t)$ is, by definition supported on the set $B(r) := \{x \mid d(x, B) \leq r\}$. By definition of r it is easy to check that $B(r) \subset A^c$, i.e., it has no intersection with A. This implies

$$\mathrm{tr}\left[P_r(t)(\rho_0 - \rho_1) \otimes \Phi\right] = 0 \,. \tag{6}$$

Eqs. (5) and (6) are compatible only if $t \geq \Omega(d^{\frac{1}{D-1}})$, as claimed. □

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