Dual QCD fits to the masses of the $c\bar{c}$ and $b\bar{b}$ systems

M. Baker
University of Washington, Seattle, Washington 98105

James S. Ball
University of Utah, Salt Lake City, Utah 84112

F. Zachariasen
California Institute of Technology, Pasadena, California 91125

(Received 19 September 1991)

We use the static potentials between heavy quarks derived from the classical approximation to the dual QCD field equations to calculate most of the low-lying states of the $c\bar{c}$ and $b\bar{b}$ systems. The agreement, achieved with only four parameters, two of which are roughly determined in advance, is better than 1%. The spin-orbit force cannot yet be calculated from dual QCD; if we use the Breit–Fermi form for it, all the $c\bar{c}$ and $b\bar{b}$ levels can be predicted. We also predict the masses of the lightest $c\bar{b}$ states.

PACS number(s): 12.38.Aw; 11.15.Kc; 12.40.Qq; 14.40.Jz

I. INTRODUCTION

In two recent papers [1, 2] we have calculated the static spin-independent and spin-dependent potentials between a quark and an antiquark, using the classical approximation to dual QCD. This note is devoted to reporting how well these potentials fit the energy levels of charmonium and bottomonium.

The potentials predicted by dual QCD are the central potential $V(R)$ (R is the separation of the quark and antiquark), the tensor force $V_T(R)$, and the spin-spin potential $V_S(R)$ proportional to $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$. Analytic fits to the numerical dual QCD predictions of these potentials are

$$V(R) = -\frac{4}{3}\frac{\alpha_s}{R}e^{-0.643x}+1.0324R(-\tilde{F}_0^2)-0.5164\sqrt{-\frac{\tilde{F}_0^2}{\lambda}}\,, \tag{1.1}$$

$$V_S(R) = \frac{8\pi\alpha_s}{9m_q^2}\delta^3(\mathbf{R}) + \frac{\alpha_s\pi}{m_q^2}(-\lambda\tilde{F}_0^2)^{3/2}(0.0311-0.00361x)\frac{e^{-0.530x}}{x}, \tag{1.2}$$

and

$$V_T(R) = \frac{\pi\alpha_s}{m_q^2}(-\lambda\tilde{F}_0^2)^{3/2}(\,0.1018+0.0534x - 0.00761x^2)\frac{e^{-0.360x}}{x^3}\,. \tag{1.3}$$

In these formulas α_s is the conventional QCD coupling constant, λ and $-\tilde{F}_0^2$ are the two standard parameters of dual QCD [3], and x is the dimensionless length $x=\sqrt{-\lambda\tilde{F}_0^2}R$. The entire static potential is

$$V \equiv V(R)+V_T(R)(3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{R}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{R}}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)+V_S(R)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2, \tag{1.4}$$

where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the quark and antiquark spins. We may use the potentials to calculate all S states and all singlet-spin states of the $c\bar{c}$ and $b\bar{b}$ systems. We can also compute the masses of the ${}^3P$ combinations $\frac{1}{2}({}^3P_2+{}^3P_1)$ and $\frac{1}{3}(2{}^3P_2+{}^3P_0)$, since these are independent of the spin-orbit force V_{LS}.

A best fit to all thirteen of the levels independent of V_{LS} is made and the parameters of dual QCD, namely, λ and $\sqrt{-\tilde{F}_0^2}$, are determined, as are the c- and b-quark masses m_c and m_b.

We therefore have four parameters at our disposal, namely, λ, $\sqrt{-\tilde{F}_0^2}$ and the two quark masses m_c and m_b. We already have some information about λ and $\sqrt{-\tilde{F}_0^2}$ from earlier dual QCD fits [3] to the string tension and the magnetic condensate G_2; these gave $\lambda = 1.61$ and $\sqrt{-\tilde{F}_0^2} = 420$ MeV. Because of uncertainties in G_2, however, these values can only be regarded as estimates; they will be determined more precisely by the fits to charmonium and bottomonium levels described here.

The solutions to the classical field equations of dual QCD [3], on which our derivation of the potentials is based [1, 2], made the choice $g'^2 = g^2/\lambda = 5$, where g is the dual QCD coupling constant (we have $eg = 2\pi$ so that $\alpha_s = e^2/4\pi = \pi/g^2$). This value was determined [3] by fitting the experimental string tension and the vacuum energy density as given through the trace anomaly. The field equations of dual QCD describing a flux tube were solved for this value of g', and therefore, all of the fields C_0, C, B, and B_3 are calculated for $g'^2 = 5$. Since we do not wish to recalculate the solutions to the dual QCD field equations, we shall continue to use $g'^2 = 5$. Thus g' is not an independent parameter.

We are as yet unable to calculate the spin-orbit force from dual QCD (although we hope to be able to do so in the near future) because of the difficulty of handling the interaction of moving quark sources with the dual potentials. Extending our fit to incorporate the remaining three independent combinations of the known P-wave states of the $c\bar{c}$ and $b\bar{b}$ systems requires adding a spin-orbit potential without introducing additional parameters. For this reason we will use the Breit–Fermi [4] form which expresses the spin-dependent potentials in terms of the vector and scalar components of the spin-independent potential:

$$V_{LS}(R) = \frac{1}{2m_q^2 R}\mathbf{S}\cdot\mathbf{L}\left(3\frac{\partial V_v}{\partial R} - \frac{\partial V_s}{\partial R}\right), \tag{1.5a}$$

$$V_T(R) = \frac{1}{12m_q^2}S_{12}\left(\frac{1}{R}\frac{\partial V_v}{\partial R} - \frac{\partial^2 V_v}{\partial R^2}\right). \tag{1.5b}$$

If we examine what V_T would be produced by Eq. (1.5b) from various terms given in Eq. (1.1), we see that the string tension would lead to a $1/R$ term not present in our V_T. On the other hand, if we treat the first term as V_v, the V_T obtained is of the same form as that given in Eq. (1.3) and has very nearly the same small-r limit. In Fig. 1 we compare our V_T with that calculated from Eq. (1.5b) assuming V_v is the first term in Eq. (1.1). While the exponential falloff is somewhat different, these potentials will give nearly identical results for our perturbation calculation. For lack of any better method of obtaining V_{LS} we will use Eq. (1.5a) with the same assumption for V_v and simply ignore the V_s term. The reader should bear in mind that these assumptions only affect the masses of three states.

We may now proceed to solve the Schrödinger equation

$$\left(-\frac{1}{2\mu}\nabla^2 + V\right)\psi_n = E_n\psi_n \tag{1.6}$$

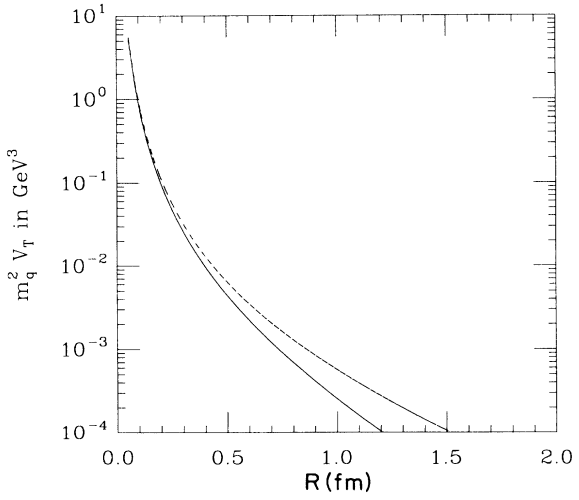

FIG. 1. The spin-dependent potential $m_q^2\, V_T(R)$ plotted as a function of quark separation R. The dashed curve is our empirical form and the solid curve is the result using the Breit-Fermi formula.

in terms of the parameters λ, $\sqrt{-\tilde{F}_0^2}$ and μ, for both the $c\bar{c}$ system ($\mu = m_c/2$) and the $b\bar{b}$ system ($\mu = m_b/2$).

II. SOLUTION OF THE SCHRÖDINGER EQUATION AND THE METHOD OF FITTING THE DATA

The calculation of the mass of a particular state proceeds as follows: First the Schrödinger equation is solved with the spin-independent potential given in Eq. (1.1), and the eigenvalue E is determined by the boundary conditions that the wave function be finite at the origin and vanish exponentially as $r \to \infty$. After determining the normalized wave function, the energy shifts ΔE_S produced by the the spin-dependent potential are then calculated perturbatively. The mass of the state is then

$$m = 2m_q + E + \Delta E_S\ . \tag{2.1}$$

Since the wave function depends only on the orbital angular momentum L, the Schrödinger equation need only be solved for the required values of L and the principal quantum number n. The determination of the masses of all of the known $c\bar{c}$ states and $b\bar{b}$ states below threshold requires nine solutions to the Schrödinger equation. Since our goal is to conduct a computer controlled search in our space of four parameters to obtain a best least-squares fit to the masses, it is important that our treatment of the Schrödinger equation be as automatic as is possible. The method we developed worked very well and in the interest of completeness we will describe it below.

Procedures which focus on the boundary conditions at the origin suffer from the "stiffness" of the differential equation being solved. The fact that numerical integration will always diverge for sufficiently large radius even for the exact eigenvalue means that judgment beyond that of a computer may be required. At large radius any

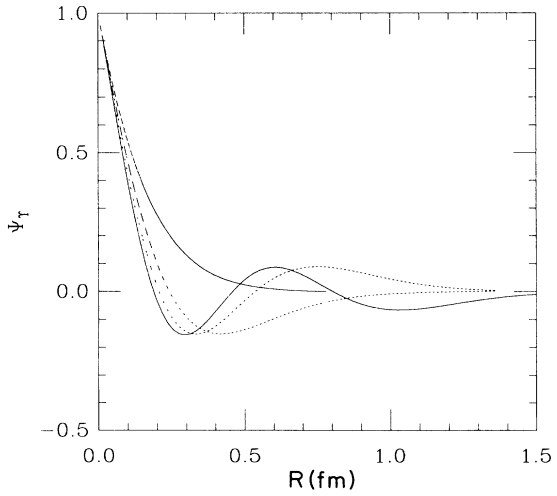

FIG. 2. The wave functions for the $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$, and $\Upsilon(4S)$ plotted as a function of quark separation R.

TABLE I. Predicted and experimental masses [5] of the V_{LS} independent $c\bar{c}$ and $b\bar{b}$ states.

State	Pred. mass (GeV)	Exper. mass (GeV)
η_c (1S)	2.979	2.980
ψ (1S)	3.118	3.097
ψ (2S)	3.713	3.686
$\frac{1}{2}[\chi_{c_2}(1P)+\chi_{c_1}(1P)]$	3.503	3.533
$\frac{1}{3}[2\chi_{c_2}(1P)+\chi_{c_0}(1P)]$	3.490	3.509
$\Upsilon(1S)$	9.462	9.460
$\Upsilon(2S)$	10.003	10.023
$\Upsilon(3S)$	10.343	10.355
$\Upsilon(4S)$	10.622	10.580
$\frac{1}{2}[\chi_{b_2}(1P)+\chi_{b_1}(1P)]$	9.906	9.902
$\frac{1}{3}[2\chi_{b_2}(1P)+\chi_{b_0}(1P)]$	9.900	9.895
$\frac{1}{2}[\chi_{b_2}(2P)+\chi_{b_1}(2P)]$	10.254	10.262
$\frac{1}{3}[2\chi_{b_2}(2P)+\chi_{b_0}(2P)]$	10.249	10.258

solution is a linear combination of a growing and dying exponential. If one starts at large r with the boundary condition that $(\partial\Psi/\partial r)/\Psi$ is negative corresponding to a falling exponential and integrates inward, the "stiffness" of the equation works for you, selecting only the exponential that grows going toward the origin. This procedure automatically guarantees the correct large-r behavior of the solution. As one approaches the origin this solution will get very large indicating the presence of the solution singular at the origin. If Ψ vanishes at some small value of $r = r_0$, the strength of the singular solution must be quite small since its numerical value is very large. As the value of $r_0 \to 0$ the eigenvalue for which $\Psi(r_0) = 0$ approaches the bound-state eigenvalue. We implement this method as follows: We begin at large r and integrate the Schrödinger equation and the second-order differential equation satisfied by $\partial\Psi/\partial E$ into r_0. This procedure is then iterated using the Newton-Raphson method to determine the required value of E. In practice this was a two-step process. The iteration procedure was first carried out for $r_0 = 0.05$ GeV^{-1} to obtain an approximate value of E. This value was used as the starting value for another iteration with $r_0 = 0.0001$ GeV^{-1} which determined the final eigenvalue. Once the eigenvalue was known, the normalized wave function was obtained by integrating out from the origin to a value of r at which the integrals used to calculate ΔE_S had converged. The desired principal quantum number was obtained by selecting the initial trial eigenvalue in the correct range. The wave functions for the four Υ states are shown in Fig. 2.

TABLE II. Predicted masses of the singlet $c\bar{c}$ and $b\bar{b}$ states.

State	Pred. mass (GeV)
$\eta_c(2S)$	3.613
$(^1P_1)c\bar{c}$ $(n=1)$	3.496
$(^1D_2)c\bar{c}$ $(n=1)$	3.796
$\eta_b(1S)$	9.359
$\eta_b(2S)$	9.957
$\eta_b(3S)$	10.306
$\eta_b(4S)$	10.590
$(^1P_1)b\bar{b}$ $(n=1)$	9.903
$(^1P_1)b\bar{b}$ $(n=2)$	10.252
$(^1D_2)b\bar{b}$ $(n=1)$	10.142
$(^1D_2)b\bar{b}$ $(n=2)$	10.436

The search utilized the minimization program MINUIT developed at CERN to minimize an effective χ^2 defined as follows:

$$\chi^2 = \sum \left(\frac{(\text{experiment} - \text{theory})}{(0.01 \times \text{experiment})} \right)^2 . \tag{2.2}$$

This would be the actual χ^2 if the experimental statistical error was in fact 1% or equivalently what might be expected to be equal to the number of degrees of freedom if the theory was good to 1%. Without using V_{LS} we fit thirteen states with four parameters, obtaining a $\chi^2 = 2.24$ corresponding to an average least-square error of 0.4%. By including V_{LS} masses of sixteen states could be fitted. In this case the $\chi^2 = 2.74$ leaving the average error unchanged from the previous fit.

III. RESULTS

The minimization technique described in Sec. II results in the following, when applied to the states of $c\bar{c}$ and $b\bar{b}$ listed in Table I (these are the states independent of V_{LS}).

A. Values of parameters

(i) $\lambda = 1.7443$.
(ii) $\sqrt{-\tilde{F}_0^2} = 435$ MeV.
(iii) $m_c = 1.352$ GeV.
(iv) $m_b = 4.770$ GeV.

It is gratifying to see that the values of λ and $\sqrt{-(\tilde{F}_0)^2}$ have not changed very much from those they had in our earlier, much cruder, fits to the string tension and to G_2.

These parameters result in $\alpha_s = 0.3602$, a string tension $\sigma = 0.956$ GeV^2, a flux-tube radius of 0.5 fm, a

TABLE III. Predicted and experimental widths [5] of the V_{LS} independent $c\bar{c}$ and $b\bar{b}$ states.

State	$\Gamma_{e^+e^-}$ (Theor.) (keV)	$\Gamma_{e^+e^-}$ (Exper.) (keV)
$\psi(1S)$	9.553	4.72
$\psi(2S)$	7.015	2.14
$\Upsilon(1S)$	2.489	1.34
$\Upsilon(2S)$	1.128	0.586
$\Upsilon(3S)$	0.885	0.44
$\Upsilon(4S)$	0.778	0.24

parameter $\sqrt{-\lambda\tilde{F}_0^2} = 575$ MeV, and the constant potential in $V(R)$, Eq. (1.1), comes out to be -170 MeV.

B. Fits to masses of states

The results of our fit without V_{LS} are shown in Table I.

The greatest disagreement lies in the $n = 1$ ^3P_J states of $c\bar{c}$, where the maximum error is 30 MeV, or about 1%. Therefore, especially in view of the small number of parameters, we regard the fit as excellent.

We can, of course, also predict as yet unobserved (nearly) stable states of these systems; these are shown in Table II. Note that since these levels are all singlet-spin states, the spin-orbit potential does not affect them. They are therefore unambiguous predictions of dual QCD. We have listed the singlet D $c\bar{c}$ state although its mass is above the $D\bar{D}$ threshold because the D-wave phase space should be small, minimizing the effect of the open channel.

Further predictions which are trivial to make are the levels of toponium, if the top quark is ever found. Since the top quark will certainly be very heavy, the levels should of course closely approximate those of the hydrogen atom. We shall not, therefore, bother to list the results for various guesses as to the value of m_t here.

C. Fits to electron-position decay widths of states

Another quantity which is readily computed from the value of the wave function at the origin is the decay width into e^+e^- pairs, using the Van Royen–Weisskopf formula [6]. We find the results shown in Table III.

The theoretical predictions are all evidently too large, by factors of between 2 and 3. This is also a feature of phenomenological fits to $\Gamma_{e^+e^-}$ [7] and is usually attributed to theoretical limitations of the Van Royen–Weisskopf formula [7]. The ratios of widths may be a better test; we quote these next, in Table IV.

The agreement, while slightly better than for the widths themselves, is still not particularly good. A better theoretical understanding of $\Gamma_{e^+e^-}$ is required.

D. Fits including the Breit-Fermi spin-orbit force

Finally, if we now add a spin-orbit force V_{LS}, we can predict all ^3P_J states of $c\bar{c}$ and $b\bar{b}$ individually, resulting in Table V. We emphasize, however, that this last set of results is not a consequence of dual QCD; for it to be so, we must await the derivation of V_{LS} from first principles within the dual QCD framework. The fit made here is a best fit to all of the known $c\bar{c}$ and $b\bar{b}$ states, and does not simply use the parameters we found in constructing Table I. The parameters are therefore slightly changed from those listed in A, to wit:

(i) $\lambda = 1.7444$.
(ii) $\sqrt{-\tilde{F}_0^2} = 437$ MeV.
(iii) $m_c = 1.350$ GeV.
(iv) $m_b = 4.769$ GeV.

With these values, which we note are very little changed from those without the spin-orbit force, we obtain Table V. The predicted masses for the singlet states are essentially unchanged from those given in Table II. The new predictions for the triplet states near or below threshold are given in Table VI. Only one candidate for these states has been observed, the $\psi(3770)$ [5], which could be the $^3D_1 c\bar{c}$. Its orbital angular momentum is not known and its width of 24 MeV indicates the importance of the open $D\bar{D}$ channel.

We would expect that the linear combinations of the P states used in our earlier fit and corresponding combinations of D states that are independent of V_{LS} are more reliable predictions of dual QCD than the individual masses given above.

E. Prediction for the masses of $c\bar{b}$ states

Our potentials should be applicable to any state composed of sufficiently heavy quarks. The only required changes are the use of the correct reduced mass and the replacement of m_q^2 by the product of the quark masses.

TABLE IV. Predicted and experimental ratios of widths [5] of the V_{LS} independent $c\bar{c}$ and $b\bar{b}$ states.

Ratio	Theory	Experiment
$\frac{\Gamma_{e^+e^-}(\psi(1S))}{\Gamma_{e^+e^-}(\psi(2S))}$	1.362	2.206
$\frac{\Gamma_{e^+e^-}(\Upsilon(1S))}{\Gamma_{e^+e^-}(\Upsilon(2S))}$	2.206	2.287
$\frac{\Gamma_{e^+e^-}(\Upsilon(1S))}{\Gamma_{e^+e^-}(\Upsilon(3S))}$	2.812	3.045
$\frac{\Gamma_{e^+e^-}(\Upsilon(1S))}{\Gamma_{e^+e^-}(\Upsilon(4S))}$	3.199	4.323

TABLE V. Predicted and experimental masses [5] of all observed $c\bar{c}$ and $b\bar{b}$ states below threshold.

State	Pred. mass (GeV)	Exper. mass (GeV)
$\eta_c(1S)$	2.978	2.980
$\psi(1S)$	3.118	3.097
$\psi(2S)$	3.716	3.686
$\chi_{c_0}(1P)$	3.422	3.415
$\chi_{c_1}(1P)$	3.482	3.511
$\chi_{c_2}(1P)$	3.527	3.556
$\Upsilon(1S)$	9.461	9.460
$\Upsilon(2S)$	10.004	10.023
$\Upsilon(3S)$	10.345	10.355
$\Upsilon(4S)$	10.626	10.580
$\chi_{b_0}(1P)$	9.866	9.860
$\chi_{b_1}(1P)$	9.895	9.892
$\chi_{b_2}(1P)$	9.917	9.913
$\chi_{b_0}(2P)$	10.221	10.235
$\chi_{b_1}(2P)$	10.246	10.255
$\chi_{b_2}(2P)$	10.265	10.269

The results for the lowest mass $c\bar{b}$ states are $\eta_{c\bar{b}} = 6.287$ GeV and $^3S_1 = 6.372$ GeV. The higher states could also be calculated but in the absence of any experimental candidates this seems premature.

IV. CONCLUSIONS

We have found that the classical approximation to dual QCD gives an excellent fit to the (nearly) stable states of the heavy-quark–antiquark system, in terms of only four parameters, two of which (λ and $\sqrt{-\tilde{F}_0^2}$) are nearly known already. It should be emphasized to the reader that this is in a sense a calculation from first principles in QCD, and *not* simply one more in a plethora of phenomenological calculations with little if any theoretical foundation. The calculation presented here is the solution (albeit in the classical approximation) of a well-defined Lagrangian which is supposed to represent QCD at long range, and from which one can make many other predictions as well, such as the existence and radius of flux tubes (which comes out to be 0.5 fm), the string tension, the value of the gluon condensate [which comes out to be (377 MeV)^4 for our optimal parameters], the deconfinement transition temperature and the chiral transition temperature. Furthermore it provides a simple physical understanding of confinement, in that it describes QCD as a non-Abelian dual superconductor. The only flaw it has, which we do not yet know how to remedy while still preserving the dual superconductivity feature [8], is that in higher orders (beyond the classical, or tree, level) the quantum field theory described by the dual QCD Lagrangian is not unitary. In addition there is, not a flaw, but an as yet unsolved problem, which is how to couple color-electric quarks to dual potentials beyond the Abelian level.

TABLE VI. Predicted masses of the triplet $c\bar{c}$ and $b\bar{b}$ states.

State	Pred. mass (GeV)
$(^3D_1)c\bar{c}$ $(n=1)$	3.777
$\chi_{b_0}(3P)$	10.510
$\chi_{b_1}(3P)$	10.533
$\chi_{b_2}(3P)$	10.550
$(^3D_1)b\bar{b}$ $(n=1)$	10.134
$(^3D_2)b\bar{b}$ $(n=1)$	10.142
$(^3D_3)b\bar{b}$ $(n=1)$	10.149
$(^3D_1)b\bar{b}$ $(n=2)$	10.429
$(^3D_2)b\bar{b}$ $(n=2)$	10.437
$(^3D_3)b\bar{b}$ $(n=2)$	10.444

ACKNOWLEDGMENTS

The work of N.B. was supported in part by the U.S. Dept. of Energy under Contract No. DOE/ER/40614. The work of J.S.B. was supported in part by the U.S. National Science Foundation Grant No. PHY 9008482. The work of F.Z. was supported in part by the U.S. Dept. of Energy under Contract No. DEAC-03-81ER40050.

[1] M. Baker, J.S. Ball, and F. Zachariasen, Phys. Rev. D **44**, 3328 (1991).
[2] M. Baker, J.S. Ball, and F. Zachariasen, Phys. Rev. D **44**, 3949 (1991).
[3] M. Baker, J.S. Ball, and F. Zachariasen, Phys. Rev. D **41**, 2612 (1990).
[4] H. Schnitzer, Phys. Rev. D **13**, 74 (1976); D. Gromes, Nucl. Phys. **B131**, 80 (1977).
[5] Particle Data Group, J.J. Hernández *et al.*, Phys. Lett. B **239**, 1 (1990).
[6] R. Van Royen and V. Weisskopf, Nuovo Cimento **50**, 617 (1967).
[7] E. Eichten *et al.*, Phys. Rev. D **21**, 203 (1980).
[8] M. Baker, James S. Ball, and F. Zachariasen, Phys. Rev. D **44**, 2578 (1991).