THE PHYSICS OF FLUIDS VOLUME 3, NUMBER 5 SEPTEMBER-OCTOBER, 1960

Growth of Vapor Bubbles in a Rapidly Heated Liquid

S. A. Zwick*
California Institute of Technology, Pasadena, California
(Received February 29, 1960)

The earlier theory of the growth of vapor bubbles in superheated liquids is extended to the situation in which the rate of temperature rise of the liquid is large. Numerical solutions are presented for the early stages of bubble growth for various rates of liquid temperature rise. The asymptotic behavior of a bubble is found explicitly for a temperature rise of the liquid which is linear in time. In this case the bubble radius grows initially as $t^{1/2}$, as in asymptotic solutions found previously for small rates of temperature rise, but then deviates toward a late $t^{3/2}$ variation.

I. INTRODUCTION

THEORETICAL treatments of the problem of vapor bubble growth in a superheated liquid at constant external pressure have been given by several authors. Assuming the liquid temperature to increase slowly in comparison with changes in the parameters describing bubble behavior, they arrive at a law of the form

$$R \sim (K_b \,\Delta T)(t - \tau)^{1/2}, \quad \text{as} \quad t \to \infty \tag{1}$$

for the asymptotic phase of bubble growth.[1–3] Here R is the bubble radius at a time t, after the time τ when the bubble begins significant growth, $\Delta T = T_0 - T_b$ denotes the superheat (difference between the initial temperature T_0 and boiling temperature T_b) of the liquid when the bubble growth begins, and K_b is a constant, characteristic of the liquid at its boiling point. The asymptotic radius-time relation, Eq. (1), has been verified by experiments involving rates of superheat on the order of 1°C/min or less in water and other common liquids.[4] Generally, it applies after the bubble radius is an order of magnitude larger than its actual value at time τ and is useful for further orders of magnitude of growth, depending on the rate of superheat and the amount of bubble translation through the liquid (assumed negligible in the analysis). In the present paper the limitation of low superheat rates is removed, and a modification of Eq. (1) is obtained which applies equally well when the liquid temperature climbs rapidly.

Rapid heating of the liquid may be expected to produce two changes in the growth of a vapor bubble. The first is a relative shortening of the early phase of growth from unstable equilibrium, due to the change in the effective initial impulse. This amounts to a negative shift of τ, in Eq. (1), by an amount proportional to the logarithm of the superheat rate. Numerical data and graphs describing the early growth are presented in Sec. II for superheat rates up to 2000°C/sec in water.

The second effect of a rapid heating of the liquid is an increase in the asymptotic growth rate, resulting from the enhanced heat transfer to the bubble wall. Although the early bubble growth is a dynamic process, involving pressure forces, surface tension, and liquid inertia, heat transfer becomes the controlling factor as soon as an appreciable evaporation rate (growth rate) has been attained, in virtue of the latent heat requirements for evaporation. By the time the asymptotic phase of growth is reached the dynamic balance will have been restored. Thus the equation of growth for the asymptotic phase states simply that the vapor pressure in the bubble must equal the pressure applied by the liquid (surface tension forces being negligible by this time), i.e., that the temperature at the bubble wall has dropped to the boiling point of the liquid. Subsequent heat transfer to the bubble wall then becomes sensitive to the temperature of the bulk liquid, which we suppose to be increasing rapidly.

An analysis of the asymptotic growth rate for the case of rapid heating of the liquid will be given in Sec. III. Application is made in Sec. IV to the problem of volume boiling.

II. SUMMARY OF EARLY GROWTH DATA

Results of calculations covering the early phase of bubble growth are summarized here in tabular and graphical form. Figures 1–8 give curves of

---

* Now with the Lawrence Radiation Laboratory, Livermore, California.
[1] M. S. Plesset and S. A. Zwick, J. Appl. Phys. **25**, 493 (1954).
[2] H. K. Forster and N. Zuber, J. Appl. Phys. **25**, 474 (1954).
[3] G. Birkhoff, R. S. Margulies, and W. A. Horning, Phys. Fluids **1**, 201 (1958).
[4] P. Dergarabedian, J. Appl. Mech. **20**, 537 (1953).

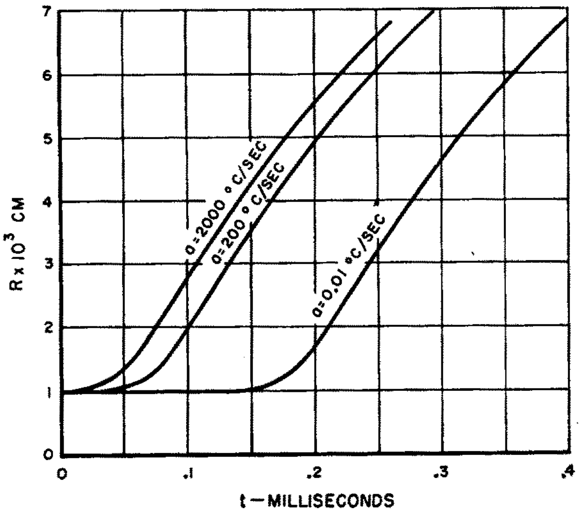

FIG. 1. Calculated bubble radius versus time for the initial and early asymptotic phases of growth. The curves refer to bubble growth in water at 1 atm external pressure, with the liquid temperature rising at the rates $a = 0.01°$, $200°$, $2000°$C/sec. In each case the bubble begins to grow at 103°C.

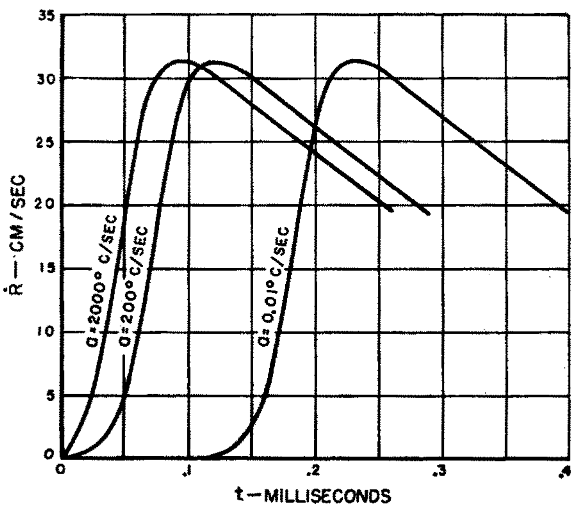

FIG. 2. Radial velocities corresponding to the growth curves of Fig. 1.

bubble radius and radial velocity as functions of time, for water at 1 atm and initial temperature 103°, 106°C, and at 18.8 atm (boiling point 210°C) and initial temperature 213°, 216°C. For each initial situation three rates of temperature rise of the bulk liquid were considered: $a = 0.01$°C/sec, 200°C/sec, 2000°C/sec. The time $t = 0$ in the plots refers to the beginning of bubble growth from unstable equilibrium, which takes place at the instant the liquid temperature passes the quoted initial temperature. The corresponding time $t = \tau$ of Eq. (1) would then be reached near the knee of each radius-time curve.

The curves in Figs. 1–8 were calculated from formulas given previously for the growth of a pure vapor bubble from its radius R_0 of unstable equilibrium. These may be written in parametric form as[5]

$$\left.\begin{aligned} R = R_0 e^{w/3},\ t \sim t_0 + \frac{1}{\alpha\beta}\left[\ln\left(\frac{w}{3}\right) + \left(a_1 - \frac{4}{3}\right)w\right.\\ \left. + \left(a_2 - \frac{4}{3}a_1 + \frac{8}{9}\right)\frac{w^2}{2} + \cdots\right],\\ t_0 = \frac{1}{\alpha\beta}\ln\left[\frac{\alpha\beta\zeta(3\beta^2+1)}{2a\mu}\right], \end{aligned}\right\} \quad (2)$$

valid for $t > 1/\alpha\beta$ but $R < 10R_0$, roughly; and,[1,6]

$$\begin{aligned} R = R_0 z^{\frac{1}{2}},\quad z \sim \frac{2}{\pi\mu}u^{\frac{1}{2}}\left[1 + \frac{b_1}{u^{1/6}} + \frac{b_2}{u^{2/6}} + \cdots\right],\\ t \sim t_1 + \frac{1}{\alpha}\int_{u_1}^{u}\frac{dv}{z^{4/3}(v)}, \end{aligned} \quad (3)$$

which applies when $R \gg R_0$. Values of R_0, α, β, ζ, μ and the expansion coefficients a_1, a_2, b_1, b_2, b_3 are given in Table I. These, and the constant $K_b\,\Delta T$ of Eq. (1) depend on the initial value T_0 of the liquid temperature, but not on its rate of rise. The "delay period" t_0, which does depend on the superheat rate a, is compared with $1/\alpha\beta$ in Table II. Not tabulated is t_1 (or u_1) in Eq. (3), an arbitrary constant chosen to provide a best fit of solution (3) with solution (2) by graphical means at a point where both are reasonably accurate.

One finds the transition from Eqs. (2) to (3) adequately represented if the time parameter τ of Eq. (1) [which designates the constant terms involving t_1, u_1, in the expression for t in Eq. (3)] is estimated by t_0. The transition zone occupies roughly a time interval $10/\alpha\beta$ centered about $t = t_0$. By comparison, the departure from solution (3) due to the finite rate of temperature rise in the liquid becomes appreciable by about the time $t = \tau + \Delta T/5a$ ($q \approx 0.2$ in Fig. 9). Using the data in Table II, we conclude that over the time intervals portrayed the asymptotic bubble growth in Figs. 1–8 is unaffected by superheat rates up to $a = 2000$°C/sec. However, bubbles starting to grow at lower initial superheats (say, 2°C or below) would be affected by such rapid heating soon after reaching their asymptotic phase.

[5] S. A. Zwick, Report 21-19, Hydro. Lab., California Institute of Technology, December 1954. See pp. 82–84.
[6] Reference 5, pp. 84–88.

TABLE I. Constants for the determination of bubble growth in water for initial and early asymptotic phases.

T_b(°C)	100°	100°	210°	210°
ΔT(°C)	3°	6°	3°	6°
R_0(cm)	1.019×10^{-3}	4.83×10^{-4}	6.12×10^{-5}	2.96×10^{-5}
α(sec^{-1})	3.39×10^{5}	1.035×10^{6}	1.93×10^{7}	5.74×10^{7}
ζ(°C)	1.023	0.888	7.19	6.35
μ	0.341	0.148	2.44	1.093
β	0.263	0.618	5.93×10^{-3}	2.96×10^{-2}
a_1	2.032	1.929	2.109	2.108
a_2	2.055	1.853	2.184	2.183
b_1	−0.910	−0.689	−1.754	−1.341
b_2	−0.462	−2.506	−1.184	−0.696
b_3	−1.481	−1.972	1.185	0.526
$K_b\Delta T$(cm sec$^{-\frac{1}{2}}$)	0.707	1.415	0.0428	0.0857

TABLE II. Time parameters for the delay period of bubble growth for various values of temperature rise rates (a).

	$1/\alpha\beta$ (sec)	t_0 (sec)
$T_b=100$°C, $\Delta T=3$°C		
$a=0.01$°C/sec	1.12×10^{-5}	1.86×10^{-4}
$a=200$		7.50×10^{-5}
$a=2000$		4.92×10^{-5}
$T_b=100$°C, $\Delta T=6$°C		
$a=0.01$°C/sec	1.56×10^{-6}	3.10×10^{-5}
$a=200$		1.55×10^{-5}
$a=2000$		1.19×10^{-5}
$T_b=210$°C, $\Delta T=3$°C		
$a=0.01$°C/sec	8.76×10^{-6}	1.46×10^{-4}
$a=200$		5.90×10^{-5}
$a=2000$		3.88×10^{-5}
$T_b=210$°C, $\Delta T=6$°C		
$a=0.01$°C/sec	5.90×10^{-7}	1.18×10^{-5}
$a=200$		5.97×10^{-6}
$a=2000$		4.61×10^{-6}

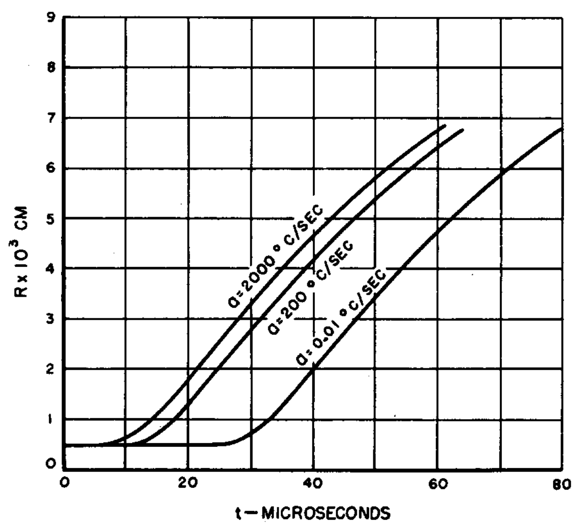

FIG. 3. Bubble radius versus time for bubbles starting to grow at 106°C in water at 1 atm.

III. ASYMPTOTIC BUBBLE GROWTH

By the asymptotic phase of growth, the latent heat requirements for evaporation have resulted in a cooling of the bubble wall to the boiling point of the liquid (assumed to be under a constant applied pressure). Further bubble growth is a comparatively slow process, dependent almost entirely on the rate of heat transfer to the bubble wall. Analysis of the asymptotic growth therefore depends upon the solution of the time-dependent heat convection problem in the liquid.

The latter has been obtained under the assumption of a "thin thermal boundary layer" in the liquid surrounding the bubble, through which the major portion of the temperature drop occurs from the the value T in the liquid far from the bubble to the value θ at the bubble wall. Let L, k, and D denote the latent heat of evaporation, thermal conductivity, and thermal diffusivity of the liquid; these are slowly varying functions of temperature, which may be given their value at the boiling temperature

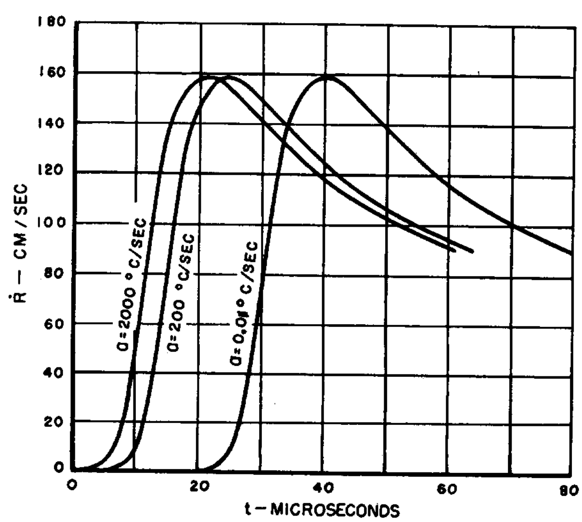

FIG. 4. Radial velocity curves for Fig. 3.

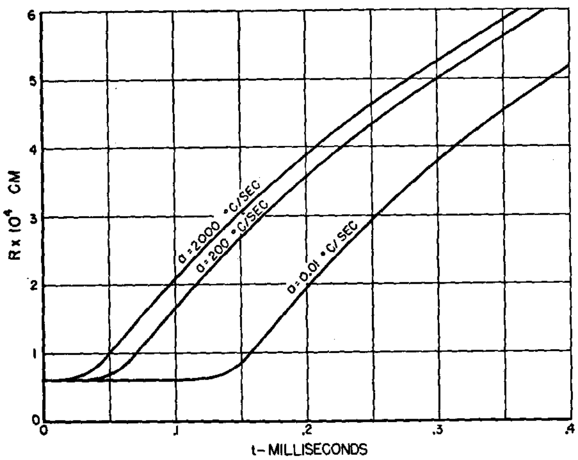

FIG. 5. Radius-time curves for bubbles starting to grow at 213°C in water at 18.8 atm (boiling point 210°C).

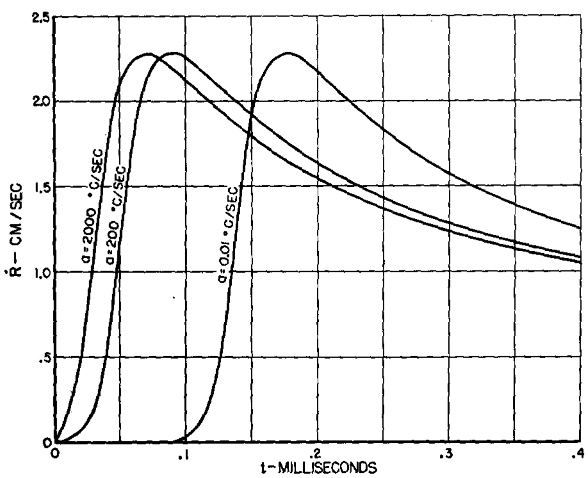

FIG. 6. Radial velocity curves corresponding to the growth curves of Fig. 5.

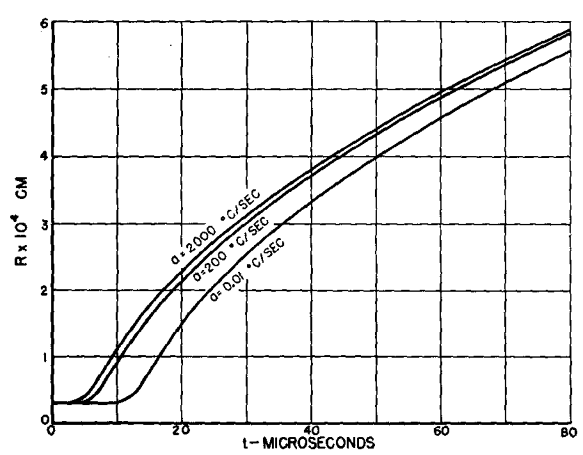

FIG. 7. Radius-time curves calculated for bubbles starting growth at 216°C in water at 18.8 atm.

T_b for the present problem. Then the relation for the temperature θ at the bubble wall becomes[7]

$$\theta = T - \left[\frac{L}{3k}\left(\frac{D}{\pi}\right)^{\frac{1}{2}}\right]_{T_b} \cdot \int_0^t \left\{\frac{d}{dx}\left(\rho_v(x)R^3(x)\right) \Big/ \left(\int_x^t R^4(y)\,dy\right)^{\frac{1}{2}}\right\} dx. \quad (4)$$

Within the integrand, $R(t)$ is the bubble radius at time t, and ρ_v denotes the equilibrium vapor density, a known (not necessarily slowly varying) function of the temperature at the bubble wall. The lower limit of integration $x = 0$ in Eq. (4) refers to the time when bubble growth begins, but may be assigned to any earlier instant since the integrand will remain negligible (in comparison with its later values) until this time. To standardize the subsequent discussion we shall assign the time $t = 0$ to the instant when the liquid temperature increases past the boiling point.

In order to construct an asymptotic solution equivalent to Eq. (1), we imagine the bubble to be a point until the time $t = \tau$ when suddenly it begins to grow. The bubble wall temperature θ then drops abruptly to the boiling temperature T_b. To account for the temperature rise in the liquid over the history of bubble growth we shall take

$$\left.\begin{aligned} T &= T_b + at, \\ \text{so that} \quad \Delta T &= T_0 - T_b = a\tau. \end{aligned}\right\} \quad (5)$$

(For bubble histories so long that the temperature rise is not adequately linear—such as might occur with bubbles starting to grow at very small initial superheats—one may consider the rate-of-rise coefficient a as an appropriate average value.) Thus we obtain as the description of the asymptotic phase of bubble growth

$$at = \left[\frac{L}{3k}\left(\frac{D}{\pi}\right)^{\frac{1}{2}}\right]_{T_b} \cdot \int_k^t \left\{\frac{d}{dx}\left(\rho_v(x)R^3(x)\right) \Big/ \left(\int_x^t R^4(y)\,dy\right)^{\frac{1}{2}}\right\} dx,$$

$$(t > \tau),\ R(\tau) = 0, \quad (6)$$

an implicit equation for the bubble radius R as a function of time. Here the lower limit of integration has been raised to $x = \tau$ since the integrand is supposed to be identically zero for $x < \tau$. It may

[7] M. S. Plesset and S. A. Zwick, J. Appl. Phys. 23, 95 (1952). An extended discussion of the heat problem may be found in reference 5, pp. 38–68.

be noted that the integral in Eq. (6) (representing the temperature difference between the "surrounding" liquid and the bubble wall) will not vanish as $t \to \tau^+$ because the integrand is now singular at both limits of integration.

It is possible to invert Eq. (6) to obtain an equation that does not involve derivatives of R. Put

$$\left.\begin{aligned} u(t) &= \lambda^2 \int_\tau^t R^4(y)\, dy, \\ \text{so that} \quad & \\ t &= \tau + \frac{1}{\lambda^2} \int_0^u \frac{dv}{R^4(v)}, \end{aligned}\right\} \quad (7)$$

where λ is a constant introduced for convenience. Then Eq. (6) becomes

$$at = \lambda\left[\frac{L}{3k}\left(\frac{D}{\pi}\right)^{\frac{1}{2}}\right]_{T_b} \int_0^u \frac{(d/dv)(\rho_v R^3)\, dv}{(u - v)^{\frac{1}{2}}}. \quad (8)$$

Next multiply by $du/(w - u)^{\frac{1}{2}}$ and integrate from $u = 0$ to $u = w$. The left side of Eq. (8) may be integrated by parts to give

$$\int_0^w at(u) \frac{du}{(w - u)^{\frac{1}{2}}} = -2a \int_0^w t(u)\, d(w - u)^{\frac{1}{2}}$$
$$= 2a\tau w^{\frac{1}{2}} + 2a \int_0^w (w - u)^{\frac{1}{2}}\left(\frac{dt}{du}\right) du.$$

On the right, an interchange in the order of integration yields

$$\int_0^w \frac{du}{(w - u)^{\frac{1}{2}}} \int_0^u \frac{(d/dv)(\rho_v R^3)\, dv}{(u - v)^{\frac{1}{2}}} = \int_0^w \frac{d}{dv}(\rho_v R^3)\, dv$$
$$\cdot \int_v^w \frac{du}{(w - u)^{\frac{1}{2}}(u - v)^{\frac{1}{2}}} = \pi \rho_v(w) R^3(w)$$

in virtue of the initial condition on R. Here $\rho_v(w)$ is simply the value of the equilibrium vapor density at the boiling point. Substituting from Eq. (7) for dt/du, and replacing w by u, one has

$$\left.\begin{aligned} R^3(u) &= u^{\frac{1}{2}} + \frac{a}{\lambda^2\, \Delta T} \int_0^u \frac{(u - v)^{\frac{1}{2}}\, dv}{R^4(v)} \\ \text{on choosing} \quad & \\ \lambda &= \Delta T\left(\frac{6k}{L\rho_v(\pi D)^{\frac{1}{2}}}\right)_{T_b}. \end{aligned}\right\} \quad (9)$$

For $a \to 0$, with ΔT held constant, Eq. (9) gives $R^3 = u^{\frac{1}{2}}$. This may be used in Eq. (7) to recover the asymptotic solution (1) with

$$K_b = \frac{\lambda}{3^{\frac{1}{2}}\, \Delta T} = 2\left(\frac{3}{\pi}\right)^{\frac{1}{2}}\left(\frac{k}{L\rho_v\, D^{\frac{1}{2}}}\right)_{T_b}. \quad (10)$$

To find a solution to Eq. (9) for finite a it is convenient to make a further change of variables. We shall put

$$\left.\begin{aligned} \phi &= R^3/u^{\frac{1}{2}}, \qquad s = (7a/\lambda^2\, \Delta T)u^{\frac{1}{2}}, \\ \text{and set} \quad & \\ r &= (7a/\lambda^2\, \Delta T)v^{\frac{1}{2}} \end{aligned}\right\} \quad (11)$$

in the integral of Eq. (9), so that the equation reduces to

$$\begin{aligned} \phi &= 1 + \frac{3}{7}\int_0^s \frac{dr}{\phi^{4/3}(r)}\left[1 - \frac{r^3}{s^3}\right]^{\frac{1}{2}} \\ &= 1 + \frac{3}{7}\int_0^s \frac{dr}{\phi^{4/3}(r)} \\ &\quad - \frac{3}{7}\int_0^s \frac{dr}{\phi^{4/3}(r)}\left\{1 - \left[1 - \frac{r^3}{s^3}\right]^{\frac{1}{2}}\right\}, \end{aligned} \quad (12)$$

and the physical variables become

$$R = \lambda\left(\frac{\tau}{7}\right)^{\frac{1}{2}} \phi^{\frac{1}{3}} s^{\frac{1}{3}}, \qquad t = \tau\left\{1 + \frac{3}{7}\int_0^s \frac{dr}{\phi^{4/3}(r)}\right\}. \quad (13)$$

One may verify from Eq. (12) that $\phi \sim 1$ for small s, while for $s \to \infty$, $\phi \sim$ (const) $s^{3/7}$. Hence the factor $\phi^{-4/3}$ in the last integral of Eq. (12) is slowly varying in comparison with the brace, which is essentially zero until $r \to s$ when it rises abruptly to one. The brace thus acts as a δ function at $r = s$ relative to $\phi^{-4/3}$, and we may approximate Eq. (12) by

$$\left.\begin{aligned} \phi(s) &= 1 + \frac{3}{7}\int_0^s \frac{dr}{\phi^{4/3}(r)} - \frac{3\kappa}{7}\frac{s}{\phi^{4/3}(s)}, \\ & \text{where the constant } \kappa \text{ is given by} \\ \kappa &= \int_0^1 \{1 - [1 - y^3]^{\frac{1}{2}}\}\, dy \\ &= 1 - \frac{1}{3}B\left(\frac{3}{2}, \frac{1}{3}\right) = 0.159 \end{aligned}\right\} \quad (14)$$

to three decimal places.

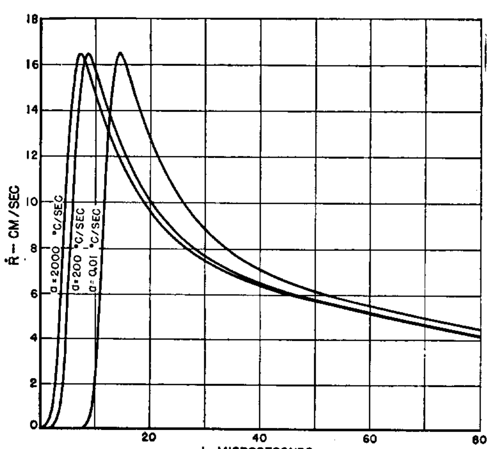

FIG. 8. Radial velocity versus time for the growth curves of Fig. 7.

By inspection, the solution to Eq. (14) is

$$s = \frac{1}{C}\,[\phi^{7/3} - \phi^{-\nu}], \tag{15}$$

where C and ν are constants. Substitution into Eq. (14) yields

$$C = 1 - \frac{3}{7}\,\kappa = 0.932, \quad \nu = \frac{4}{3}\frac{\kappa}{1-\kappa} = 0.252. \tag{16}$$

The approximation (15) could in principle be used in Eq. (12) to provide higher-order correction terms. Here we shall terminate the analysis with an estimate of the asymptotic errors. From Eqs. (15) and (16) one obtains

$$\phi^{7/3} \sim \begin{cases} 1 + (1-\kappa)s = 1 + 0.841s & \text{as } s \to 0, \\ Cs = 0.932s & \text{as } s \to \infty. \end{cases}$$

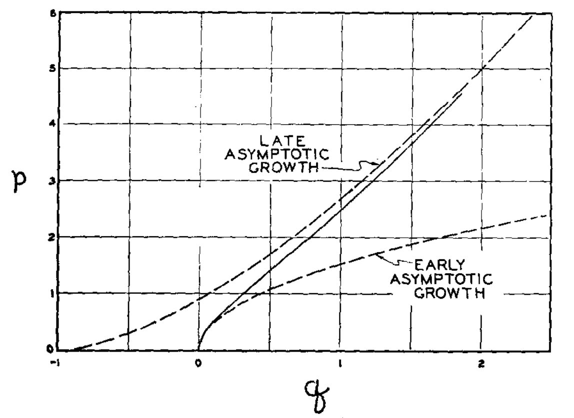

FIG. 9. Asymptotic solution for bubble growth, plotted in terms of the reduced radius variable $p = (7C/\tau)^{\frac{1}{2}}(R/\lambda)$ and time variable $q = C(t-\tau)/\tau$. The dashed curves give the early and late asymptotic forms of the solution. (The latter goes to zero at $q = -0.932$.)

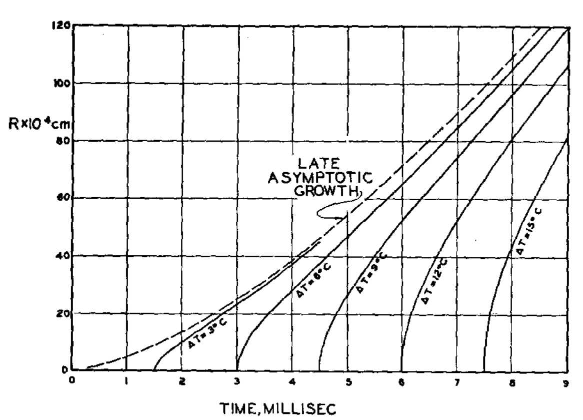

FIG. 10. A family of growth curves, obtained from Fig. 9 for the case of bubble growth in water at 18.8 atm ($T_b = 210$°C) under a temperature rise of 2000°C/sec. Each curve is labeled by the value of the superheat ΔT at which the corresponding bubble begins to grow. The late asymptotic form of solution is indicated by the dashed curve in the figure.

The correct behavior may be found by assuming $\phi \to 1$, or $\phi \to \infty$ as (const) $s^{3/7}$, in the integral of Eq. (12). The first iteration gives

$$\phi^{7/3} \sim \begin{cases} 1 + \frac{1}{3}B\left(\frac{3}{2}, \frac{1}{3}\right)s = 1 + 0.841s & \text{as } s \to 0, \\ \frac{1}{7}B\left(\frac{3}{2}, \frac{1}{7}\right)s = 0.922s & \text{as } s \to \infty. \end{cases}$$

The approximation (15) is therefore accurate near $s = 0$ $(t = \tau)$, but in error by about 1% as $s \to \infty$ $(t \to \infty)$.

Using Eq. (15) in Eq. (13), one can write the asymptotic radius-time solution (15) as a pair of parametric relations. These are conveniently put into dimensionless form in terms of reduced radius and time variables p and q; thus,

$$\begin{aligned} p &\equiv (R/\lambda)(7C/\tau)^{\frac{1}{2}} = \phi^{\frac{1}{2}}[\phi^{7/3} - \phi^{-\nu}]^{\frac{1}{2}}, \\ q &\equiv C[(t - \tau/\tau)] = \phi - C - (1-C)\phi^{-(\nu+4/3)}, \end{aligned} \tag{17}$$

valid for $\phi > 1$. Relation (17), together with the leading factors in the expansions about $\phi = 0, \infty$ obtained from Eq. (17)

$$p \sim \begin{cases} \left(\frac{7}{3}\,q\right)^{\frac{1}{2}}\left[1 + \frac{14}{3(3\nu+7)}\,q + \cdots\right] & \text{as } q \to 0, \\ (q+C)^{\frac{3}{2}}\left[1 - \frac{(3C-2)}{2(q+C)^{\nu+7/3}} + \cdots\right] & \text{as } q \to \infty, \end{cases} \tag{18}$$

have been plotted in Fig. 9. The solution follows the early asymptotic form $p \sim (7q/3)^{\frac{1}{2}}$ from reduced time $q = 0$ to about $q = 0.2$, when it begins to deviate toward the late asymptotic form $p = (q+C)^{3/2}$. Here the indicated reduced time $q = -C$ corresponds to the instant $t = 0$ [see Eq. (17)] when the liquid temperature passes the boiling point. The asymptotic bubble growth actually begins at $q = 0$ $(t = \tau = \Delta T/a)$.

In terms of R and t, the early and late asymptotic forms become

$$R \sim \begin{cases} 2\left(\frac{3}{\pi}\right)^{\frac{1}{2}}\left(\frac{k}{L\rho_v D^{\frac{1}{2}}}\right)_{T_b} \Delta T(t-\tau)^{\frac{1}{2}}, & (t \to \tau^+), \\ \frac{6C}{(7\pi)^{\frac{1}{2}}}\left(\frac{k}{L\rho_v D^{\frac{1}{2}}}\right)_{T_b} a t^{\frac{3}{2}}, & (t \to \infty). \end{cases} \tag{19}$$

One notes from Eq. (19) that bubbles starting to grow at later times τ (larger initial superheat $\Delta T = \tau/a$) than a given reference bubble in the liquid will at first grow relatively more rapidly than the reference bubble. They would in fact overtake the reference bubble if the early form in Eq. (19) were followed over the complete history of growth.

Actually, the growth rate of the reference bubble increases in such a way as to keep the reference bubble always larger than the late-starters. Eventually all bubbles approach the same late asymptotic rate.

This process is depicted graphically in Fig. 10 for the case of bubble growth in water at 18.8 atm under a superheat rate $a = 2000°C/sec$. Each member of the family of curves is identical to that in Fig. 9, except for the change of scale obtained by varying τ while keeping $R/\tau^{3/2}$ and t/τ constant in passing from one curve to another. The dashed curve is the late asymptotic form, corresponding to $p = (q + C)^{3/2}$, which evidently is a common envelope to members of the family.

IV. VOLUME BOILING IN A RAPIDLY HEATED LIQUID

The asymptotic solution (17) describes the growth of an isolated bubble in the liquid. In the present section we consider the summation of such solutions with respect to a distribution function giving the number of bubbles in the liquid which start to grow per unit time, in order to estimate the rate of production of vapor.

Such a procedure assumes that the conditions requisite to Eq. (17) apply—that the temperature and pressure can be considered uniform in the neighborhood of the bubble, with the pressure remaining constant—and also that bubbles present grow as though they were isolated and at rest in the liquid. Regarding the latter conditions, we may note that the growing vapor bubbles do not themselves cool the surrounding liquid,[8] and they grow so as to maintain a constant pressure. Moreover, bubble translation through the liquid due to buoyancy is not important here, since the time interval of interest is presumably on the order of a few hundredths of a second, depending on the rate of heating, during which time the bubbles are small and growing rapidly [see Fig. (10)]. Therefore, until bubbles begin to grow into one another, the assumption of bubble isolation may be considered valid.

The primary effect of the volume boiling is to cause a uniform expansion of the liquid-bubble mixture, so long as the pressure stays constant and the supposed rate of temperature rise in the liquid can be maintained. This dilation is due of course to the fact that a negligible amount of liquid is required to create a bubble. In effect, a number of balloons are being inflated in the liquid, which therefore must be displaced. Impediments to the flow, or volume constraints on the mixture, would increase the pressure and change the problem.

Let $V_\tau(t)$ be the volume at time t of a bubble appearing in the liquid at time τ, and let $dn(\tau)$ give the number of bubbles per cubic centimeter (cc) which start to grow between times τ and $\tau + d\tau$ in some region of the liquid. Then $s(t) = \int V_\tau(t)dn(\tau)$, summed from $\tau = 0$ to $\tau = t$, gives the volume of bubbles generated by each cc of liquid in the region by the time t. The total volume occupied by an original cc at this time will be $1 + s(t)$.

According to Eq. (17),

$$V_\tau(t) = \frac{4}{3}\pi R^3 = \begin{cases} 0 & \text{for } \tau > t, \\ \dfrac{4\pi}{3(7C)^{3/2}}\left(\dfrac{\lambda}{\tau}\right)^3 p^3 \tau^{9/2} & \text{for } 0 < \tau < t, \end{cases} \tag{20}$$

in which the factor λ/τ does not actually depend on τ. Summation with respect to $dn(\tau)$ yields, for the vapor produced per cc,

$$\left.\begin{aligned} s(t) &= \int_0^t V_\tau(t)\, dn(\tau) \\ &= \frac{4\pi}{3}\left(\frac{\lambda C}{7^{1/2}\tau}\right)^3 t^{9/2} \int_{x=0}^{x=1} p^3\left(\frac{x}{C}\right)^{9/2} dn(xt), \\ &\text{where} \\ x &= \frac{C}{\phi - (1 - C)\phi^{-(\nu+4/3)}}, \\ p^3 &= \phi[\phi^{7/3} - \phi^{-\nu}]^{3/2}. \end{aligned}\right\} \tag{21}$$

One may consider Eq. (21) to give a valid description of volume boiling until $s(t)$ becomes of order unity, provided the assumptions leading to Eq. (17) continue to apply and bubble translation relative to the liquid remains negligible.

In order to carry out the integration in Eq. (21), it is necessary to know the function $n(t)$, which counts the number of bubbles per cc that have appeared in the liquid by time t. An equivalent count would be the number of bubble nuclei larger than a given size (air content) in a cc of liquid under standard conditions. In practice, neither quantity is well known or easy to measure. On the other hand, there is some possibility of measuring the rate of vapor production in volume boiling, so that instead of using Eq. (21) to predict $s(t)$, one would use it to infer the behavior of $n(t)$.

In general, the inversion of Eq. (21) to give $n(t)$ in terms of $s(t)$ is a problem in linear transform

[8] Except that in immediate contact. Half the temperature drop in water can be shown to occur within about one-quarter of a bubble radius from the bubble wall.

theory. If we set

$$\left.\begin{aligned} \sigma(z) &= \int_0^\infty t^{-z-1}[t^{-9/2}s(t)]\,dt, \\ \beta(z) &= \frac{z}{B}\int_0^1 x^{z-1}\left(\frac{x}{C}\right)^{9/2} p^3(x)\,dx, \end{aligned}\right\} \qquad (22)$$

where $1/B \equiv (4\pi/3)(\lambda C/7^{1/2}\tau)^3$ is the constant factor in Eq. (21), then we obtain

$$n(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} t^z\left[\frac{\sigma(z)}{\beta(z)}\right] dz \qquad (\gamma > 0). \qquad (23)$$

Here the contour of integration is a line $Re(z) = \gamma$, parallel to the imaginary axis, along which the integral for $\sigma(z)$ converges.[9] The result (23) is an inversion formula for the relation

$$\sigma(z) = \frac{\beta(z)}{z}\int_0^\infty t^{-z}\,dn(t)$$

which follows from Eq. (21).

For the particular case that the vapor production is described by a power law, $t^{-9/2}s(t) = (\text{const})t^k$, the bubble count will follow the same law. In this instance, Eqs. (21) or (23) reduce to

$$t^{-9/2}s(t) = (\text{const})t^k = n(t)\beta(k),$$

where $\beta(k)$ is the function defined by Eq. (22) at $z = k$. An interpolation formula for $\beta(k)$ which is correct at $k = 0$ and asymptotically correct for large k is given by

$$B\beta(k) \approx (1 + 0.308k)^{-3/2}.$$

Hence if we define k by $d\ln(t^{-9/2}s)/d\ln t$ when $s(t)$ does not actually obey a power law, we obtain the interpolation formula for $n(t)$

$$n(t) \approx Bt^{-9/2}s(t)\left\{1 + 0.308\,\frac{d\ln[t^{-9/2}s(t)]}{d\ln t}\right\}^{3/2}, \qquad (24)$$

with $B = (3/4\pi)(7^{1/2}\tau/\lambda C)^3 = 1.051/a^3K_b^3$.

To afford an example of the use of Eq. (24), we may note that for $s(t) = V_\tau(t)$ the correct value of $n(t)$ is zero for $t < \tau$ and one (bubble per cc) for $t > \tau$. For this choice of s, Eq. (24) indicates $n = 0$ for $t < \tau$ and $n \sim 1$ for large t; however, it gives an overshoot to 1.38 at $t = \tau^+$ before dropping toward one with increasing t.

[9] $\sigma(z)$ will exist in a right half-plane if $s(t)$ is defined to vanish for $t < \epsilon$, say, where ϵ is small but finite.

ACKNOWLEDGMENTS

The author wishes to thank Professor M. S. Plesset for suggesting this problem and for many helpful discussions. He also wishes to thank Miss Zora Lindberg for her help with the numerical computations.

This work was supported by the Office of Naval Research.