Unitarity Limits on the Mass and Radius of Dark-Matter Particles

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Using partial-wave unitarity and the observed density of the Universe, we show that a stable elementary particle which was once in thermal equilibrium cannot have a mass greater than 340 TeV. An extended object which was once in thermal equilibrium cannot have a radius less than 7.5×10^{-7} fm. A lower limit to the relic abundance of such particles is also found.

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The idea that the dark matter known to exist in galactic halos consists of some, as yet undiscovered, stable massive particle has received a great deal of attention in recent years. Dozens of particle candidates have been suggested and new ones are constantly being proposed. Most of these dark-matter candidates have relic abundances which are calculated in the "Lee-Weinberg" manner and model parameters are typically adjusted to allow their density today to be near critical density, $\Omega_X \approx 1$. Unfortunately, as a result of the plethora of models and limited experimental information, the properties of consistent dark-matter candidates are, to a large extent, unconstrained, thereby hindering the verification or exclusion of the particle solution to the dark-matter problem. Clearly, "model-independent" constraints are desirable. In this Letter we present a model-independent constraint on the mass of elementary-particle dark-matter candidates. For almost any such particle which was once in thermal equilibrium, partial-wave unitarity of the S matrix bounds the annihilation cross section in the early Universe, which in turn bounds the relic abundance and the mass of the particle. In general, we find that stable elementary particles with masses greater than around 340 TeV are very likely excluded. Extended objects[1] with radii less than 7.5×10^{-7} fm are also very likely excluded. While the mass upper limit we find is not rigorous and rather high, we still feel it may be of some interest because of its general nature.

The relic abundance of a particle species X which was once in thermal equilibrium is determined by its total thermally averaged annihilation cross section $\langle\sigma(X\bar{X}\to \text{all})v_{\rm rel}\rangle$ at freezeout. At high temperatures the number density of X's is roughly the same as the number density of photons, but as the temperature drops below the mass of the X, their number density drops exponentially. This continues until the total annihilation cross section is no longer large enough to maintain equilibrium and the X number density then "freezes out." The number density today is given roughly by[2]

$$\Omega_X h^2 = \frac{1.07\times10^9 (n+1) x_f^{n+1}\ \text{GeV}^{-1}}{g_*^{1/2} m_{\rm Pl} \langle\sigma v_{\rm rel}\rangle'_f} \approx \frac{3\times10^{-27}\ \text{cm}^3/\text{sec}}{\langle\sigma v_{\rm rel}\rangle_f}, \qquad (1)$$

where $\Omega_X = \rho_X/\rho_{\rm crit}$ is the present average density of X's divided by the critical density, $\frac{1}{2}\le h \le 1$ is the Hubble constant in units of 100 km/sec Mpc, $x_f = m_X/T_f$, T_f is the freezeout temperature, $g_* \approx 107$ is the effective number of degrees of freedom at T_f, and $m_{\rm Pl} = 1.22 \times 10^{19}$ GeV.

Since the $X\bar{X}$ annihilations at freezeout occur at nonrelativistic velocities ($v \approx \frac{1}{4} \ll 1$), one can expand the cross section in powers of $v^2 \equiv v_{\rm rel}^2/4$ and keep only the first (or first two) terms. In thermal averaging one replaces $\langle v_{\rm rel}^2\rangle$ by $6/x_f$ and so in Eq. (1) the cross section is written $\langle\sigma v_{\rm rel}\rangle = \langle\sigma v_{\rm rel}\rangle' x^{-n}$, where n parameterizes the dependence of the cross section on x. The freezeout temperature is given roughly by[2]

$$x_f = \ln B - (n+\tfrac{1}{2}) \ln\ln B, \qquad (2)$$

where $B = 0.038 g m_{\rm Pl} m_X \langle\sigma v_{\rm rel}\rangle'/\sqrt{g^*}$ and g is the number of degrees of freedom of the X particle. Typically $x_f \approx 25$ corresponding to $v_{\rm rel}^2/4 \approx \frac{1}{16}$ at freeze out. Please note that $v_{\rm rel}$ is not really a velocity, but is related to the flux factor. It is defined as $v_{\rm rel} = 2v$, and so $0 \le v_{\rm rel}^2/4 \le 1$.

From Eq. (1) we see that if $\langle\sigma v_{\rm rel}\rangle_f \ll 3\times10^{-27}$ cm^3/sec, then $\Omega_X h^2 \gg 1$, which would be inconsistent with the "observation," $\Omega_{\rm tot} h^2 \le 1$, obtained from the age of the Universe. Any particle model which predicts an annihilation cross section smaller than this critical value at $v_{\rm rel}^2/4 \approx \frac{1}{16}$ is therefore inconsistent with cosmology.[3] We will now show that partial-wave unitarity provides a maximum possible cross section and therefore a minimum possible $\Omega_X h^2$. Extremely massive elementary particles and very small extended objects violate these bounds and therefore are inconsistent with cosmology.

Consider the process $a+b\rightarrow c+d$ and the scattering matrix

$$\langle f|S|i\rangle=\langle f|i\rangle+i(2\pi)^4\delta^4(P_f-P_i)\langle f|T|i\rangle\,, \qquad (3)$$

where $P_i=p_a+p_b$ and $P_f=p_c+p_d$. The T matrix can be expanded in partial waves using the helicity formalism,[4]

$$\langle\lambda_c\lambda_d|T(s,\Omega)|\lambda_a\lambda_b\rangle =8\pi s^{1/2}e^{i\phi(\lambda-\lambda')}\sum_J(2J+1)d^J_{\lambda\lambda'}(\theta)\langle\lambda_c\lambda_d|T_J(s)|\lambda_a\lambda_b\rangle\,, \qquad (4)$$

where $\lambda_a,\ldots,\lambda_d$ are the helicities of particles $a,\ldots,d$, $\lambda=\lambda_a-\lambda_b$, $\lambda'=\lambda_c-\lambda_d$, s is the Mandelstam variable, $\Omega=(\theta,\phi)$ is the center-of-mass scattering angle, and $d^J_{\lambda\lambda'}$ are the Wigner functions.

Using matrix notation[5] $\langle\lambda_c\lambda_d|T_J(s)|\lambda_a\lambda_b\rangle=(T_J)_{if}$ and $\tilde p_k=\mathrm{diag}(p_1,p_2,\ldots)$, where p_k is the center-of-mass three-momentum of particle system i, f, etc., partial-wave unitarity of the S matrix can be written[5]

$$T_J-T_J^\dagger=2iT_J\tilde pT_J^\dagger\,. \qquad (5)$$

Defining $S_J=1+2i\tilde p^{1/2}T_J\tilde p^{1/2}$, we see that partial-wave unitarity can also be written $S_JS_J^\dagger=1$ or

$$|S_{\mathrm{el},J}|^2+\sum_f|S_{i\neq f,J}|^2=1\,, \qquad (6)$$

where $S_{\mathrm{el},J}$ stands for the elastic channel, $i=f$. The next step is to define $S_{\mathrm{el},J}=\eta_Je^{2i\delta_J}$, where δ_J is a real phase shift and η_J is an inelasticity factor, $0\le\eta_J\le1$. Then $|S_{\mathrm{el},J}|^2=\eta_J^2$, and $\sum_f|S_{i\neq f,J}|^2=1-\eta_J^2$. Finally, using $T_{\mathrm{el},J}=(S_{\mathrm{el},J}-1)/2ip$ and $T_{f\neq i,J}=S_{f\neq i,J}/2i(p_ip_f)^{1/2}$, and the standard formula for the unpolarized cross section in terms of partial waves $\sigma=\sum\sigma_J$, where

$$\sigma_J=\frac{4\pi(2J+1)}{(2s_a+1)(2s_b+1)}\sum_\lambda\sum_f\frac{p_f}{p_i}|T_{if,J}|^2\,, \qquad (7)$$

we find the result of Pilkuhn[5]

$$\sigma_{r,J}=4\pi\frac{2J+1}{(2s_a+1)(2s_b+1)}\sum_\lambda\sum_{f\neq i}\frac{p_f}{p_i}|T_{if,J}|^2 =\frac{\pi(2J+1)(1-\eta_J^2)}{p_i^2}\,. \qquad (8)$$

Here $\sigma_{r,J}$ is the "reaction" cross section, that is, the total cross section minus the elastic piece. It has a maximum when $\eta_J=0$, so we conclude that

$$\sigma_J(a+b\rightarrow c+d)\le\pi(2J+1)/p_i^2\,. \qquad (9)$$

In the early Universe,

$$p_i^2=E^2-m_X^2=\frac{m_X^2v_{\mathrm{rel}}^2}{4(1-v_{\mathrm{rel}}^2/4)}\approx\frac{m_X^2v_{\mathrm{rel}}^2}{4}\,,$$

so $\sigma_Jv_{\mathrm{rel}}\le(\sigma_J)_{\max}v_{\mathrm{rel}}$, where

$$(\sigma_J)_{\max}v_{\mathrm{rel}}\approx\frac{4\pi(2J+1)}{m_X^2v_{\mathrm{rel}}} \approx\frac{3\times10^{-22}(2J+1)\,\mathrm{cm^3/sec}}{[m_X/(1\ \mathrm{TeV})]^2}\,. \qquad (10)$$

In order to apply the limits of Eq. (10) to the annihilation in the early Universe we need to determine which partial waves contribute. After summing over helicities, the angular dependence, $\cos\theta$, which indicates the partial wave, enters the cross section only through the Mandelstam variable

$$t=m_a^2+m_c^2-2E_aE_c+2p_cp_a\cos\theta =m_a^2+m_c^2-2E_aE_c+2p_c\cos\theta\, m_Xv_{\mathrm{rel}}/2 +O(v_{\mathrm{rel}}^2/4)\,. \qquad (11)$$

So there is a factor of v_{rel} appearing with every factor of $\cos\theta$. In the expansion of the annihilation cross section in powers of $v_{\mathrm{rel}}^2/4\approx\frac{1}{16}$, the lowest-order term $O((v_{\mathrm{rel}}^2/4)^0)$ therefore has no angular dependence and must be a $J=0$ partial wave. The $J=1$ partial wave is smaller by a factor $v_{\mathrm{rel}}^2/4$, and the higher partial waves are further suppressed. In fact, since partial-wave unitarity must hold for any value of $v_{\mathrm{rel}}^2/4$, and when $v_{\mathrm{rel}}^2/4$ increases, the maximum cross section, Eq. (10), decreases, the $J=0$ bound, taken when $v_{\mathrm{rel}}^2/4\approx\frac{1}{16}$, is not as stringent as possible. The $J=1$ maximum cross section also decreases for larger $v_{\mathrm{rel}}^2/4$, and more importantly, the term in the actual cross section of order $v_{\mathrm{rel}}^2/4$ increases. If the $J=1$ bound is satisfied for a larger value of $v_{\mathrm{rel}}^2/4$, for instance $v_{\mathrm{rel}}^2/4\approx\frac{1}{2}$, then the $J=1$ partial wave is below the bound by a factor of $8^{-3/2}\approx0.04$ by freezeout. We conclude that it is more than adequate to use only the $J=0$ partial wave in finding a bound.

Now we use Eqs. (1), (2), and (10) to bound Ω_Xh^2 and m_X. Including only the $n=0$ part of the cross section and replacing v_{rel} by $(6/x_f)^{1/2}$, we find that

$$\Omega_Xh^2\ge1.7\times10^{-6}\sqrt{x_f}[m_X/(1\ \mathrm{TeV})]^2 \qquad (12)$$

for a Majorana fermion with $g=2$. For a Dirac fermion, Ω_Xh^2 is a factor of 2 larger. Now using $\Omega_Xh^2\le1$, we find the mass limit

$$m_X\le340\ \mathrm{TeV}\,, \qquad (13)$$

and $x_f\approx28$. Equation (13) was found for a Majorana fermion. The limit for a scalar particle is similar, while for a Dirac fermion it is about a factor of $\sqrt2$ smaller, that is, $m_X\le240$ TeV. This is the main result of this Letter.

Another, more conservative, way of finding the mass bound is to assume that the cross section Eq. (10) holds throughout the period of annihilation and freezeout. In this case, the v_{rel}^{-1} factor affects the thermal averaging and the integration from freezeout to today. The thermally averaged maximum cross section becomes

$$\langle (\sigma_J)_{\max} v_{\rm rel} \rangle \approx \frac{4\pi(2J+1)}{m_X^2} \left(\frac{x^{1/2}}{\sqrt{\pi}} \right), \qquad (14)$$

and the relic abundance is given by Eq. (1), with $n = -\frac{1}{2}$,

$$\Omega_X h^2 \geq \frac{6.0\times 10^{-7}}{2J+1} x_f^{1/2} \left(\frac{m_X}{1\ \mathrm{TeV}} \right)^2 . \qquad (15)$$

The freezeout temperature is the same as before with $\langle \sigma v_{\rm rel} \rangle'$ multiplied by a factor of $(6/\pi)^{1/2}$. [We set $n = -\frac{1}{2}$ in Eq. (2), both now and before, since the $x_f^{1/2}$ here is just an algebraic factor.] Using these formulas, the mass limit becomes $m_X < 550$ TeV. This is probably an overly conservative bound since one does not expect $\sigma v_{\rm rel} \propto v_{\rm rel}^{-1}$ for annihilation channels in a nonrelativistic expansion.

However, we do not claim that the derivation leading to Eq. (13) is rigorous, or that exceptions cannot occur. For example, elastic scattering via t-channel exchange of a massless particle gives rise to a term in the matrix element proportional to $t^{-1} \propto v_{\rm rel}^{-2}(1-\cos\theta)^{-1}$. Naively expanding this would suggest that all partial waves contribute to the term of lowest order in $v_{\rm rel}^2/4$. The problem, in this case, is that we are outside the Lehmann ellipse of convergence, and the partial-wave expansion is not valid. Fortunately, in annihilation, the mass of the annihilation product must be less than m_X, and the partial-wave expansion converges, giving nicely the results we claim above. Another possible exception, which we do not consider very likely, is that the coefficients of the partial-wave expansion contain factors of $(s-4m_X^2)^{-1} \propto v_{\rm rel}^{-2}$, in just such a way as to cancel the $v_{\rm rel}^2$ factors associated with the $\cos^2\theta$ factors. For elastic scattering, it can be proved that this cannot occur (Ref. 5, p. 291), but we have been unable to complete the proof for the inelastic case. This may be related to the possibility of s-channel poles, which can cause another possible exception to our limit. A factor of $(s-m_i^2)^{-1}$, with $m_i = 2m_X$, will give an additional factor of $v_{\rm rel}^{-2}$, in which case partial waves up to $J=2$ need to be included in our maximum cross section, and the mass limit weakens. However, we feel that such a pole is unlikely. It requires not only an exchange particle of precisely twice the mass of the X, but also that the exchange particle be nearly stable. The width of the exchange particle will dominate the pole unless it is very small, and since the exchanged particle is more massive than the X, and has decay channels into lighter particles, we consider this possibility remote.

We note that the mass limit, Eq. (13), involves a mass somewhat higher than typically considered in particle-dark-matter model building. But since the bound is model independent we feel it may be of some use. We can immediately apply it to candidates which appear in the literature. For example, Brahm and Hall[6] have recently found that SU(2)-singlet fermions with an additional U(1)′ symmetry make suitable dark-matter candidates as long as their mass does not exceed 40 TeV. A cosmological upper bound on the mass of another dark-matter candidate, the neutralino, has also been recently found.[7] We believe that both of these mass limits are examples of the unitarity mass limit, Eq. (13).

One may also consider applying the mass limit to the "original" cold-dark-matter candidate, the Dirac neutrino. Dolgov and Zeldovich[8] claimed a range of neutrino masses, 3 GeV $< m_\nu <$ 3 TeV, as being cosmologically acceptable. Their upper bound, consistent with ours, was based on neutrino annihilation into fermions through Z-boson exchange whose cross section is proportional to m_ν^{-2} in the high-mass limit. However, Enqvist, Kainulainen, and Maalampi[9] correctly noted that the W^+W^- channels, among others, open up for very massive neutrinos, and that these new channels dominate the cross section in the high-mass limit. In fact, they claimed that because the matrix element keeps growing as m_ν increases, there is *no* upper limit from cosmology on Dirac neutrino masses. This claim is at variance with our result for Dirac fermions. We believe that the solution[10] to this apparent paradox is that as $m_\nu \to \infty$, the neutrino Yukawa coupling becomes large and the perturbative calculation of Enqvist, Kainulainen, and Maalampi is not applicable. In fact, by using unitarity to bound the largest eigenvalue of the scattering matrix, Chanowitz, Furman, and Hinchliffe[11] showed that the breakdown of perturbation theory occurs at around $m_\nu \approx 1$ TeV, consistent with, but far below, the limit we set. Above this mass we enter murky waters and it is not clear that the neutrino is a viable dark matter candidate or that it would remain a "neutrino." The breakdown of perturbation theory suggests that it becomes "strongly interacting" and would not exist as a free, stable state. If, on the other hand, the neutrino for some reason (unknown to us) stays "elementary," we argue that our limit applies, giving an upper limit on the neutrino mass from cosmology.

Finally, we should comment on the applicability of these bounds to stable extended objects,[1] should such states exist. For these objects, higher partial waves will generally contribute to the nonrelativistic cross section, and the cosmological mass bound, Eq. (13) does not apply; however, partial-wave unitarity may still be used to limit the total annihilation cross section, and cosmology provides a constraint on the size of such objects. Consider an extended object with spin 0 and radius R_X. The highest partial wave that can contribute to the particle-antiparticle collision is roughly $J_{\max} = 2m_X v_{\rm rel} R_X$, resulting in a maximum total cross section,

$$(\sigma v_{\rm rel})_{\max} \approx \frac{4\pi}{m_X^2 v_{\rm rel}} \sum_{J=0}^{J_{\max}} (2J+1) \approx 16\pi R_X^2 v_{\rm rel}\,, \quad (16)$$

4 times the geometric cross section. Using Eqs. (1), (2), and (16), we can now bound $\Omega_X h^2$ and R_X. We find that

$$\Omega_X h^2 \geq \frac{4\times 10^{-15} x_f^{3/2}}{[R_X/(1\ \mathrm{fm})]^2}, \tag{17}$$

which leads to the bound

$$R_X \geq 7.5\times 10^{-7}\ \mathrm{fm}. \tag{18}$$

Here we used $x_f = 27$ which was obtained from Eq. (2) using $m_X = 1000$ TeV; the radius limit, Eq. (18), varies only logarithmically with m_X. The limit for spin-$\frac{1}{2}$ particles is more stringent by a factor of $\sqrt{2}$.

We point out that Eq. (16) is valid only if $J_{\max} \gg 1$. On the other hand, if $J_{\max} \ll 1$, the cross section is bound by Eq. (9) with $J=0$. Since freezeout occurs when $v_{\mathrm{rel}} \approx \frac{1}{2}$, Eq. (18) is reliable only when $R_X \gg 1/m_X$, while an object with $R_X \ll 1/m_X$ must be considered pointlike and its mass limited by Eq. (13). Furthermore, we note that there is no major discontinuity in the overlap region, $R_X \sim 1/m_X$, since the mass limit for pointlike particles, Eq. (13), is very nearly that which we would have obtained from the radius limit, Eq. (18), had we used the Compton wavelength of the particle for R_X.

Of course, if some process such as a quark-hadron or electroweak phase transition, out-of-equilibrium decay of a massive particle, or inflation produces a significant amount of entropy after freezeout, the relic abundance is diluted and our limits are weakened accordingly. Nevertheless, although our derivation is not rigorous, and exceptions may exist, we believe that the limit on mass, Eq. (13), radius, Eq. (18), and relic abundance, Eq. (12), is of great interest and applies to many (if not most) dark-matter candidates.

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[1]"Elementary particles" refers to fundamental pointlike particles (for example, leptons), while "extended objects" describes composite particles such as protons or neutrons.

[2]E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, 1989).

[3]Note that if a cosmic asymmetry exists between X's and $\bar{X}$'s, the above formulas do not apply. However, the relic density is always *larger* in this case and so all the bounds just strengthen.

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[9]K. Enqvist, K. Kainulainen, and J. Maalampi, Nucl. Phys. **B317**, 647 (1989).

[10]We would like to thank David Seckel for pointing this puzzle out to us and Scott Willenbrock for indicating its solution. Actually, Dolgov and Zeldovich (Ref. 8) give a somewhat similar caveat to ours.

[11]M. S. Chanowitz, M. A. Furman, and I. Hinchliffe, Nucl. Phys. **B153**, 402 (1979).