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Rényi generalizations of the conditional quantum mutual information

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The conditional quantum mutual information $I(A;B|C)$ of a tripartite state ρ_{ABC} is an information quantity which lies at the center of many problems in quantum information theory. Three of its main properties are that it is non-negative for any tripartite state, that it decreases under local operations applied to systems A and B, and that it obeys the duality relation $I(A;B|C) = I(A;B|D)$ for a four-party pure state on systems $ABCD$. The conditional mutual information also underlies the squashed entanglement, an entanglement measure that satisfies all of the axioms desired for an entanglement measure. As such, it has been an open question to find Rényi generalizations of the conditional mutual information, that would allow for a deeper understanding of the original quantity and find applications beyond the traditional memoryless setting of quantum information theory. The present paper addresses this question, by defining different α-Rényi generalizations $I_\alpha(A;B|C)$ of the conditional mutual information, some of which we can prove converge to the conditional mutual information in the limit $\alpha \to 1$. Furthermore, we prove that many of these generalizations satisfy non-negativity, duality, and monotonicity with respect to local operations on one of the systems A or B (with it being left as an open question to prove that monotonicity holds with respect to local operations on both systems). The quantities defined here should find applications in quantum information theory and perhaps even in other areas of physics, but we leave this for future work. We also state a conjecture regarding the monotonicity of the Rényi conditional mutual informations defined here with respect to the Rényi parameter α. We prove that this conjecture is true in some special cases and when α is in a neighborhood of one. [http://dx.doi.org/10.1063/1.4908102]

I. INTRODUCTION

How much correlation do two parties have from the perspective of a third? This kind of correlation is what the conditional quantum mutual information (CQMI) quantifies. Indeed, let ρ_{ABC} be a density operator corresponding to a quantum state shared between three parties, say, Alice, Bob, and Charlie. Then the conditional quantum mutual information is defined as

$$I(A;B|C)_\rho \equiv H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho, \tag{1.1}$$

where $H(F)_\sigma \equiv -\mathrm{Tr}\{\sigma_F \log \sigma_F\}$ is the von Neumann entropy of a state σ_F on system F and we unambiguously let $\rho_C \equiv \mathrm{Tr}_{AB}\{\rho_{ABC}\}$ denote the reduced density operator on system C, for example. Refs. 19 and 74 provided a compelling operational interpretation of the conditional quantum mutual information in terms of the quantum state redistribution protocol: given many independent copies of a four-party pure state ψ_{ADBC}, with a sender possessing the D and B systems, a receiver possessing the C systems, and the sender and receiver sharing noiseless entanglement

before communication begins, the optimal rate of quantum communication necessary to transfer the B systems to the receiver is given by $\frac{1}{2}I(A;B|C)_\psi$.

It is a nontrivial fact, known as strong subadditivity of quantum entropy,[45,46] that the conditional quantum mutual information of any tripartite quantum state is non-negative. This can be viewed as a general constraint imposed on the marginal entropy values of arbitrary tripartite quantum states. Strong subadditivity also implies that the conditional mutual information can never increase under local quantum operations performed on the systems A and B,[13] so that $I(A;B|C)_\rho$ is a sensible measure of the correlations present between systems A and B, from the perspective of C. That is, the following inequality holds

$$I(A;B|C)_\rho \geq I(A';B'|C)_\omega, \tag{1.2}$$

where $\omega_{A'B'C} \equiv (\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'})(\rho_{ABC})$ with $\mathcal{N}_{A\to A'}$ and $\mathcal{M}_{B\to B'}$ arbitrary local quantum operations performed on the input systems A and B, leading to output systems A' and B', respectively. Inequalities like these are extremely useful in applications, with nearly all coding theorems in quantum information theory invoking the strong subadditivity inequality in their proofs.

One of the most fruitful avenues of research in quantum information theory has been the program of generalizing entropies beyond those that are linear combinations of the von Neumann entropy.[52,54,17,62,71,50,72,20] Not only is this interesting from a theoretical perspective but more importantly, these generalizations have found application in operational settings in which there is no assumption of many independent and identically distributed (i.i.d.) systems, so that the law of large numbers does not come into play. In particular, the family of Rényi entropies has proved to possess a wide variety of applications in these non-i.i.d. settings. More recently, researchers have shown that nearly all of the known information quantities being employed in the non-i.i.d. setting are special cases of a Rényi family of quantum entropies.[50,5]

However, in spite of this aforementioned progress, it has been a vexing open question to determine a Rényi generalization of the conditional quantum mutual information that can be useful in applications. On the one hand, a potential Rényi generalization of the conditional mutual information of a tripartite state ρ_{ABC} consists of simply taking a linear combination of Rényi entropies. For example, in analogy with the definition in (1.1), one could define a Rényi generalization of the conditional mutual information as follows:

$$I'_\alpha(A;B|C)_\rho \equiv H_\alpha(AC)_\rho + H_\alpha(BC)_\rho - H_\alpha(C)_\rho - H_\alpha(ABC)_\rho, \tag{1.3}$$

where $H_\alpha(F)_\sigma \equiv [1-\alpha]^{-1}\log \operatorname{Tr}\{\sigma_F^\alpha\}$ is the Rényi entropy of a state σ_F on system F, with parameter $\alpha \in (0,1)\cup(1,\infty)$ (with the Rényi entropy being defined for $\alpha \in \{0,1,\infty\}$ in the limit as α approaches 0, 1, and ∞, respectively). Although this quantity is non-negative in some very special cases,[2] in general, $I'_\alpha(A;B|C)_\rho$ can be negative, and in fact there are some simple examples of states for which this occurs. Furthermore, the results of Ref. 47 imply that there are generally no linear inequality constraints on the marginal Rényi entropies of a multiparty quantum state other than non-negativity when $\alpha \in (0,1)\cup(1,\infty)$. This implies that monotonicity under local quantum operations generally does not hold for $I'_\alpha(A;B|C)_\rho$, and Ref. 47 provides many examples of four-party states ρ_{ABCD} such that $I'_\alpha(A;BD|C)_\rho < I'_\alpha(A;B|C)_\rho$. For these reasons, we feel that formulas like that in (1.3) should not be considered as Rényi generalizations of the conditional quantum mutual information, given that non-negativity and monotonicity under local operations are two of the basic properties of the conditional quantum mutual information, which are consistently employed in applications. However, one could certainly argue that the case $\alpha = 2$ is useful for the class of Gaussian quantum states, as done in Ref. 2.

On the other hand, the standard approach for generalizing information quantities such as entropy, conditional entropy, and mutual information beyond the von Neumann setting begins with the realization that these quantities can be written in terms of the Umegaki relative entropy $D(\rho\|\sigma)$[70]

$$H(A)_\rho = -D(\rho_A\|I_A), \tag{1.4}$$

$$H(A|B)_\rho \equiv H(AB)_\rho - H(B)_\rho = -\min_{\sigma_B} D(\rho_{AB}\|I_A\otimes\sigma_B), \tag{1.5}$$

$$I(A;B)_\rho \equiv H(A)_\rho + H(B)_\rho - H(AB)_\rho = \min_{\sigma_B} D(\rho_{AB}\|\rho_A\otimes\sigma_B), \tag{1.6}$$

where

$$D(\rho\|\sigma) \equiv \begin{cases} [\mathrm{Tr}\{\rho\}]^{-1}[\mathrm{Tr}\{\rho\log\rho\} - \mathrm{Tr}\{\rho\log\sigma\}] & \text{if } \mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma) \\ +\infty & \text{otherwise} \end{cases}. \tag{1.7}$$

Note that the unique optimum σ_B in (1.5) and (1.6) turns out to be the reduced density operator ρ_B. The Rényi relative entropy of order $\alpha \in [0,1) \cup (1,\infty)$ is defined as[52]

$$D_\alpha(\rho\|\sigma) \equiv \begin{cases} \frac{1}{\alpha-1}\log\mathrm{Tr}\left\{[\mathrm{Tr}\{\rho\}]^{-1}\rho^\alpha\sigma^{1-\alpha}\right\} & \text{if } \mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma) \text{ or } (\alpha \in [0,1) \text{ and } \rho \not\perp \sigma) \\ +\infty & \text{otherwise} \end{cases}, \tag{1.8}$$

with the support conditions established in Ref. 65. Using this quantity, one can easily define Rényi generalizations of entropy, conditional entropy, and mutual information in analogy with the above formulations

$$H_\alpha(A)_\rho = -D_\alpha(\rho_A\|I_A), \tag{1.9}$$

$$H_\alpha(A|B)_\rho \equiv -\min_{\sigma_B} D_\alpha(\rho_{AB}\|I_A \otimes \sigma_B), \tag{1.10}$$

$$I_\alpha(A;B)_\rho \equiv \min_{\sigma_B} D_\alpha(\rho_{AB}\|\rho_A \otimes \sigma_B). \tag{1.11}$$

Since the Rényi relative entropy obeys monotonicity under quantum operations for $\alpha \in [0,1) \cup (1,2]$,[52] in the sense that $D_\alpha(\rho\|\sigma) \geq D_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$ for a quantum operation $\mathcal{N}$, the above generalizations have proven useful in several applications (see Refs. 41, 48, and 62 and references therein).

II. OVERVIEW OF RESULTS

The main purpose of the present paper is to develop Rényi generalizations of the conditional quantum mutual information that satisfy the aforementioned properties of non-negativity, monotonicity under local quantum operations, and duality. We come close to achieving this goal by showing that non-negativity, duality, and monotonicity under local operations on one of the systems A or B hold for many of our Rényi generalizations. Numerical evidence has not falsified monotonicity under local operations holding for both systems A and B, but it remains an open question to determine if this holds for both systems A and B. Nevertheless, we think the quantities defined here should be useful in applications in quantum information theory, and they might even find use in other areas of physics.[30,9,29,36,39,27,38]

After establishing some notation and recalling definitions in Sec. III, our starting point is in Sec. IV, where we recall that the conditional quantum mutual information of a tripartite state ρ_{ABC} can be written in terms of the relative entropy as follows (see "*Proof of (1.5)*" in Ref. 46):

$$I(A;B|C)_\rho = D\left(\rho_{ABC}\|\exp\{\log\rho_{AC} + \log\rho_{BC} - \log\rho_C\}\right). \tag{2.1}$$

We then recall the following generalized Lie-Trotter product formula from Ref. 60, with the particular form below being inspired from developments in Ref. 43

$$\exp\{\log\rho_{AC} + \log\rho_{BC} - \log\rho_C\} = \lim_{\alpha\to 1}\left[\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\rho_{BC}^{1-\alpha}\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\right]^{1/(1-\alpha)}, \tag{2.2}$$

where we assume that the operators ρ_{AC}, ρ_{BC}, and ρ_C are invertible. The relation above suggests a number of Rényi generalizations of the relative entropy formulation in (2.1), one of which is

$$D_\alpha\left(\rho_{ABC}\left\|\left[\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\rho_{BC}^{1-\alpha}\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\right]^{1/(1-\alpha)}\right.\right) = \frac{1}{\alpha-1}\log\mathrm{Tr}\left\{\rho_{ABC}^\alpha\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\rho_{BC}^{1-\alpha}\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\right\}. \tag{2.3}$$

We prove that several of these Rényi conditional mutual informations are non-negative for $\alpha \in [0,1) \cup (1,2]$ and obey monotonicity under local quantum operations on one of the systems A or

B in the same range of α (with the proof following from the Lieb concavity theorem[44] and the Ando convexity theorem[4]). Our proof for monotonicity under local operations depends on operator orderings in the particular Rényi generalization of the conditional mutual information. For example, we can show that monotonicity under operations on the B system holds for the quantity defined in (2.3), due to the fact that ρ_{BC} is "placed in the middle." We also consider several limiting cases, the most important of which is the limit as $\alpha \to 1$. We prove that some of the α-Rényi conditional mutual informations converge to $I(A;B|C)_\rho$ in this limit. Note that classical and quantum quantities related to these have been explored in prior work.[6,21]

The sandwiched Rényi relative entropy[50,72] is another variant of the Rényi relative entropy which has found a number of applications recently in the context of strong converse theorems.[72,49,26,15,67] It is defined for $\alpha \in (0,1) \cup (1,\infty)$ as follows:

$$\widetilde{D}_\alpha(\rho\|\sigma) \equiv \begin{cases} \frac{1}{\alpha-1}\log\left[[\mathrm{Tr}\{\rho\}]^{-1}\mathrm{Tr}\left\{\left(\sigma^{(1-\alpha)/2\alpha}\rho\sigma^{(1-\alpha)/2\alpha}\right)^\alpha\right\}\right] & \text{if } \mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma) \text{ or } (\alpha \in (0,1) \text{ and } \rho \not\perp \sigma) \\ +\infty & \text{otherwise} \end{cases}. \tag{2.4}$$

In Sec. VI, we use this sandwiched Rényi relative entropy to establish a number of sandwiched Rényi generalizations of the conditional mutual information, one of which is

$$\begin{aligned}\widetilde{D}_\alpha\left(\rho_{ABC}\left\|\left[\rho_{AC}^{(1-\alpha)/2\alpha}\rho_C^{(\alpha-1)/2\alpha}\rho_{BC}^{(1-\alpha)/\alpha}\rho_C^{(\alpha-1)/2\alpha}\rho_{AC}^{(1-\alpha)/2\alpha}\right]^{\alpha/(1-\alpha)}\right.\right) \\ = \frac{1}{\alpha-1}\log\mathrm{Tr}\left\{\left(\rho_{ABC}^{1/2}\rho_{AC}^{(1-\alpha)/2\alpha}\rho_C^{(\alpha-1)/2\alpha}\rho_{BC}^{(1-\alpha)/\alpha}\rho_C^{(\alpha-1)/2\alpha}\rho_{AC}^{(1-\alpha)/2\alpha}\rho_{ABC}^{1/2}\right)^\alpha\right\},\end{aligned} \tag{2.5}$$

where the equality follows from the fact that

$$\mathrm{Tr}\left\{\left(\sigma^{(1-\alpha)/2\alpha}\rho\sigma^{(1-\alpha)/2\alpha}\right)^\alpha\right\} = \mathrm{Tr}\left\{\left(\rho^{1/2}\sigma^{(1-\alpha)/\alpha}\rho^{1/2}\right)^\alpha\right\}. \tag{2.6}$$

Although both Rényi generalizations of the conditional mutual information feature "operator sandwiches," we give this particular generalization the epithet "sandwiched" because it is derived from the sandwiched Rényi relative entropy. We prove that several of these sandwiched Rényi conditional mutual informations are non-negative for all $\alpha \in [1/2,1) \cup (1,\infty)$ and that they are monotone under local quantum operations on one of the systems A or B for the same range of α (with the proof following from recent work in Refs. 31 and 24). We can prove that some of them converge to $I(A;B|C)_\rho$ in the limit as $\alpha \to 1$, and there are other interesting quantities to consider for $\alpha = 1/2$ or $\alpha = \infty$, leading to a min- and max-version of conditional mutual information, respectively. There are certainly other possible definitions for Rényi conditional mutual information that one could consider and we discuss these in the conclusion.

One of the most curious non-classical properties of the conditional quantum mutual information is that it obeys a duality relation.[19,74] That is, for a four-party pure state ψ_{ABCD}, the following equality holds

$$I(A;B|C)_\psi = I(A;B|D)_\psi. \tag{2.7}$$

In Sec. VII, we prove that some variants of the Rényi conditional mutual information obey duality relations analogous to the above one.

A well known property of both the traditional and the sandwiched Rényi relative entropies is that they are monotone non-decreasing in α. That is, for $0 \le \alpha \le \beta$, we have the following inequalities:[65,50]

$$D_\alpha(\rho\|\sigma) \le D_\beta(\rho\|\sigma), \qquad \widetilde{D}_\alpha(\rho\|\sigma) \le \widetilde{D}_\beta(\rho\|\sigma). \tag{2.8}$$

Section VIII states an open conjecture, that the Rényi generalizations of the conditional mutual information obey a similar monotonicity. We prove that this conjecture is true in some special cases, we prove that it is true when α is in a neighborhood of one, and numerical evidence indicates that it is true in general. We finally conclude in Sec. IX with a summary of our results and a discussion of directions for future research.

III. NOTATION AND DEFINITIONS

A. Norms, states, channels, and measurements

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators acting on a Hilbert space $\mathcal{H}$. We restrict ourselves to finite-dimensional Hilbert spaces throughout this paper. For $\alpha \geq 1$, we define the α-norm of an operator X as

$$\|X\|_\alpha \equiv \mathrm{Tr}\{(\sqrt{X^\dagger X})^\alpha\}^{1/\alpha}, \tag{3.1}$$

and we use the same notation even for the case $\alpha \in (0,1)$, when it is not a norm. Let $\mathcal{B}(\mathcal{H})_+$ denote the subset of positive semi-definite operators, and let $\mathcal{B}(\mathcal{H})_{++}$ denote the subset of positive definite operators. We also write $X \geq 0$ if $X \in \mathcal{B}(\mathcal{H})_+$ and $X > 0$ if $X \in \mathcal{B}(\mathcal{H})_{++}$. An operator ρ is in the set $\mathcal{S}(\mathcal{H})$ of density operators (or states) if $\rho \in \mathcal{B}(\mathcal{H})_+$ and $\mathrm{Tr}\{\rho\} = 1$, and an operator ρ is in the set $\mathcal{S}(\mathcal{H})_{++}$ of strictly positive definite density operators if $\rho \in \mathcal{B}(\mathcal{H})_{++}$ and $\mathrm{Tr}\{\rho\} = 1$. The tensor product of two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ is denoted by $\mathcal{H}_A \otimes \mathcal{H}_B$ or $\mathcal{H}_{AB}$. Given a multipartite density operator $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we unambiguously write $\rho_A = \mathrm{Tr}_B\{\rho_{AB}\}$ for the reduced density operator on system A. We use ρ_{AB}, σ_{AB}, τ_{AB}, ω_{AB}, etc., to denote general density operators in $\mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$, while ψ_{AB}, φ_{AB}, ϕ_{AB}, etc., denote rank-one density operators (pure states) in $\mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ (with it implicit, clear from the context, and the above convention implying that ψ_A, φ_A, ϕ_A may be mixed if ψ_{AB}, φ_{AB}, ϕ_{AB} are pure). In expressions like that in (2.3) and (2.5), an identity operator is implicit when not written (and should be clear from the context), so that, for example, the expression $\rho_{BC}^{1-\alpha}$ in (2.3) should be interpreted as $I_A \otimes \rho_{BC}^{1-\alpha}$.

The trace distance between two quantum states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ is equal to $\|\rho - \sigma\|_1$. It has a direct operational interpretation in terms of the distinguishability of these states. That is, if ρ or σ are prepared with equal probability and the task is to distinguish them via some quantum measurement, then the optimal success probability in doing so is equal to $(1 + \|\rho - \sigma\|_1/2)/2$. Throughout the paper, for technical convenience and simplicity, some of our statements apply only to states in $\mathcal{S}(\mathcal{H})_{++}$. This might seem restrictive, but in the following sense, it is physically reasonable. Given any state $\omega \in \mathcal{S}(\mathcal{H}) \setminus \mathcal{S}(\mathcal{H})_{++}$, there is a state $\omega(\xi) = (1-\xi)\omega + \xi I/\dim(\mathcal{H})$ for a constant $\xi > 0$, so that $\omega(\xi) \in \mathcal{S}(\mathcal{H})_{++}$ and $\|\omega - \omega(\xi)\|_1 \leq 2\xi$. Thus, the bias in distinguishing ω from $\omega(\xi)$ is no more than $\xi/2$, so that $\omega(\xi)$ can "mask" as ω.

Throughout this paper, we take the usual convention that $f(A) = \sum_i f(a_i)|i\rangle\langle i|$ when given a function f and a Hermitian operator A with spectral decomposition $A = \sum_i a_i |i\rangle\langle i|$. So this means that A^{-1} is interpreted as a generalized inverse, so that $A^{-1} = \sum_{i:a_i \neq 0} a_i^{-1}|i\rangle\langle i|$, $\log(A) = \sum_{i:a_i>0} \log(a_i)|i\rangle\langle i|$, $\exp(A) = \sum_i \exp(a_i)|i\rangle\langle i|$, etc. Throughout the paper, we interpret log as the natural logarithm. The above convention for $f(A)$ leads to the convention that A^0 denotes the projection onto the support of A, i.e., $A^0 = \sum_{i:a_i \neq 0}|i\rangle\langle i|$. We employ the shorthand $\mathrm{supp}(A)$ and $\ker(A)$ to refer to the support and kernel of an operator A, respectively.

A linear map $\mathcal{N}_{A\to B} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is positive if $\mathcal{N}_{A\to B}(\sigma_A) \in \mathcal{B}(\mathcal{H}_B)_+$ whenever $\sigma_A \in \mathcal{B}(\mathcal{H}_A)_+$. A linear map $\mathcal{N}_{A\to B} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is strictly positive if $\mathcal{N}_{A\to B}(\sigma_A) \in \mathcal{B}(\mathcal{H}_B)_{++}$ whenever $\sigma_A \in \mathcal{B}(\mathcal{H}_A)_{++}$. Let id_A denote the identity map acting on a system A. A linear map $\mathcal{N}_{A\to B}$ is completely positive if the map $\mathrm{id}_R \otimes \mathcal{N}_{A\to B}$ is positive for a reference system R of arbitrary size. A linear map $\mathcal{N}_{A\to B}$ is trace-preserving if $\mathrm{Tr}\{\mathcal{N}_{A\to B}(\tau_A)\} = \mathrm{Tr}\{\tau_A\}$ for all input operators $\tau_A \in \mathcal{B}(\mathcal{H}_A)$. If a linear map is completely positive and trace-preserving (CPTP), we say that it is a quantum channel or quantum operation. A positive operator-valued measure (POVM) is a set $\{\Lambda^m\}$ of positive semi-definite operators such that $\sum_m \Lambda^m = I$.

B. Relative entropies

We defined the relative entropy $D(P\|Q)$ between $P, Q \in \mathcal{B}(\mathcal{H})_+$ in (1.7), with $P \neq 0$. The definition is consistent with the following limit, so that

$$\lim_{\xi \searrow 0} [\mathrm{Tr}\{P\}]^{-1} \mathrm{Tr}\{P[\log P - \log(Q + \xi I)]\} = D(P\|Q), \tag{3.2}$$

where I is the identity operator acting on $\mathcal{H}$. The statement in (3.2) follows because the quantity

$$\lim_{\xi \searrow 0} \operatorname{Tr}\{P \log(Q + \xi I)\} \tag{3.3}$$

is finite and equal to $\operatorname{Tr}\{P \log Q\}$ if $\operatorname{supp}(P) \subseteq \operatorname{supp}(Q)$. Otherwise, (3.3) is infinite. The relative entropy $D(P\|Q)$ is non-negative if $\operatorname{Tr}\{P\} \geq \operatorname{Tr}\{Q\}$, a result known as Klein's inequality.[42] Thus, for density operators ρ and σ, the relative entropy is non-negative, and furthermore, it is equal to zero if and only if $\rho = \sigma$.

We defined the Rényi relative entropy in (1.8). This definition is consistent with the following limit, so that for $\alpha \in [0,1) \cup (1,\infty)$

$$\lim_{\xi \searrow 0} \frac{1}{\alpha - 1} \log \operatorname{Tr}\left\{[\operatorname{Tr}\{P\}]^{-1} P^{\alpha} (Q + \xi I)^{1-\alpha}\right\} = D_{\alpha}(P\|Q), \tag{3.4}$$

as can be checked by a proof similar to Ref. 50, Lemma 13. The quantity obeys the following monotonicity inequality for all $\alpha \in [0,1) \cup (1,2]$:

$$D_{\alpha}(P\|Q) \geq D_{\alpha}(\mathcal{N}(P)\|\mathcal{N}(Q)), \tag{3.5}$$

where $P, Q \in \mathcal{B}(\mathcal{H})_{+}$ and $\mathcal{N}$ is a CPTP map.[52] Thus, by applying this, we find that $D_{\alpha}(P\|Q)$ is non-negative for all $\alpha \in [0,1) \cup (1,2]$ whenever $\operatorname{Tr}\{P\} \geq \operatorname{Tr}\{Q\}$, so that it is always non-negative for density operators ρ and σ. Furthermore, it is equal to zero if and only if $\rho = \sigma$.

We also defined the sandwiched Rényi relative entropy in (2.4). Similar to the above quantities, the definition is consistent with the following limit, so that

$$\lim_{\xi \searrow 0} \frac{1}{\alpha - 1} \log \left[[\operatorname{Tr}\{P\}]^{-1} \operatorname{Tr}\left\{\left[(Q + \xi I)^{(1-\alpha)/2\alpha} P (Q + \xi I)^{(1-\alpha)/2\alpha}\right]^{\alpha}\right\}\right] = \widetilde{D}_{\alpha}(P\|Q), \tag{3.6}$$

as proved in Ref. 50, Lemma 13. Whenever $\operatorname{supp}(P) \subseteq \operatorname{supp}(Q)$ or ($\alpha \in (0,1)$ and $P \not\perp Q$), it admits the following alternate forms:

$$\widetilde{D}_{\alpha}(P\|Q) \equiv \frac{1}{\alpha - 1} \log \left[[\operatorname{Tr}\{P\}]^{-1} \operatorname{Tr}\left\{\left(Q^{(1-\alpha)/2\alpha} P Q^{(1-\alpha)/2\alpha}\right)^{\alpha}\right\}\right] \tag{3.7}$$

$$= \frac{\alpha}{\alpha - 1} \log \left\|Q^{(1-\alpha)/2\alpha} P Q^{(1-\alpha)/2\alpha}\right\|_{\alpha} - \frac{1}{\alpha - 1} \log \operatorname{Tr}\{P\} \tag{3.8}$$

$$= \frac{\alpha}{\alpha - 1} \log \left\|P^{1/2} Q^{(1-\alpha)/\alpha} P^{1/2}\right\|_{\alpha} - \frac{1}{\alpha - 1} \log \operatorname{Tr}\{P\}. \tag{3.9}$$

It obeys the following monotonicity inequality for all $\alpha \in [1/2,1) \cup (1,\infty)$:

$$\widetilde{D}_{\alpha}(P\|Q) \geq \widetilde{D}_{\alpha}(\mathcal{N}(P)\|\mathcal{N}(Q)), \tag{3.10}$$

where $P, Q \in \mathcal{B}(\mathcal{H})_{+}$ and $\mathcal{N}$ is a CPTP map[24] (see also Refs. 7, 49, 72, and 50 for other proofs of this for more limited ranges of α). Thus, by applying this, we find that $\widetilde{D}_{\alpha}(P\|Q)$ is non-negative for all $\alpha \in [1/2,1) \cup (1,\infty)$ whenever $\operatorname{Tr}\{P\} \geq \operatorname{Tr}\{Q\}$, so that it is always non-negative for density operators ρ and σ. Furthermore, it is equal to zero if and only if $\rho = \sigma$.

IV. CONDITIONAL QUANTUM MUTUAL INFORMATION BASED ON VON NEUMANN ENTROPY

In this section, we prove that the conditional quantum mutual information has many seemingly different representations in terms of a relative-entropy-like quantity (however all of them being equal). This paves the way for designing different Rényi generalizations of the conditional quantum mutual information. Furthermore, we give a conceptually different proof of the fact that the conditional quantum mutual information $I(A;B|C)$ is monotone under local quantum operations on systems A and B. This alternate proof will be the basis for similar proofs when we consider Rényi generalizations in Secs. V and VI. Finally, we discuss how representing $I(A;B|C)$ as we do in Proposition 2 allows for a straightforward comparison of it with the minimum relative entropy "distance" to quantum Markov states, a quantity originally considered in Ref. 32.

A. Various formulations of the conditional quantum mutual information

One of the core quantities that we consider in this paper is the following function of four density operators $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)$:

$$\Delta(\rho_{ABC}, \tau_{AC}, \theta_{BC}, \omega_C) \equiv \operatorname{Tr}\{\rho_{ABC}[\log \rho_{ABC} - \log \tau_{AC} - \log \theta_{BC} + \log \omega_C]\}, \tag{4.1}$$

where logarithms of density operators are understood in the usual sense described in Sec. III. Let I_{ABC} denote the identity operator acting on $\mathcal{H}_{ABC}$. A sufficient condition for

$$\lim_{\xi \searrow 0} \Delta(\rho_{ABC}, \tau_{AC} + \xi I_{ABC}, \theta_{BC} + \xi I_{ABC}, \omega_C + \xi I_{ABC}) \tag{4.2}$$

to be finite and equal to (4.1) is that

$$\operatorname{supp}(\rho_{ABC}) \subseteq \operatorname{supp}(\tau_{AC}), \operatorname{supp}(\theta_{BC}), \operatorname{supp}(\omega_C), \tag{4.3}$$

for the same reason given after (3.2). When comparing with $\operatorname{supp}(\rho_{ABC})$, it is implicit throughout this paper that $\operatorname{supp}(\tau_{AC}) \equiv \operatorname{supp}(I_B \otimes \tau_{AC})$, $\operatorname{supp}(\theta_{BC}) \equiv \operatorname{supp}(I_A \otimes \theta_{BC})$, and $\operatorname{supp}(\omega_C) \equiv \operatorname{supp}(I_{AB} \otimes \omega_C)$. The condition in (4.3) is equivalent to $\operatorname{supp}(\rho_{ABC})$ being in the intersection of the supports of τ_{AC}, θ_{BC}, and ω_C. Note that there are more general support conditions which lead to a finite value for (4.2), but for simplicity, we focus exclusively on the above support condition. If the support condition in (4.3) holds, then by inspection we can write

$$\Delta(\rho_{ABC}, \tau_{AC}, \theta_{BC}, \omega_C) = D(\rho_{ABC} \| \exp\{\log \tau_{AC} + \log \theta_{BC} - \log \omega_C\}). \tag{4.4}$$

Furthermore, observe that

$$\lim_{\xi \searrow 0} \Delta(\rho_{ABC}, \rho_{AC} + \xi I_{ABC}, \rho_{BC} + \xi I_{ABC}, \rho_C + \xi I_{ABC}) \tag{4.5}$$

is finite and equal to (4.1) because the support condition in (4.3) holds when choosing τ_{AC}, θ_{BC}, and ω_C as the marginals of ρ_{ABC} (see, e.g., Ref. 54, Lemma B.4.1).

Lemma 1. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)$ and suppose that the support condition in (4.3) holds. Then

$$\Delta(\rho_{ABC}, \tau_{AC}, \theta_{BC}, \omega_C) = I(A;B|C)_\rho + D(\rho_{AC} \| \tau_{AC}) + D(\rho_{BC} \| \theta_{BC}) - D(\rho_C \| \omega_C). \tag{4.6}$$

Proof. This follows simply by adding to and subtracting from $\Delta(\rho_{ABC}, \tau_{AC}, \theta_{BC}, \omega_C)$ each of $\operatorname{Tr}\{\rho_{ABC} \log \rho_{AC}\}$, $\operatorname{Tr}\{\rho_{ABC} \log \rho_{BC}\}$, and $\operatorname{Tr}\{\rho_{ABC} \log \rho_C\}$. We then apply the definitions of $I(A;B|C)_\rho$, $D(\rho_{AC} \| \tau_{AC})$, $D(\rho_{BC} \| \theta_{BC})$, and $D(\rho_C \| \omega_C)$. ∎

For the mutual information, there are four seemingly different ways of writing it as a relative entropy.[14] However, for the conditional mutual information, there are many ways of doing so, as summarized in the following proposition. The significance of Proposition 2 is that it paves the way for designing many different Rényi generalizations of the conditional mutual information.

Proposition 2. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$. Then

$$I(A;B|C)_\rho = \Delta(\rho_{ABC}, \rho_{AC}, \rho_{BC}, \rho_C) = \inf_{\tau_{AC}} \Delta(\rho_{ABC}, \tau_{AC}, \rho_{BC}, \rho_C) \tag{4.7}$$

$$= \inf_{\theta_{BC}} \Delta(\rho_{ABC}, \rho_{AC}, \theta_{BC}, \rho_C) = \sup_{\omega_C} \Delta(\rho_{ABC}, \rho_{AC}, \rho_{BC}, \omega_C) \tag{4.8}$$

$$= \inf_{\tau_{AC}} \Delta(\rho_{ABC}, \tau_{AC}, \rho_{BC}, \tau_C) = \inf_{\tau_{AC}} \sup_{\omega_C} \Delta(\rho_{ABC}, \tau_{AC}, \rho_{BC}, \omega_C) \tag{4.9}$$

$$= \inf_{\theta_{BC}} \Delta(\rho_{ABC}, \rho_{AC}, \theta_{BC}, \theta_C) = \inf_{\theta_{BC}} \sup_{\omega_C} \Delta(\rho_{ABC}, \rho_{AC}, \theta_{BC}, \omega_C) \tag{4.10}$$

$$= \inf_{\sigma_{ABC}} \Delta(\rho_{ABC}, \sigma_{AC}, \sigma_{BC}, \rho_C) = \inf_{\tau_{AC}, \theta_{BC}} \Delta(\rho_{ABC}, \tau_{AC}, \theta_{BC}, \rho_C) \tag{4.11}$$

$$= \inf_{\sigma_{ABC}} \Delta(\rho_{ABC}, \sigma_{AC}, \sigma_{BC}, \sigma_C) = \inf_{\tau_{AC}, \theta_{BC}} \Delta(\rho_{ABC}, \tau_{AC}, \theta_{BC}, \tau_C) \tag{4.12}$$

$$= \inf_{\tau_{AC}, \theta_{BC}} \Delta(\rho_{ABC}, \tau_{AC}, \theta_{BC}, \theta_C) = \inf_{\sigma_{ABC}} \sup_{\omega_C} \Delta(\rho_{ABC}, \sigma_{AC}, \sigma_{BC}, \omega_C) \tag{4.13}$$

$$= \inf_{\tau_{AC}, \theta_{BC}} \sup_{\omega_C} \Delta(\rho_{ABC}, \tau_{AC}, \theta_{BC}, \omega_C), \tag{4.14}$$

where the optimizations are over states on the indicated Hilbert spaces obeying the support condition in (4.3) and over σ_{ABC} for which $\operatorname{supp}(\rho_{ABC}) \subseteq \operatorname{supp}(\sigma_{ABC})$. *The infima and suprema can be interchanged in all of the above cases, are achieved by the marginals of ρ_{ABC}, and can thus be replaced by minima and maxima.*

Proof. We only prove two of these relations, noting that the rest follow from similar ideas. We first prove (4.14). Invoking Lemma 1, we have that

$$\inf_{\tau_{AC},\theta_{BC}} \sup_{\omega_C} \Delta(\rho_{ABC},\tau_{AC},\theta_{BC},\omega_C) = I(A;B|C)_\rho + \inf_{\tau_{AC}} D(\rho_{AC}\|\tau_{AC}) + \inf_{\theta_{BC}} D(\rho_{BC}\|\theta_{BC}) - \inf_{\omega_C} D(\rho_C\|\omega_C). \tag{4.15}$$

Invoking the fact that the relative entropy is minimized and equal to zero when its first argument is equal to its second, we see that the right hand side is equal to $I(A;B|C)_\rho$.

We now prove the first equality in (4.12). Let σ_{ABC} denote some tripartite state for which $\operatorname{supp}(\rho_{ABC}) \subseteq \operatorname{supp}(\sigma_{ABC})$. By Lemma 1, we have that

$$\Delta(\rho_{ABC},\sigma_{AC},\sigma_{BC},\sigma_C) = I(A;B|C)_\rho + D(\rho_{AC}\|\sigma_{AC}) + D(\rho_{BC}\|\sigma_{BC}) - D(\rho_C\|\sigma_C). \tag{4.16}$$

But it is known that the relative entropy is monotone under a partial trace, so that

$$D(\rho_{AC}\|\sigma_{AC}) \geq D(\rho_C\|\sigma_C). \tag{4.17}$$

Thus, we have that

$$D(\rho_{AC}\|\sigma_{AC}) + D(\rho_{BC}\|\sigma_{BC}) - D(\rho_C\|\sigma_C) \geq 0. \tag{4.18}$$

This implies that

$$\inf_{\sigma_{ABC}} \Delta(\rho_{ABC},\sigma_{AC},\sigma_{BC},\sigma_C) = I(A;B|C)_\rho + \inf_{\sigma_{ABC}} \left[D(\rho_{AC}\|\sigma_{AC}) + D(\rho_{BC}\|\sigma_{BC}) - D(\rho_C\|\sigma_C)\right]. \tag{4.19}$$

The three rightmost terms are non-negative (as shown above), so that we can minimize them (to their absolute minimum of zero) by picking a state σ_{ABC} such that

$$\sigma_{AC} = \rho_{AC}, \quad \log\sigma_{BC} - \log\sigma_C = \log\rho_{BC} - \log\rho_C, \tag{4.20}$$

or by symmetry, one such that

$$\sigma_{BC} = \rho_{BC}, \quad \log\sigma_{AC} - \log\sigma_C = \log\rho_{AC} - \log\rho_C. \tag{4.21}$$

One clear choice satisfying this is $\sigma_{ABC} = \rho_{ABC}$, but there could be others. ■

Remark 3. A priori, we require infima and suprema in the above proposition because the sets over which the optimizations occur are not compact. More explicitly, suppose that $\rho_{ABC} = \omega_{AB} \otimes \theta_C$ for $\omega_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$ and $\theta_C \in \mathcal{S}(\mathcal{H}_C)$. Then the sequence of states,

$$\omega_{AB}(n) \equiv \frac{1}{n}\frac{\omega^0_{AB}}{\operatorname{Tr}\{\omega^0_{AB}\}} + \left(1 - \frac{1}{n}\right)\frac{I_{AB} - \omega^0_{AB}}{\operatorname{Tr}\{I_{AB} - \omega^0_{AB}\}}, \tag{4.22}$$

is such that $\operatorname{supp}(\rho_{ABC}) \subseteq \operatorname{supp}(\omega_{AB}(n))$ *for all $n \geq 1$, but* $\operatorname{supp}(\rho_{ABC}) \nsubseteq \operatorname{supp}(\omega_{AB}(\infty))$.

Corollary 4. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$. Then there is a Pinsker-like lower bound on the conditional mutual information $I(A;B|C)_\rho$

$$I(A;B|C)_\rho \geq \tfrac{1}{4}\left\|\rho_{ABC} - \exp\{\log\rho_{AC} + \log\rho_{BC} - \log\rho_C\}\right\|_1^2. \tag{4.23}$$

Proof. The corollary results from the following chain of inequalities:

$$\begin{aligned} &D(\rho_{ABC}\|\exp\{\log\rho_{AC} + \log\rho_{BC} - \log\rho_C\}) \\ &\geq D_{1/2}(\rho_{ABC}\|\exp\{\log\rho_{AC} + \log\rho_{BC} - \log\rho_C\}) \qquad (4.24) \\ &= -2\log\operatorname{Tr}\left\{\sqrt{\rho_{ABC}}\sqrt{\exp\{\log\rho_{AC} + \log\rho_{BC} - \log\rho_C\}}\right\} \qquad (4.25) \end{aligned}$$

$$\geq -2\log\left(1-\tfrac{1}{2}\left\|\sqrt{\rho_{ABC}}-\sqrt{\exp\{\log\rho_{AC}+\log\rho_{BC}-\log\rho_C\}}\right\|_2^2\right) \tag{4.26}$$

$$\geq \left\|\sqrt{\rho_{ABC}}-\sqrt{\exp\{\log\rho_{AC}+\log\rho_{BC}-\log\rho_C\}}\right\|_2^2 \tag{4.27}$$

$$\geq \tfrac{1}{4}\left\|\rho_{ABC}-\exp\{\log\rho_{AC}+\log\rho_{BC}-\log\rho_C\}\right\|_1^2. \tag{4.28}$$

The first step follows from monotonicity of the Rényi relative entropy with respect to the Rényi parameter (see (2.8)). The rest are from a line of reasoning similar to that in the proofs of Ref. 75, Theorem 2.1 and Corollary 2.2, which in turn follows from some of the development in Ref. 11. ■

B. Monotonicity of the conditional quantum mutual information under local quantum operations

In this section, we show that the Δ quantity in (4.1) obeys monotonicity under tensor-product quantum operations acting on the systems A and B, thus establishing it as a fundamental information measure upon which the conditional mutual information is based. Later, we also establish a Rényi generalization of this quantity, which is the core quantity underlying our various Rényi generalizations of conditional mutual information.

Lemma 5. Let $\rho_{ABC}\in\mathcal{S}(\mathcal{H}_{ABC})$, $\tau_{AC}\in\mathcal{S}(\mathcal{H}_{AC})$, $\theta_{BC}\in\mathcal{S}(\mathcal{H}_{BC})$, and $\omega_C\in\mathcal{S}(\mathcal{H}_C)$ and suppose that the support condition in (4.3) holds. Let $\mathcal{N}_{A\to A'}$ and $\mathcal{M}_{B\to B'}$ be CPTP maps acting on the systems A and B, respectively. Then the following monotonicity inequality holds

$$\Delta(\rho_{ABC},\tau_{AC},\theta_{BC},\omega_C)\geq\Delta((\mathcal{N}_{A\to A'}\otimes\mathcal{M}_{B\to B'})(\rho_{ABC}),\mathcal{N}_{A\to A'}(\tau_{AC}),\mathcal{M}_{B\to B'}(\theta_{BC}),\omega_C). \tag{4.29}$$

Proof. We first prove the inequality

$$\Delta(\rho_{ABC},\tau_{AC},\theta_{BC},\omega_C)\geq\Delta(\mathcal{N}_{A\to A'}(\rho_{ABC}),\mathcal{N}_{A\to A'}(\tau_{AC}),\theta_{BC},\omega_C). \tag{4.30}$$

To prove this, we simply expand out the terms

$$\Delta(\rho_{ABC},\tau_{AC},\theta_{BC},\omega_C)=D(\rho_{ABC}\|\tau_{AC}\otimes I_B)-\operatorname{Tr}\{\rho_{BC}\log\theta_{BC}\}+\operatorname{Tr}\{\rho_C\log\omega_C\}. \tag{4.31}$$

Noting that $\operatorname{supp}(\mathcal{N}_{A\to A'}(\rho_{ABC}))\subseteq\operatorname{supp}(\mathcal{N}_{A\to A'}(\tau_{AC}))$ if $\operatorname{supp}(\rho_{ABC})\subseteq\operatorname{supp}(\tau_{AC})$ (see, e.g., Ref. 54, Lemma B.4.2), we similarly have that

$$\begin{aligned}\Delta(\mathcal{N}_{A\to A'}(\rho_{ABC}),\mathcal{N}_{A\to A'}(\tau_{AC}),\theta_{BC},\omega_C)=D(\mathcal{N}_{A\to A'}(\rho_{ABC})\|\mathcal{N}_{A\to A'}(\tau_{AC})\otimes I_B)\\ -\operatorname{Tr}\{\rho_{BC}\log\theta_{BC}\}+\operatorname{Tr}\{\rho_C\log\omega_C\}.\end{aligned} \tag{4.32}$$

Then the inequality in (4.30) follows from the ordinary monotonicity of relative entropy

$$D(\rho_{ABC}\|\tau_{AC}\otimes I_B)\geq D(\mathcal{N}_{A\to A'}(\rho_{ABC})\|\mathcal{N}_{A\to A'}(\tau_{AC})\otimes I_B). \tag{4.33}$$

An essentially identical approach gives us the following inequality:

$$\Delta(\rho_{ABC},\tau_{AC},\theta_{BC},\omega_C)\geq\Delta(\mathcal{M}_{B\to B'}(\rho_{ABC}),\tau_{AC},\mathcal{M}_{B\to B'}(\theta_{BC}),\omega_C). \tag{4.34}$$

Combining this one with (4.30) gives us the inequality in the statement of the lemma. ■

One of the crucial properties of the conditional quantum mutual information is that it is monotone under CPTP maps acting on the systems A and B, respectively. That is,

$$I(A;B|C)_\rho\geq I(A';B'|C)_\xi, \tag{4.35}$$

where $\xi_{A'B'C}\equiv(\mathcal{N}_{A\to A'}\otimes\mathcal{M}_{B\to B'})(\rho_{ABC})$. From the statement of Lemma 5, we can conclude with a conceptually different proof (other than directly making use of strong subadditivity as done in Ref. 13, Proposition 3) that the conditional mutual information is monotone under tensor-product maps acting on systems A and B. The following theorem is a straightforward consequence of Lemma 5 and the fact that $I(A;B|C)_\rho=\Delta(\rho_{ABC},\rho_{AC},\rho_{BC},\rho_C)$.

Theorem 6 (Ref. 13, Proposition 3). *Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$, $\mathcal{N}_{A\to A'}$ and $\mathcal{M}_{B\to B'}$ be CPTP maps acting on the systems A and B, respectively, and $\xi_{A'B'C} \equiv (\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'})(\rho_{ABC})$. Then the following inequality holds*

$$I(A;B|C)_\rho \geq I(A';B'|C)_\xi. \tag{4.36}$$

C. Comparison with the minimum relative entropy to quantum Markov states

In classical information theory, a tripartite probability distribution $p_{A,B,C}(a,b,c)$ has conditional mutual information $I(A;B|C)$ equal to zero if and only if it can be written as a Markov distribution $p_C(c)\, p_{A|C}(a|c)\, p_{B|C}(b|c)$. Equivalently, it is equal to zero if and only if the distribution $p_{A,B,C}(a,b,c)$ is recoverable after marginalizing over the random variable A, that is, if there exists a classical channel $q(a|c)$ such that $p_{A,B,C}(a,b,c) = q(a|c)\, p_{B,C}(b,c)$. Furthermore, the classical conditional mutual information of $p_{A,B,C}$ can be written as the relative entropy distance between $p_{A,B,C}$ and the nearest Markov distribution [Ref. 32, Sec. II].

The generalization of these ideas to quantum information theory is not so straightforward, and we briefly review what is known from Refs. 28 and 32. Our main aim in doing so is to set the stage for establishing a Rényi generalization of conditional mutual information and the subsequent discussion in Sec. VIII D.

An important class of quantum states is the quantum Markov states, introduced in Ref. 1 and studied for finite-dimensional tripartite states in Ref. 28. Following Ref. 28, we define a state ρ_{ABC} to be a quantum Markov state if $I(A;B|C)_\rho = 0$. Let $\mathcal{M}_{A-C-B}$ denote this class of states. The main result of Ref. 28 is that such a state has the following explicit form:

$$\rho_{ABC} = \bigoplus_j q(j)\, \sigma_{AC_j^L} \otimes \sigma_{C_j^R B}, \tag{4.37}$$

for some probability distribution $q(j)$, density operators $\{\sigma_{AC_j^L}, \sigma_{C_j^R B}\}$, and a decomposition of the Hilbert space for C as $\mathcal{H}_C = \bigoplus_j \mathcal{H}_{C_j^L} \otimes \mathcal{H}_{C_j^R}$. We also know that a state ρ_{ABC} is a quantum Markov state if any of the following conditions hold:[53,55]

$$\rho_{ABC} = \rho_{AC}^{1/2} \rho_C^{-1/2} \rho_{BC} \rho_C^{-1/2} \rho_{AC}^{1/2}, \tag{4.38}$$

$$\rho_{ABC} = \rho_{BC}^{1/2} \rho_C^{-1/2} \rho_{AC} \rho_C^{-1/2} \rho_{BC}^{1/2}, \tag{4.39}$$

$$\rho_{ABC} = \exp\{\log \rho_{AC} + \log \rho_{BC} - \log \rho_C\}. \tag{4.40}$$

Interestingly, if ρ_C is positive definite, then the map $(\cdot) \to \rho_{AC}^{1/2} \rho_C^{-1/2} (\cdot)\, \rho_C^{-1/2} \rho_{AC}^{1/2}$ is a quantum channel from system C to AC, as one can verify by observing that it is completely positive and trace preserving. Otherwise, the map is trace non-increasing. These same statements also obviously apply to the map $(\cdot) \to \rho_{BC}^{1/2} \rho_C^{-1/2} (\cdot)\, \rho_C^{-1/2} \rho_{BC}^{1/2}$. See Refs. 33 and 34 for more conditions for a tripartite state to be a quantum Markov state.

Let $M(\rho_{ABC})$ denote the relative entropy "distance" to quantum Markov states[32]

$$M(\rho_{ABC}) \equiv \inf_{\sigma_{ABC} \in \mathcal{M}_{A-C-B}} D(\rho_{ABC} \| \sigma_{ABC}), \tag{4.41}$$

where $\mathcal{M}_{A-C-B}$ is the set of quantum Markov states defined above. Clearly, it suffices to restrict the above infimum to the set of Markov states σ_{ABC} for which $\operatorname{supp}(\rho_{ABC}) \subseteq \operatorname{supp}(\sigma_{ABC})$. We can now easily compare $I(A;B|C)$ with $M(\rho_{ABC})$, as done in Ref. 32. First, since every quantum Markov state satisfies the condition $\sigma_{ABC} = \exp\{\log \sigma_{AC} + \log \sigma_{BC} - \log \sigma_C\}$, we see that this formula is equivalent to

$$M(\rho_{ABC}) = \inf_{\sigma_{ABC} \in \mathcal{M}_{A-C-B}} D(\rho_{ABC} \| \exp\{\log \sigma_{AC} + \log \sigma_{BC} - \log \sigma_C\}), \tag{4.42}$$

from which we obtain the following inequality:

$$M(\rho_{ABC}) \geq \inf_{\omega_{ABC}} D(\rho_{ABC} \| \exp\{\log \omega_{AC} + \log \omega_{BC} - \log \omega_C\}) \tag{4.43}$$
$$= \inf_{\omega_{ABC}} \Delta(\rho_{ABC}, \omega_{AC}, \omega_{BC}, \omega_C) \tag{4.44}$$
$$= I(A; B|C)_\rho, \tag{4.45}$$

where the infimum is over all states ω_{ABC} satisfying $\operatorname{supp}(\rho_{ABC}) \subseteq \operatorname{supp}(\omega_{ABC})$. The above inequality was already stated in Ref. 32, Theorem 4 (and with the simpler proof along the lines above given by Jenčová at the end of Ref. 32), but one of the main contributions of Ref. 32 was to show that there are tripartite states ω_{ABC} for which there is a strict inequality $M(\omega_{ABC}) > I(A; B|C)_\omega$, and in fact Ref. 32, Sec. VI showed that the gap can be arbitrarily large.

Thus, from the results in Ref. 32, we can already conclude that taking the Rényi relative entropy distance to quantum Markov states will not lead to a useful Rényi generalization of conditional mutual information as one might hope. After the completion of the present paper, we were informed that this matter was pursued independently in Ref. 22.

V. RÉNYI CONDITIONAL MUTUAL INFORMATION

In this section, we establish many Rényi generalizations of the conditional mutual information that bear some properties similar to its properties. Furthermore, we can prove that some of these generalizations converge to it in the limit as the Rényi parameter $\alpha \to 1$. We are motivated to define a Rényi conditional mutual information by considering the generalized Lie-Trotter product formula[60]

$$\exp\{\log \tau_{AC} + \log \theta_{BC} - \log \omega_C\} = \lim_{\alpha \to 1} \left[\tau_{AC}^{(1-\alpha)/2} \omega_C^{(\alpha-1)/2} \theta_{BC}^{1-\alpha} \omega_C^{(\alpha-1)/2} \tau_{AC}^{(1-\alpha)/2}\right]^{1/(1-\alpha)}, \tag{5.1}$$

where the equality holds when $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})_{++}$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})_{++}$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)_{++}$. By plugging the RHS above (before the limit is taken) into the Rényi relative entropy formula defined in (1.8), we obtain the following expression:

$$\frac{1}{\alpha - 1} \log \operatorname{Tr}\left\{\rho_{ABC}^\alpha \tau_{AC}^{(1-\alpha)/2} \omega_C^{(\alpha-1)/2} \theta_{BC}^{1-\alpha} \omega_C^{(\alpha-1)/2} \tau_{AC}^{(1-\alpha)/2}\right\}. \tag{5.2}$$

We can evaluate the above expression even in the case when $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)$ (considering instead the generalized inverse mentioned in Sec. III). With this, we consider the formula in (5.2) to be a Rényi generalization of the formula in (4.4).

The development above motivates some other core quantities that we consider in this paper. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)$. We define the following quantities for $\alpha \in [0,1) \cup (1,\infty)$:

$$Q_\alpha(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}) \equiv \operatorname{Tr}\left\{\rho_{ABC}^\alpha \tau_{AC}^{(1-\alpha)/2} \omega_C^{(\alpha-1)/2} \theta_{BC}^{1-\alpha} \omega_C^{(\alpha-1)/2} \tau_{AC}^{(1-\alpha)/2}\right\}, \tag{5.3}$$
$$\Delta_\alpha(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}) \equiv \frac{1}{\alpha - 1} \log Q_\alpha(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}). \tag{5.4}$$

We stress that the formula in (5.4) is to be interpreted in the sense of generalized inverse, so that it is always finite if

$$\rho_{ABC} \not\perp \left|\tau_{AC}^{(1-\alpha)/2} \omega_C^{(\alpha-1)/2} \theta_{BC}^{(1-\alpha)/2}\right|^2. \tag{5.5}$$

The non-orthogonality condition in (5.5) is satisfied, e.g., if the support condition in (4.3) holds, so that (5.5) is satisfied when $\tau_{AC} = \rho_{AC}$, $\omega_C = \rho_C$, and $\theta_{BC} = \rho_{BC}$. It remains largely open to determine support conditions under which

$$\lim_{\xi \searrow 0} \Delta_\alpha(\rho_{ABC}, \tau_{AC} + \xi I_{ABC}, \omega_C + \xi I_{ABC}, \theta_{BC} + \xi I_{ABC}) \tag{5.6}$$

is finite and equal to (5.4), with complications being due to the fact that (5.3) features the multiplication of several non-commuting operators which can interact in non-trivial ways. We can also consider five other different operator orderings for the last three arguments of Q_α, i.e.,

$$Q_\alpha(\rho_{ABC}, \theta_{BC}, \omega_C, \tau_{AC}) \equiv \operatorname{Tr}\left\{\rho_{ABC}^\alpha \theta_{BC}^{(1-\alpha)/2} \omega_C^{(\alpha-1)/2} \tau_{AC}^{1-\alpha} \omega_C^{(\alpha-1)/2} \theta_{BC}^{(1-\alpha)/2}\right\}, \tag{5.7}$$

$$Q_\alpha\left(\rho_{ABC},\omega_C,\tau_{AC},\theta_{BC}\right) \equiv \operatorname{Tr}\left\{\rho_{ABC}^{\alpha}\omega_C^{(\alpha-1)/2}\tau_{AC}^{(1-\alpha)/2}\theta_{BC}^{1-\alpha}\tau_{AC}^{(1-\alpha)/2}\omega_C^{(\alpha-1)/2}\right\}, \tag{5.8}$$

$$Q_\alpha\left(\rho_{ABC},\omega_C,\theta_{BC},\tau_{AC}\right) \equiv \operatorname{Tr}\left\{\rho_{ABC}^{\alpha}\omega_C^{(\alpha-1)/2}\theta_{BC}^{(1-\alpha)/2}\tau_{AC}^{1-\alpha}\theta_{BC}^{(1-\alpha)/2}\omega_C^{(\alpha-1)/2}\right\}, \tag{5.9}$$

$$Q_\alpha\left(\rho_{ABC},\tau_{AC},\theta_{BC},\omega_C\right) \equiv \operatorname{Tr}\left\{\rho_{ABC}^{\alpha}\tau_{AC}^{(1-\alpha)/2}\theta_{BC}^{(1-\alpha)/2}\omega_C^{\alpha-1}\theta_{BC}^{(1-\alpha)/2}\tau_{AC}^{(1-\alpha)/2}\right\}, \tag{5.10}$$

$$Q_\alpha\left(\rho_{ABC},\theta_{BC},\tau_{AC},\omega_C\right) \equiv \operatorname{Tr}\left\{\rho_{ABC}^{\alpha}\theta_{BC}^{(1-\alpha)/2}\tau_{AC}^{(1-\alpha)/2}\omega_C^{\alpha-1}\tau_{AC}^{(1-\alpha)/2}\theta_{BC}^{(1-\alpha)/2}\right\}. \tag{5.11}$$

In the above, we are abusing notation by always having the power $(\alpha-1)/2$ associated with ω_C and the power $(1-\alpha)/2$ associated with τ_{AC} and θ_{BC}, but we take the convention that the different Q_α quantities are uniquely identified by the operator ordering of its last three arguments. These different Q_α functions lead to different Δ_α quantities, again uniquely identified by the operator ordering of the last three arguments.

We can then use the above observations, the observation in Proposition 2, and the definition of the Rényi relative entropy to define Rényi generalizations of the conditional mutual information. There are many definitions that we could take for a Rényi conditional mutual information by using the different optimizations summarized in Proposition 2 and the different orderings of operators as suggested above.

In spite of the many possibilities suggested above, we choose to define the *Rényi conditional mutual information* as the following quantity because it obeys some additional properties (beyond those satisfied by many of the above generalizations) which we would expect to hold for a Rényi generalization of the conditional mutual information.

Definition 7. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$. The Rényi conditional mutual information of ρ_{ABC} is defined for $\alpha \in [0,1)\cup(1,\infty)$ as

$$I_\alpha(A;B|C)_\rho \equiv \inf_{\sigma_{BC}} \Delta_\alpha\left(\rho_{ABC},\rho_{AC},\rho_C,\sigma_{BC}\right), \tag{5.12}$$

where the optimization is over density operators σ_{BC} such that $\operatorname{supp}(\rho_{ABC}) \subseteq \operatorname{supp}(\sigma_{BC})$.

Note that unlike the conditional mutual information, this definition is not symmetric with respect to A and B. Thus one might also call it the Rényi information that B has about A from the perspective of C. Note also that, for trivial C, the definition reduces to the usual definition of Rényi mutual information in (1.11).

One advantage of the above definition is that we can identify an explicit form for the minimizing σ_{BC} and thus for $I_\alpha(A;B|C)_\rho$, as captured by the following proposition:

Proposition 8. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$. The Rényi conditional mutual information of ρ_{ABC} has the following explicit form for $\alpha \in (0,1)\cup(1,\infty)$:

$$I_\alpha(A;B|C)_\rho = \frac{\alpha}{\alpha-1}\log\operatorname{Tr}\left\{\left(\rho_C^{(\alpha-1)/2}\operatorname{Tr}_A\left\{\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\right\}\rho_C^{(\alpha-1)/2}\right)^{1/\alpha}\right\}. \tag{5.13}$$

This follows because the infimum in (5.12) can be replaced by a minimum and the minimum σ_{BC} is unique with an explicit form.

A proof of Proposition 8 appears in Appendix A.

A. Limit of the Rényi conditional mutual information as $\alpha \to 1$

In this section, we consider the limit of the Δ_α quantity as the Rényi parameter $\alpha \to 1$. This allows us to prove that some variations of the Rényi conditional mutual information converge to the conditional mutual information in the limit as $\alpha \to 1$.

Theorem 9. *Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)$ and suppose that the support condition in (4.3) holds. Then*

$$\lim_{\alpha\to 1}\Delta_\alpha\left(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}\right) = \Delta\left(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}\right). \tag{5.14}$$

The same limiting relation holds for the other Δ_α quantities defined from (5.7) to (5.11).

Proof. We will consider L'Hôpital's rule in order to evaluate the limit of Δ_α as $\alpha \to 1$, due to the presence of the denominator term $\alpha - 1$ in Δ_α. To this end, we compute the following derivative with respect to α

$$\begin{aligned}\frac{d}{d\alpha}Q_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) &= \operatorname{Tr}\left\{(\log\rho_{ABC})\,\rho_{ABC}^{\alpha}\tau_{AC}^{(1-\alpha)/2}\omega_C^{(\alpha-1)/2}\theta_{BC}^{1-\alpha}\omega_C^{(\alpha-1)/2}\tau_{AC}^{(1-\alpha)/2}\right\}\\ &\quad-\frac{1}{2}\operatorname{Tr}\left\{\rho_{ABC}^{\alpha}(\log\tau_{AC})\,\tau_{AC}^{(1-\alpha)/2}\omega_C^{(\alpha-1)/2}\theta_{BC}^{1-\alpha}\omega_C^{(\alpha-1)/2}\tau_{AC}^{(1-\alpha)/2}\right\}\\ &\quad+\frac{1}{2}\operatorname{Tr}\left\{\rho_{ABC}^{\alpha}\tau_{AC}^{(1-\alpha)/2}(\log\omega_C)\,\omega_C^{(\alpha-1)/2}\theta_{BC}^{1-\alpha}\omega_C^{(\alpha-1)/2}\tau_{AC}^{(1-\alpha)/2}\right\}\\ &\quad-\operatorname{Tr}\left\{\rho_{ABC}^{\alpha}\tau_{AC}^{(1-\alpha)/2}\omega_C^{(\alpha-1)/2}(\log\theta_{BC})\,\theta_{BC}^{1-\alpha}\omega_C^{(\alpha-1)/2}\tau_{AC}^{(1-\alpha)/2}\right\}\\ &\quad+\frac{1}{2}\operatorname{Tr}\left\{\rho_{ABC}^{\alpha}\tau_{AC}^{(1-\alpha)/2}\omega_C^{(\alpha-1)/2}\theta_{BC}^{1-\alpha}(\log\omega_C)\,\omega_C^{(\alpha-1)/2}\tau_{AC}^{(1-\alpha)/2}\right\}\\ &\quad-\frac{1}{2}\operatorname{Tr}\left\{\rho_{ABC}^{\alpha}\tau_{AC}^{(1-\alpha)/2}\omega_C^{(\alpha-1)/2}\theta_{BC}^{1-\alpha}\omega_C^{(\alpha-1)/2}(\log\tau_{AC})\,\tau_{AC}^{(1-\alpha)/2}\right\}. \qquad (5.15)\end{aligned}$$

Thus, the function $Q_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC})$ is differentiable for $\alpha \in (0,\infty)$. Applying L'Hôpital's rule, we consider

$$\lim_{\alpha\to1}\Delta_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) = \lim_{\alpha\to1}\frac{1}{Q_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC})}\frac{d}{d\alpha}Q_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}). \qquad (5.16)$$

We can evaluate the limits separately to find that

$$\lim_{\alpha\to1}Q_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) = \operatorname{Tr}\left\{\rho_{ABC}\tau_{AC}^0\omega_C^0\theta_{BC}^0\omega_C^0\tau_{AC}^0\right\}, \qquad (5.17)$$

$$\begin{aligned}\lim_{\alpha\to1}\frac{d}{d\alpha}Q_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) &= \operatorname{Tr}\left\{(\log\rho_{ABC})\,\rho_{ABC}\tau_{AC}^0\omega_C^0\theta_{BC}^0\omega_C^0\tau_{AC}^0\right\}\\ &\quad-\frac{1}{2}\operatorname{Tr}\left\{\rho_{ABC}(\log\tau_{AC})\,\tau_{AC}^0\omega_C^0\theta_{BC}^0\omega_C^0\tau_{AC}^0\right\}+\frac{1}{2}\operatorname{Tr}\left\{\rho_{ABC}\tau_{AC}^0(\log\omega_C)\,\omega_C^0\theta_{BC}^0\omega_C^0\tau_{AC}^0\right\}\\ &\quad-\operatorname{Tr}\left\{\rho_{ABC}\tau_{AC}^0\omega_C^0(\log\theta_{BC})\,\theta_{BC}^0\omega_C^0\tau_{AC}^0\right\}+\frac{1}{2}\operatorname{Tr}\left\{\rho_{ABC}\tau_{AC}^0\omega_C^0\theta_{BC}^0(\log\omega_C)\,\omega_C^0\tau_{AC}^0\right\}\\ &\quad-\frac{1}{2}\operatorname{Tr}\left\{\rho_{ABC}\tau_{AC}^0\omega_C^0\theta_{BC}^0\omega_C^0(\log\tau_{AC})\,\tau_{AC}^0\right\}. \qquad (5.18)\end{aligned}$$

Since by assumption $\operatorname{supp}(\rho_{ABC})$ is contained in each of $\operatorname{supp}(\tau_{AC})$, $\operatorname{supp}(\omega_C)$, and $\operatorname{supp}(\theta_{BC})$, we exploit the relations $\rho_{ABC} = \rho_{ABC}^0\rho_{ABC}\rho_{ABC}^0$, $\rho_{ABC}^0\tau_{AC}^0 = \rho_{ABC}^0$, $\rho_{ABC}^0\theta_{BC}^0 = \rho_{ABC}^0$, $\rho_{ABC}^0\omega_C^0 = \rho_{ABC}^0$ and their Hermitian conjugates to find that

$$\lim_{\alpha\to1}Q_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) = 1, \qquad (5.19)$$

$$\lim_{\alpha\to1}\frac{d}{d\alpha}Q_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) = \Delta(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}), \qquad (5.20)$$

which when combined with (5.16) leads to (5.14). Essentially the same proof establishes the limiting relation for the other Δ_α quantities defined from (5.7) to (5.11). ■

Corollary 10. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$. Then the following limiting relation holds

$$\lim_{\alpha\to1}\Delta_\alpha(\rho_{ABC},\rho_{AC},\rho_C,\rho_{BC}) = I(A;B|C)_\rho. \qquad (5.21)$$

Proof. This follows from the fact that $\operatorname{supp}(\rho_{ABC}) \subseteq \operatorname{supp}(\rho_{AC}), \operatorname{supp}(\rho_C), \operatorname{supp}(\rho_{BC})$ (see, e.g., Ref. 54, Lemma B.4.1), from the above theorem, and by recalling that $\Delta(\rho_{ABC},\rho_{AC},\rho_C,\rho_{BC}) = I(A;B|C)_\rho$. ■

Theorem 11. *Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$. Then the Rényi conditional mutual information converges to the conditional mutual information in the limit as $\alpha \to 1$*

$$\lim_{\alpha\to1}I_\alpha(A;B|C)_\rho = I(A;B|C)_\rho. \qquad (5.22)$$

The idea behind the proof of Theorem 11 is the same as that behind the proof of Theorem 9. However, we have the explicit form for $I_\alpha(A;B|C)_\rho$ from Proposition 8, which allows us to evaluate the limit without the need for uniform convergence of $\Delta_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC})$ in τ_{AC}, ω_C, and θ_{BC} as $\alpha \to 1$. A proof of Theorem 11 appears in Appendix B.

Remark 12. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)$ and suppose that the support condition in (4.3) holds. If $\Delta_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC})$ converges uniformly in τ_{AC}, ω_C, and θ_{BC} to $\Delta(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC})$ as $\alpha \to 1$, then we could conclude that all Rényi generalizations of the conditional mutual information (as proposed at the beginning of Sec. V) converge to it in the limit as $\alpha \to 1$.

B. Monotonicity with respect to local quantum operations on one system

The following lemma is the critical one which will allow us to conclude that the Rényi conditional mutual information is monotone non-increasing with respect to local quantum operations acting on one system for $\alpha \in [0,1) \cup (1,2]$.

Lemma 13. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)$ and suppose that the non-orthogonality condition in (5.5) holds. Let $\mathcal{N}_{A\to A'}$ and $\mathcal{M}_{B\to B'}$ denote quantum operations acting on systems A and B, respectively. Then the following monotonicity inequalities hold for $\alpha \in [0,1) \cup (1,2]$:

$$\Delta_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) \geq \Delta_\alpha(\mathcal{M}_{B\to B'}(\rho_{ABC}),\tau_{AC},\omega_C,\mathcal{M}_{B\to B'}(\theta_{BC})), \tag{5.23}$$

$$\Delta_\alpha(\rho_{ABC},\omega_C,\tau_{AC},\theta_{BC}) \geq \Delta_\alpha(\mathcal{M}_{B\to B'}(\rho_{ABC}),\omega_C,\tau_{AC},\mathcal{M}_{B\to B'}(\theta_{BC})), \tag{5.24}$$

$$\Delta_\alpha(\rho_{ABC},\omega_C,\theta_{BC},\tau_{AC}) \geq \Delta_\alpha(\mathcal{N}_{A\to A'}(\rho_{ABC}),\omega_C,\theta_{BC},\mathcal{N}_{A\to A'}(\tau_{AC})), \tag{5.25}$$

$$\Delta_\alpha(\rho_{ABC},\theta_{BC},\omega_C,\tau_{AC}) \geq \Delta_\alpha(\mathcal{N}_{A\to A'}(\rho_{ABC}),\theta_{BC},\omega_C,\mathcal{N}_{A\to A'}(\tau_{AC})). \tag{5.26}$$

Proof. We begin by proving (5.23). Consider that $Q_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC})$ is jointly concave in ρ_{ABC} and θ_{BC} when $\alpha \in [0,1)$. This is a result of Lieb's concavity theorem,[44] a special case of which is the statement that the function

$$(S,R) \in \mathcal{B}(\mathcal{H})_+ \times \mathcal{B}(\mathcal{H})_+ \to \operatorname{Tr}\left\{S^\lambda X R^{1-\lambda} X^\dagger\right\} \tag{5.27}$$

is jointly concave in S and R when $\lambda \in [0,1]$. (We apply the theorem by choosing $S = \rho_{ABC}$, $R = \theta_{BC}$, and $X = \tau_{AC}^{(1-\alpha)/2}\omega_C^{(\alpha-1)/2}$.) Furthermore, by an application of Ando's convexity theorem,[4] we know that $Q_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC})$ is jointly convex in ρ_{ABC} and θ_{BC} when $\alpha \in (1,2]$.

By a standard (well known) argument due to Uhlmann,[69] the monotonicity inequality in (5.23) holds. For completeness, we detail this standard argument here for the case when $\alpha \in [0,1)$. Note that it suffices to prove the following monotonicity under partial trace:

$$Q_\alpha\left(\rho_{AB_1B_2C},\tau_{AC},\omega_C,\theta_{B_1B_2C}\right) \leq Q_\alpha\left(\rho_{AB_1C},\tau_{AC},\omega_C,\theta_{B_1C}\right), \tag{5.28}$$

because the Q_α quantity is clearly invariant under isometries acting on system B and the Stinespring representation theorem[59] states that any quantum channel can be modeled as an isometry followed by a partial trace. To this end, let $\left\{U_{B_2}^i\right\}_{i=0}^{d_{B_2}^2-1}$ denote the set of Heisenberg-Weyl operators acting on the system B_2, with d_{B_2} the dimension of system B_2. Then

$$Q_\alpha\left(\rho_{AB_1B_2C},\tau_{AC},\omega_C,\theta_{B_1B_2C}\right) = \frac{1}{d_{B_2}^2}\sum_{i=0}^{d_{B_2}^2-1} Q_\alpha\left(U_{B_2}^i\rho_{AB_1B_2C}\left(U_{B_2}^i\right)^\dagger,\tau_{AC},\omega_C,U_{B_2}^i\theta_{B_1B_2C}\left(U_{B_2}^i\right)^\dagger\right). \tag{5.29}$$

We can then invoke the Lieb concavity theorem to conclude that

$$
\begin{aligned}
&Q_\alpha\left(\rho_{AB_1B_2C},\tau_{AC},\omega_C,\theta_{B_1B_2C}\right)\\
&\leq Q_\alpha\left(\frac{1}{d_{B_2}^2}\sum_i U_{B_2}^i\rho_{AB_1B_2C}\left(U_{B_2}^i\right)^\dagger,\tau_{AC},\omega_C,\frac{1}{d_{B_2}^2}\sum_i U_{B_2}^i\theta_{B_1B_2C}\left(U_{B_2}^i\right)^\dagger\right) && (5.30)\\
&= Q_\alpha\left(\rho_{AB_1C}\otimes\pi_{B_2},\tau_{AC},\omega_C,\theta_{B_1C}\otimes\pi_{B_2}\right) && (5.31)\\
&= Q_\alpha\left(\rho_{AB_1C},\tau_{AC},\omega_C,\theta_{B_1C}\right), && (5.32)
\end{aligned}
$$

where π is the maximally mixed state. After taking logarithms and dividing by $\alpha-1$, we can conclude the monotonicity for $\alpha\in[0,1)$. A similar development with Ando's convexity theorem gets the monotonicity for $\alpha\in(1,2]$. The inequalities in (5.24)-(5.26) follow from a similar line of reasoning. ■

Remark 14. Let $\rho_{ABC}\in\mathcal{S}(\mathcal{H}_{ABC})$, $\tau_{AC}\in\mathcal{S}(\mathcal{H}_{AC})$, $\theta_{BC}\in\mathcal{S}(\mathcal{H}_{BC})$, and $\omega_C\in\mathcal{S}(\mathcal{H}_C)$ and suppose that the non-orthogonality condition in (5.5) holds. It is an open question to determine whether the Δ_α quantities defined from (5.3), (5.7)-(5.11) are monotone non-increasing with respect to quantum operations acting on either systems A or B for $\alpha\in[0,1)\cup(1,2]$. In particular, it is an open question to determine whether $\Delta_\alpha(\rho_{ABC},\rho_{AC},\rho_C,\rho_{BC})$ and $\inf_{\theta_{BC}}\Delta_\alpha(\rho_{ABC},\rho_{AC},\rho_C,\theta_{BC})$ are monotone non-increasing with respect to quantum operations acting on system A for $\alpha\in[0,1)\cup(1,2]$.

Corollary 15. Let $\rho_{ABC}\in\mathcal{S}(\mathcal{H}_{ABC})$, $\tau_{AC}\in\mathcal{S}(\mathcal{H}_{AC})$, $\theta_{BC}\in\mathcal{S}(\mathcal{H}_{BC})$, and $\omega_C\in\mathcal{S}(\mathcal{H}_C)$. All Rényi generalizations of the conditional mutual information derived from

$$\Delta_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}),\quad \Delta_\alpha(\rho_{ABC},\omega_C,\tau_{AC},\theta_{BC}) \tag{5.33}$$

are monotone non-increasing with respect to quantum operations acting on system B, for $\alpha\in[0,1)\cup(1,2]$. All Rényi generalizations of the conditional mutual information derived from

$$\Delta_\alpha(\rho_{ABC},\omega_C,\theta_{BC},\tau_{AC}),\quad \Delta_\alpha(\rho_{ABC},\theta_{BC},\omega_C,\tau_{AC}) \tag{5.34}$$

are monotone non-increasing with respect to quantum operations acting on system A, for $\alpha\in[0,1)\cup(1,2]$. The derived Rényi generalizations are optimized with respect to τ_{AC}, ω_C, and θ_{BC} satisfying the support condition in (4.3) (which implies the non-orthogonality condition in (5.5)).

Proof. We prove that a variation derived from (4.14) obeys the monotonicity (with the others mentioned above following from similar ideas). Beginning with the inequality in Lemma 13, we find that

$$
\begin{aligned}
\sup_{\omega_C}\Delta_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) &\geq \sup_{\omega_C}\Delta_\alpha(\mathcal{M}_{B\to B'}(\rho_{ABC}),\tau_{AC},\omega_C,\mathcal{M}_{B\to B'}(\theta_{BC})) && (5.35)\\
&\geq \inf_{\tau'_{AC},\theta'_{BC}}\sup_{\omega_C}\Delta_\alpha\left(\mathcal{M}_{B\to B'}(\rho_{ABC}),\tau'_{AC},\omega_C,\theta'_{BC}\right). && (5.36)
\end{aligned}
$$

Since this inequality holds for all τ_{AC} and θ_{BC}, it holds in particular for the infimum of the first line over all such states, establishing monotonicity for the Rényi generalization of the conditional mutual information derived from (4.14). ■

Corollary 16. We can employ the monotonicity inequalities from Lemma 13 to conclude that some Rényi generalizations of the conditional mutual information derived from (5.33) and (5.34) and Proposition 2 are non-negative for all $\alpha\in[0,1)\cup(1,2]$. This includes $\Delta_\alpha(\rho_{ABC},\rho_{AC},\rho_C,\rho_{BC})$ and the one from Definition 7.

Proof. Let $\rho_{ABC}\in\mathcal{S}(\mathcal{H}_{ABC})$, $\tau_{AC}\in\mathcal{S}(\mathcal{H}_{AC})$, $\theta_{BC}\in\mathcal{S}(\mathcal{H}_{BC})$, and $\omega_C\in\mathcal{S}(\mathcal{H}_C)$ and suppose that the support condition in (4.3) holds. A common proof technique applies to reach the conclusions stated above. We illustrate with an example for

$$\inf_{\theta_{BC}}\sup_{\omega_C}\Delta_\alpha(\rho_{ABC},\rho_{AC},\omega_C,\theta_{BC}). \tag{5.37}$$

We apply Lemma 13, choosing the local map on system B to be a trace-out map, to conclude that

$$\Delta_\alpha(\rho_{ABC},\rho_{AC},\omega_C,\theta_{BC}) \geq \Delta_\alpha(\rho_{AC},\rho_{AC},\omega_C,\theta_C). \tag{5.38}$$

Then, we can conclude that

$$\begin{aligned}
\sup_{\omega_C}\Delta_\alpha(\rho_{ABC},\rho_{AC},\omega_C,\theta_{BC}) &\geq \sup_{\omega_C}\Delta_\alpha(\rho_{AC},\rho_{AC},\omega_C,\theta_C) &(5.39)\\
&\geq \Delta_\alpha(\rho_{AC},\rho_{AC},\theta_C,\theta_C) &(5.40)\\
&= \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\rho_{AC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\theta_C^{(\alpha-1)/2}\theta_C^{1-\alpha}\theta_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\right\} &(5.41)\\
&= \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\rho_{AC}\theta_C^{0}\right\} &(5.42)\\
&= 0, &(5.43)
\end{aligned}$$

with the last inequality following from the support condition $\operatorname{supp}(\rho_{ABC}) \subseteq \operatorname{supp}(\theta_{BC})$ implying the support condition $\operatorname{supp}(\rho_{AC}) \subseteq \operatorname{supp}(\theta_C)$ [Ref. 54, Lemma B.4.2]. Since the inequality holds for all θ_{BC} satisfying the support condition, we can conclude that the quantity in (5.37) is non-negative. A similar technique can be used to conclude that other Rényi generalizations of the conditional mutual information are non-negative (including the one in Definition 7). ■

Remark 17. If the system C is classical, then the Rényi conditional mutual information given in Definition 7 is monotone with respect to local operations on both A and B. This is because the optimizing state is classical on system C and then we have the commutation

$$\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\sigma_{BC}^{1-\alpha}\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2} = \sigma_{BC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\rho_{AC}^{1-\alpha}\rho_C^{(\alpha-1)/2}\sigma_{BC}^{(1-\alpha)/2}. \tag{5.44}$$

Remark 18. It is an open question to determine whether all Rényi generalizations of the conditional mutual information designed from the different optimizations in Proposition 2 and the different orderings in (5.3), (5.7)-(5.11) are non-negative for $\alpha \in [0,1)\cup(1,2]$.

VI. SANDWICHED RÉNYI CONDITIONAL MUTUAL INFORMATION

As in Sec. V, there are many ways in which we can define a sandwiched Rényi conditional mutual information. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)$. We define the following core quantities for $\alpha \in (0,1)\cup(1,\infty)$:

$$\widetilde{Q}_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) \equiv \operatorname{Tr}\left\{\left(\rho_{ABC}^{1/2}\tau_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\theta_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\tau_{AC}^{(1-\alpha)/2\alpha}\rho_{ABC}^{1/2}\right)^{\alpha}\right\}, \tag{6.1}$$

$$\widetilde{\Delta}_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) \equiv \frac{1}{\alpha-1}\log \widetilde{Q}_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}). \tag{6.2}$$

We stress again that the formula above is to be interpreted in terms of generalized inverses. By employing (3.1) and (6.1), we can write

$$\widetilde{Q}_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) = \left\|\rho_{ABC}^{1/2}\tau_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\theta_{BC}^{(1-\alpha)/2\alpha}\right\|_{2\alpha}^{2\alpha}, \tag{6.3}$$

and we see that $\widetilde{Q}_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) = 0$ if and only if

$$\rho_{ABC}^{1/2}\tau_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\theta_{BC}^{(1-\alpha)/2\alpha} = 0. \tag{6.4}$$

So $\widetilde{Q}_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) > 0$ if

$$\rho_{ABC}^{1/2} \not\perp \tau_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\theta_{BC}^{(1-\alpha)/2\alpha}. \tag{6.5}$$

The non-orthogonality condition in (6.5) is satisfied, e.g., if the support condition in (4.3) holds, so that (6.5) is satisfied when $\tau_{AC} = \rho_{AC}$, $\omega_C = \rho_C$, and $\theta_{BC} = \rho_{BC}$. It remains largely open to determine support conditions under which

$$\lim_{\xi\searrow 0}\widetilde{\Delta}_\alpha(\rho_{ABC},\tau_{AC}+\xi I_{ABC},\omega_C+\xi I_{ABC},\theta_{BC}+\xi I_{ABC}) \tag{6.6}$$

is finite and equal to (6.2), with complications being due to the fact that (6.1) features the multiplication of several non-commuting operators which can interact in non-trivial ways. As before, we define five other different $\widetilde{Q}_\alpha$ quantities, again uniquely identified by the order of the last three arguments

$$\widetilde{Q}_\alpha\left(\rho_{ABC},\theta_{BC},\omega_C,\tau_{AC}\right) \equiv \left\|\rho_{ABC}^{1/2}\theta_{BC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\tau_{AC}^{(1-\alpha)/2\alpha}\right\|_{2\alpha}^{2\alpha}, \tag{6.7}$$

$$\widetilde{Q}_\alpha\left(\rho_{ABC},\omega_C,\tau_{AC},\theta_{BC}\right) \equiv \left\|\rho_{ABC}^{1/2}\omega_C^{(\alpha-1)/2\alpha}\tau_{AC}^{(1-\alpha)/2\alpha}\theta_{BC}^{(1-\alpha)/2\alpha}\right\|_{2\alpha}^{2\alpha}, \tag{6.8}$$

$$\widetilde{Q}_\alpha\left(\rho_{ABC},\omega_C,\theta_{BC},\tau_{AC}\right) \equiv \left\|\rho_{ABC}^{1/2}\omega_C^{(\alpha-1)/2\alpha}\theta_{BC}^{(1-\alpha)/2\alpha}\tau_{AC}^{(1-\alpha)/2\alpha}\right\|_{2\alpha}^{2\alpha}, \tag{6.9}$$

$$\widetilde{Q}_\alpha\left(\rho_{ABC},\tau_{AC},\theta_{BC},\omega_C\right) \equiv \left\|\rho_{ABC}^{1/2}\tau_{AC}^{(1-\alpha)/2\alpha}\theta_{BC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\right\|_{2\alpha}^{2\alpha}, \tag{6.10}$$

$$\widetilde{Q}_\alpha\left(\rho_{ABC},\theta_{BC},\tau_{AC},\omega_C\right) \equiv \left\|\rho_{ABC}^{1/2}\theta_{BC}^{(1-\alpha)/2\alpha}\tau_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\right\|_{2\alpha}^{2\alpha}. \tag{6.11}$$

These then lead to different $\widetilde{\Delta}_\alpha$ quantities. We call the quantities above "sandwiched" because they can be viewed as having their root in the sandwiched Rényi relative entropy, i.e., for $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})_{++}$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})_{++}$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)_{++}$

$$\widetilde{\Delta}_\alpha\left(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}\right) = \widetilde{D}_\alpha\left(\rho_{ABC}\left\|\left[\tau_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\theta_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\tau_{AC}^{(1-\alpha)/2\alpha}\right]^{\alpha/(1-\alpha)}\right.\right). \tag{6.12}$$

Although there are many different possible sandwiched Rényi generalizations of the conditional mutual information, found by combining the different $\widetilde{\Delta}_\alpha$ quantities discussed above with the different optimizations summarized in Proposition 2, we choose the definition given below because it obeys many of the properties that the conditional mutual information does.

Definition 19. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$. The sandwiched Rényi conditional mutual information is defined as

$$\widetilde{I}_\alpha(A;B|C)_\rho \equiv \inf_{\sigma_{BC}}\sup_{\omega_C}\widetilde{\Delta}_\alpha\left(\rho_{ABC},\rho_{AC},\omega_C,\sigma_{BC}\right), \tag{6.13}$$

where the optimizations are over states obeying the support conditions in (4.3).

Again, unlike the conditional mutual information, this definition is not symmetric with respect to A and B. Thus one might also call it the sandwiched Rényi information that B has about A from the perspective of C. Also, for trivial C, the definition reduces to the usual definition of sandwiched Rényi mutual information (see, e.g., Refs. 72, 26, and 15).

A. Limit of the sandwiched Rényi conditional mutual information as $\alpha \to 1$

This section considers the limit of the $\widetilde{\Delta}_\alpha$ quantities as $\alpha \to 1$. For technical reasons, we restrict the development to positive definite density operators. It remains open to determine whether the following theorems hold under less restrictive conditions.

Theorem 20. *Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})_{++}$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})_{++}$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)_{++}$. Then*

$$\lim_{\alpha\to 1}\widetilde{\Delta}_\alpha\left(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}\right) = \Delta\left(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}\right). \tag{6.14}$$

The same limiting relation holds for the other $\widetilde{\Delta}_\alpha$ quantities defined from (6.7) to (6.11).

The proof of Theorem 20 is very similar to the proof of Theorem 9 and is presented in Appendix C.

Corollary 21. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$. The following limiting relation holds

$$\lim_{\alpha\to 1}\widetilde{\Delta}_\alpha\left(\rho_{ABC},\rho_{AC},\rho_C,\rho_{BC}\right) = I(A;B|C)_\rho. \tag{6.15}$$

Proof. This follows from the fact that $\operatorname{supp}(\rho_{ABC}) \subseteq \operatorname{supp}(\rho_{AC})$, $\operatorname{supp}(\rho_C)$, $\operatorname{supp}(\rho_{BC})$ (see, e.g., Ref. 54, Lemma B.4.1), Theorem 20, and by recalling that $\Delta(\rho_{ABC}, \rho_{AC}, \rho_C, \rho_{BC}) = I(A;B|C)_\rho$. ■

Remark 22. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})_{++}$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})_{++}$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)_{++}$. If $\widetilde{\Delta}_\alpha(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC})$ converges uniformly in τ_{AC}, ω_C, θ_{BC} to $\Delta(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC})$ as $\alpha \to 1$, then we could conclude that all sandwiched Rényi generalizations of the conditional mutual information (as proposed at the beginning of Sec. VI) converge to it in the limit as $\alpha \to 1$. In particular, uniform convergence implies that $\widetilde{I}_\alpha(A;B|C)_\rho$ converges to $I(A;B|C)_\rho$ as $\alpha \to 1$.

B. Monotonicity under local quantum operations on one system

This section considers monotonicity of the $\widetilde{\Delta}_\alpha$ quantities under local quantum operations. For technical reasons, we restrict the development to positive definite density operators. It remains open to determine whether the following theorems hold under less restrictive conditions.

Lemma 23. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})_{++}$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})_{++}$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)_{++}$. Let $\mathcal{N}_{A\to A'}$ and $\mathcal{M}_{B\to B'}$ denote quantum operations acting on systems A and B, respectively. Then the following monotonicity inequalities hold for all $\alpha \in [1/2,1) \cup (1,\infty)$:

$$\widetilde{\Delta}_\alpha(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}) \geq \widetilde{\Delta}_\alpha(\mathcal{M}_{B\to B'}(\rho_{ABC}), \tau_{AC}, \omega_C, \mathcal{M}_{B\to B'}(\theta_{BC})), \tag{6.16}$$

$$\widetilde{\Delta}_\alpha(\rho_{ABC}, \omega_C, \tau_{AC}, \theta_{BC}) \geq \widetilde{\Delta}_\alpha(\mathcal{M}_{B\to B'}(\rho_{ABC}), \omega_C, \tau_{AC}, \mathcal{M}_{B\to B'}(\theta_{BC})), \tag{6.17}$$

$$\widetilde{\Delta}_\alpha(\rho_{ABC}, \omega_C, \theta_{BC}, \tau_{AC}) \geq \widetilde{\Delta}_\alpha(\mathcal{N}_{A\to A'}(\rho_{ABC}), \omega_C, \theta_{BC}, \mathcal{N}_{A\to A'}(\tau_{AC})), \tag{6.18}$$

$$\widetilde{\Delta}_\alpha(\rho_{ABC}, \theta_{BC}, \omega_C, \tau_{AC}) \geq \widetilde{\Delta}_\alpha(\mathcal{N}_{A\to A'}(\rho_{ABC}), \theta_{BC}, \omega_C, \mathcal{N}_{A\to A'}(\tau_{AC})). \tag{6.19}$$

Proof. We first focus on establishing the inequality in (6.16) for $\alpha \in [1/2,1)$. From part (1) of Ref. 31, Theorem 1.1, we know that the following function is jointly concave in S and T:

$$(S,T) \in \mathcal{B}(\mathcal{H})_{++} \times \mathcal{B}(\mathcal{H})_{++} \mapsto \operatorname{Tr}\left\{\left[\Phi(S^p)^{1/2}\Psi(T^q)\Phi(S^p)^{1/2}\right]^s\right\}, \tag{6.20}$$

for strictly positive maps $\Phi(\cdot)$ and $\Psi(\cdot)$, $0 < p,q \leq 1$, and $1/2 \leq s \leq 1/(p+q)$. We can then see that $\widetilde{Q}_\alpha(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC})$ is of this form, with

$$\Psi = \tau_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}(\cdot)\,\omega_C^{(\alpha-1)/2\alpha}\tau_{AC}^{(1-\alpha)/2\alpha}, \tag{6.21}$$

$$q = \frac{1-\alpha}{\alpha}, \tag{6.22}$$

$$\Phi(\cdot) = \mathrm{id}, \tag{6.23}$$

$$p = 1, \tag{6.24}$$

$$s = \alpha. \tag{6.25}$$

For the range $\alpha \in [1/2,1)$, we have that $p \in (0,1]$ and $1/(p+q) = \alpha$, so that the conditions of part (1) of Ref. 31, Theorem 1.1 are satisfied. We conclude that $\widetilde{Q}_\alpha(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC})$ is jointly concave in θ_{BC} and ρ_{ABC}. From this, we can conclude the monotonicity in (6.16) for $\alpha \in [1/2,1)$. A similar proof establishes the inequalities in (6.17)-(6.19) for $\alpha \in [1/2,1)$.

The proof of (6.16) for $\alpha \in (1,\infty)$ is a straightforward generalization of the technique used for Ref. 24, Proposition 3. To prove (6.16), it suffices to prove that the following function

$$(\rho_{ABC}, \theta_{BC}) \in \mathcal{S}(\mathcal{H}_{ABC})_{++} \times \mathcal{S}(\mathcal{H}_{ABC})_{++} \mapsto \operatorname{Tr}\left\{\left[\rho_{ABC}^{1/2}K(\alpha)\rho_{ABC}^{1/2}\right]^\alpha\right\} \tag{6.26}$$

is jointly convex for $\alpha \in (1,\infty)$, where

$$K(\alpha) \equiv \tau_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\theta_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\tau_{AC}^{(1-\alpha)/2\alpha}. \tag{6.27}$$

To this end, consider that we can write the trace function in (6.26) as

$$\operatorname{Tr}\left\{\left[\rho_{ABC}^{1/2}K(\alpha)\rho_{ABC}^{1/2}\right]^\alpha\right\} = \sup_{H\geq 0}\alpha\operatorname{Tr}\{H\rho_{ABC}\} - (\alpha-1)\operatorname{Tr}\left\{\left[H^{1/2}L(\alpha)H^{1/2}\right]^{\alpha/(\alpha-1)}\right\}, \tag{6.28}$$

where

$$L(\alpha) \equiv \tau_{AC}^{(\alpha-1)/2\alpha}\omega_C^{(1-\alpha)/2\alpha}\theta_{BC}^{(\alpha-1)/\alpha}\omega_C^{(1-\alpha)/2\alpha}\tau_{AC}^{(\alpha-1)/2\alpha}, \tag{6.29}$$

so that $[L(\alpha)]^{-1} = K(\alpha)$. From the fact that the following map

$$S \in \mathcal{B}(\mathcal{H})_+ \mapsto \operatorname{Tr}\left\{\left[T^\dagger S^p T\right]^{1/p}\right\} \tag{6.30}$$

is concave in S for a fixed $T \in \mathcal{B}(\mathcal{H})$ and for $-1 \leq p \leq 1$ [Ref. 24, Lemma 5] and the representation formula given in (6.28), we can then conclude that the function in (6.26) is jointly convex in ρ_{ABC} and θ_{BC} for $\alpha \in (1,\infty)$.

So it remains to prove the representation formula in (6.28). Recall from the alternative proof of Ref. 24, Lemma 4 that for positive semi-definite operators X and Y and $1 < p,q < \infty$ with $1/p + 1/q = 1$, the following inequality holds

$$\operatorname{Tr}\{XY\} \leq \frac{1}{p}\operatorname{Tr}\{X^p\} + \frac{1}{q}\operatorname{Tr}\{Y^q\}, \tag{6.31}$$

with equality holding if $X^p = Y^q$. To apply the inequality in (6.31), we set

$$X = K(\alpha)^{1/2}\rho_{ABC}K(\alpha)^{1/2}, \tag{6.32}$$

$$Y = L(\alpha)^{1/2}HL(\alpha)^{1/2}, \tag{6.33}$$

$$p = \alpha, \tag{6.34}$$

$$q = \frac{\alpha}{\alpha-1}. \tag{6.35}$$

Applying (6.31), we find that

$$\operatorname{Tr}\{H\rho_{ABC}\} \leq \frac{1}{\alpha}\operatorname{Tr}\left\{\left[\rho_{ABC}^{1/2}K(\alpha)\rho_{ABC}^{1/2}\right]^\alpha\right\} + \frac{\alpha-1}{\alpha}\operatorname{Tr}\left\{\left[H^{1/2}L(\alpha)H^{1/2}\right]^{\alpha/(\alpha-1)}\right\}, \tag{6.36}$$

which can be rewritten as

$$\alpha\operatorname{Tr}\{H\rho_{ABC}\} - (\alpha-1)\operatorname{Tr}\left\{\left[H^{1/2}L(\alpha)H^{1/2}\right]^{\alpha/(\alpha-1)}\right\} \leq \operatorname{Tr}\left\{\left[\rho_{ABC}^{1/2}K(\alpha)\rho_{ABC}^{1/2}\right]^\alpha\right\}. \tag{6.37}$$

From the equality condition $X^p = Y^q$, we can see that the optimal H attaining equality is

$$L(\alpha)^{-1/2}\left[K(\alpha)^{1/2}\rho_{ABC}K(\alpha)^{1/2}\right]^{\alpha-1}L(\alpha)^{-1/2}. \tag{6.38}$$

This proves the representation formula in (6.28). A proof similar to the above one demonstrates (6.17)-(6.19) for $\alpha \in (1,\infty)$. ■

Remark 24. It is open to determine whether Lemma 23 applies to $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)$. That is, it is not clear to us whether Lemma 23 can be extended by a straightforward continuity argument as was the case in Ref. 24, Proposition 3, due to the fact that $\widetilde{\Delta}_\alpha$ features many non-commutative matrix multiplications which can interact in non-trivial ways.

Remark 25. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})_{++}$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})_{++}$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)_{++}$. It is an open question to determine whether the $\widetilde{\Delta}_\alpha$ quantities defined from (6.1), (6.7)-(6.11) are monotone non-increasing with respect to quantum operations acting on either systems A or B for $\alpha \in [1/2,1) \cup (1,\infty)$. It is also an open question to determine whether $\widetilde{I}_\alpha(A;B|C)_\rho$ is monotone non-increasing with respect to local quantum operations acting on the system A for $\alpha \in [1/2,1) \cup (1,\infty)$.

Corollary 26. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})_{++}$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})_{++}$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)_{++}$. All sandwiched Rényi generalizations of the conditional mutual information derived from

$$\widetilde{\Delta}_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}), \quad \widetilde{\Delta}_\alpha(\rho_{ABC},\omega_C,\tau_{AC},\theta_{BC}) \tag{6.39}$$

are monotone non-increasing with respect to quantum operations on system B, for $\alpha \in [1/2,1) \cup (1,\infty)$. *All sandwiched Rényi generalizations of the conditional mutual information derived from*

$$\widetilde{\Delta}_\alpha\left(\rho_{ABC},\omega_C,\theta_{BC},\tau_{AC}\right), \quad \widetilde{\Delta}_\alpha\left(\rho_{ABC},\theta_{BC},\omega_C,\tau_{AC}\right) \tag{6.40}$$

are monotone non-increasing with respect to quantum operations on system A, for $\alpha \in [1/2,1) \cup (1,\infty)$.

Proof. The argument is exactly the same as that in the proof of Corollary 15. ■

Corollary 27. We can employ the monotonicity inequalities from Lemma 13 to conclude that some Rényi generalizations of the conditional mutual information derived from (6.39) and (6.40) and Proposition 2 are non-negative for all $\alpha \in [1/2,1) \cup (1,\infty)$. *This includes* $\widetilde{\Delta}_\alpha(\rho_{ABC},\rho_{AC},\rho_C,\rho_{BC})$ *and the one from (6.13).*

Proof. The argument proceeds similarly to that in the proof of Corollary 16. ■

Remark 28. It is an open question to determine whether all sandwiched Rényi generalizations of the conditional mutual information designed from the different optimizations in Proposition 2 and the different orderings in (6.1), (6.7)-(6.11) are non-negative for $\alpha \in [1/2,1) \cup (1,\infty)$.

C. Max- and min-conditional mutual information

Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})_{++}$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})_{++}$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)_{++}$. In this section, we define a max- and min-conditional mutual information from the following two core quantities:

$$\Delta_{\max}\left(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}\right) \equiv \log\left\|\rho_{ABC}^{1/2}\tau_{AC}^{-1/2}\omega_C^{1/2}\theta_{BC}^{-1}\omega_C^{1/2}\tau_{AC}^{-1/2}\rho_{ABC}^{1/2}\right\|_\infty \tag{6.41}$$

$$= \inf\left\{\lambda : \rho_{ABC} \leq \exp\left(\lambda\right)\tau_{AC}^{1/2}\omega_C^{-1/2}\theta_{BC}\omega_C^{-1/2}\tau_{AC}^{1/2}\right\}, \tag{6.42}$$

$$\Delta_{\min}\left(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}\right) \equiv \widetilde{\Delta}_{1/2}\left(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}\right) \tag{6.43}$$

$$= -\log\left\|\sqrt{\rho_{ABC}}\sqrt{\tau_{AC}^{1/2}\omega_C^{-1/2}\theta_{BC}\omega_C^{-1/2}\tau_{AC}^{1/2}}\right\|_1^2 \tag{6.44}$$

$$= -\log F\left(\rho_{ABC},\tau_{AC}^{1/2}\omega_C^{-1/2}\theta_{BC}\omega_C^{-1/2}\tau_{AC}^{1/2}\right). \tag{6.45}$$

Also, the fidelity between $P \in \mathcal{B}(\mathcal{H})_+$ and $Q \in \mathcal{B}(\mathcal{H})_+$ is defined as $F\left(P,Q\right) \equiv \|\sqrt{P}\sqrt{Q}\|_1^2$. These quantities are inspired by the max-relative entropy from Ref. 17, defined as

$$D_{\max}\left(\rho\|\sigma\right) \equiv \inf\left\{\lambda : \rho \leq \exp\left(\lambda\right)\sigma\right\}, \tag{6.46}$$

when $\operatorname{supp}\left(\rho\right) \subseteq \operatorname{supp}\left(\sigma\right)$ and $+\infty$ otherwise, and the min-relative entropy from Ref. 40, defined as

$$D_{\min}\left(\rho\|\sigma\right) \equiv \widetilde{D}_{1/2}\left(\rho\|\sigma\right) = -\log F\left(\rho,\sigma\right). \tag{6.47}$$

We first state a generalization of the result that $\lim_{\alpha\to\infty}\widetilde{D}_\alpha\left(\rho\|\sigma\right) = D_{\max}\left(\rho\|\sigma\right)$ [Ref. 50, Theorem 5]:

Proposition 29. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})_{++}$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})_{++}$, *and* $\omega_C \in \mathcal{S}(\mathcal{H}_C)_{++}$. *Then*

$$\lim_{\alpha\to\infty}\widetilde{\Delta}_\alpha\left(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}\right) = \Delta_{\max}\left(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}\right). \tag{6.48}$$

The idea for the proof is the same as that for the proof of Ref. 50, Theorem 5, and we provide it in Appendix D. Next, we turn to monotonicity of $\Delta_{\max}$ under local quantum operations:

Proposition 30. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})_{++}$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})_{++}$, *and* $\omega_C \in \mathcal{S}(\mathcal{H}_C)_{++}$. *Let* $\mathcal{N}_{A\to A'}$ *and* $\mathcal{M}_{B\to B'}$ *denote local quantum operations acting on systems A and B, respectively. Then the following monotonicity inequalities hold:*

$$\Delta_{\max}\left(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}\right) \geq \Delta_{\max}\left(\mathcal{M}_{B\to B'}\left(\rho_{ABC}\right),\tau_{AC},\omega_C,\mathcal{M}_{B\to B'}\left(\theta_{BC}\right)\right), \tag{6.49}$$

$$\Delta_{\max}(\rho_{ABC},\omega_C,\tau_{AC},\theta_{BC}) \geq \Delta_{\max}(\mathcal{M}_{B\to B'}(\rho_{ABC}),\omega_C,\tau_{AC},\mathcal{M}_{B\to B'}(\theta_{BC})), \tag{6.50}$$

$$\Delta_{\max}(\rho_{ABC},\omega_C,\theta_{BC},\tau_{AC}) \geq \Delta_{\max}(\mathcal{N}_{A\to A'}(\rho_{ABC}),\omega_C,\theta_{BC},\mathcal{N}_{A\to A'}(\tau_{AC})), \tag{6.51}$$

$$\Delta_{\max}(\rho_{ABC},\theta_{BC},\omega_C,\tau_{AC}) \geq \Delta_{\max}(\mathcal{N}_{A\to A'}(\rho_{ABC}),\theta_{BC},\omega_C,\mathcal{N}_{A\to A'}(\tau_{AC})). \tag{6.52}$$

Proof. We begin by establishing (6.49). Let $\lambda^* = \Delta_{\max}(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC})$, so that

$$\rho_{ABC} \leq \exp(\lambda^*)\,\tau_{AC}^{1/2}\omega_C^{-1/2}\theta_{BC}\omega_C^{-1/2}\tau_{AC}^{1/2}. \tag{6.53}$$

For any CPTP map $\mathcal{M}_{B\to B'}$, the inequality in (6.53) implies the following operator inequality:

$$\mathcal{M}_{B\to B'}(\rho_{ABC}) \leq \exp(\lambda^*)\,\tau_{AC}^{1/2}\omega_C^{-1/2}\mathcal{M}_{B\to B'}(\theta_{BC})\,\omega_C^{-1/2}\tau_{AC}^{1/2}. \tag{6.54}$$

From the definition of $\Delta_{\max}$, we can conclude that

$$\lambda^* \geq \Delta_{\max}(\mathcal{M}_{B\to B'}(\rho_{ABC}),\tau_{AC},\omega_C,\mathcal{M}_{B\to B'}(\theta_{BC})), \tag{6.55}$$

which is equivalent to (6.49). The inequalities in (6.50)-(6.52) follow from a similar line of reasoning. ■

We define a max-conditional mutual information as follows:

$$I_{\max}(A;B|C)_{\rho|\rho} \equiv \Delta_{\max}(\rho_{ABC},\rho_{AC},\rho_C,\rho_{BC}). \tag{6.56}$$

This generalizes the max-mutual information, defined in Ref. 8, and its variations.[14] We define min-conditional mutual information as follows:

$$I_{\min}(A;B|C)_{\rho|\rho} \equiv \Delta_{\min}(\rho_{ABC},\rho_{AC},\rho_C,\rho_{BC}). \tag{6.57}$$

The forms given above seem quite natural, as the operators $\rho_{AC}^{1/2}\rho_C^{-1/2}\rho_{BC}\rho_C^{-1/2}\rho_{AC}^{1/2}$ appear in our review of quantum Markov states in Sec. III (however, note again that this operator is not a Markov state unless $\rho_{ABC} = \rho_{AC}^{1/2}\rho_C^{-1/2}\rho_{BC}\rho_C^{-1/2}\rho_{AC}^{1/2}$). Note that other min- and max-conditional mutual information quantities are possible by considering the other orderings and optimizations for the last three arguments to $\Delta_{\max}$ and $\Delta_{\min}$, but it is our impression that the above choice is natural.

VII. DUALITY

A fundamental property of the conditional mutual information is a duality relation: For a four-party pure state ψ_{ABCD}, the following equality holds

$$I(A;B|C)_\psi = I(A;B|D)_\psi. \tag{7.1}$$

This can easily be verified by considering Schmidt decompositions of ψ_{ABCD} for the different possible bipartite cuts of $ABCD$ (see Refs. 19 and 74 for an operational interpretation of this duality in terms of the state redistribution protocol). Furthermore, since the conditional mutual information is symmetric under the exchange of A and B, we have the following equalities:

$$I(B;A|C)_\psi = I(A;B|C)_\psi = I(A;B|D)_\psi = I(B;A|D)_\psi. \tag{7.2}$$

In this section, we prove that the Rényi conditional mutual information in Definition 7 and the sandwiched quantity in Definition 19 obey a duality relation of the above form. However, note that other (but not all) variations satisfy duality as well. In order to prove these results, we make use of the following standard lemma:

Lemma 31. For any bipartite pure state ψ_{AB}, any Hermitian operator M_A acting on system A, and the maximally entangled vector $|\Gamma\rangle_{AB} \equiv \sum_j |j\rangle_A|j\rangle_B$ (with $\{|j\rangle_A\}$ and $\{|j\rangle_B\}$ orthonormal bases), we have that

$$(M_A \otimes I_B)|\Gamma\rangle_{AB} = (I_A \otimes M_B^T)|\Gamma\rangle_{AB}, \tag{7.3}$$

$$\psi_A|\psi\rangle_{AB} = \psi_B|\psi\rangle_{AB}, \tag{7.4}$$

$$\langle\psi|M_A \otimes I_B|\psi\rangle_{AB} = \langle\psi|I_A \otimes M_B^T|\psi\rangle_{AB}, \tag{7.5}$$

where the transpose is with respect to the Schmidt basis.

Theorem 32. *The following duality relation holds for all $\alpha \in (0,1) \cup (1,\infty)$ for a pure four-party state ψ_{ABCD}:*

$$I_\alpha(A;B|C)_\psi = I_\alpha(B;A|D)_\psi. \tag{7.6}$$

Proof. Our proof exploits ideas used in the proof of Ref. 65, Lemma 6 and Ref. 64, Theorem 2. We know from Proposition 8 that

$$I_\alpha(A;B|C)_\psi = \frac{\alpha}{\alpha-1}\log \operatorname{Tr}\left\{\left(\operatorname{Tr}_A\left\{\psi_C^{(\alpha-1)/2}\psi_{AC}^{(\alpha-1)/2}\psi_{ABC}^{\alpha}\psi_{AC}^{(\alpha-1)/2}\psi_C^{(\alpha-1)/2}\right\}\right)^{1/\alpha}\right\}, \tag{7.7}$$

$$I_\alpha(B;A|D)_\psi = \frac{\alpha}{\alpha-1}\log \operatorname{Tr}\left\{\left(\operatorname{Tr}_B\left\{\psi_D^{(\alpha-1)/2}\psi_{BD}^{(\alpha-1)/2}\psi_{ABD}^{\alpha}\psi_{BD}^{(\alpha-1)/2}\psi_D^{(\alpha-1)/2}\right\}\right)^{1/\alpha}\right\}. \tag{7.8}$$

Thus, we will have proved the theorem if we can show that the eigenvalues of

$$\operatorname{Tr}_A\left\{\psi_C^{(\alpha-1)/2}\psi_{AC}^{(\alpha-1)/2}\psi_{ABC}^{\alpha}\psi_{AC}^{(\alpha-1)/2}\psi_C^{(\alpha-1)/2}\right\} \tag{7.9}$$

and

$$\operatorname{Tr}_B\left\{\psi_D^{(\alpha-1)/2}\psi_{BD}^{(\alpha-1)/2}\psi_{ABD}^{\alpha}\psi_{BD}^{(\alpha-1)/2}\psi_D^{(\alpha-1)/2}\right\} \tag{7.10}$$

are the same. To show this, consider that

$$\begin{aligned}
&\operatorname{Tr}_A\left\{\psi_C^{(\alpha-1)/2}\psi_{AC}^{(\alpha-1)/2}\psi_{ABC}^{\alpha}\psi_{AC}^{(\alpha-1)/2}\psi_C^{(\alpha-1)/2}\right\}\\
&= \operatorname{Tr}_A\left\{\psi_C^{(\alpha-1)/2}\psi_{AC}^{(\alpha-1)/2}\psi_{ABC}^{(\alpha-1)/2}\psi_{ABC}\psi_{ABC}^{(\alpha-1)/2}\psi_{AC}^{(\alpha-1)/2}\psi_C^{(\alpha-1)/2}\right\} &(7.11)\\
&= \operatorname{Tr}_{AD}\left\{\psi_C^{(\alpha-1)/2}\psi_{AC}^{(\alpha-1)/2}\psi_{ABC}^{(\alpha-1)/2}\psi_{ABCD}\psi_{ABC}^{(\alpha-1)/2}\psi_{AC}^{(\alpha-1)/2}\psi_C^{(\alpha-1)/2}\right\}. &(7.12)
\end{aligned}$$

The eigenvalues of the operator in the last line are the same as those of the operator in the first line of what follows (from the Schmidt decomposition):

$$\begin{aligned}
&\operatorname{Tr}_{BC}\left\{\psi_C^{(\alpha-1)/2}\psi_{AC}^{(\alpha-1)/2}\psi_{ABC}^{(\alpha-1)/2}\psi_{ABCD}\psi_{ABC}^{(\alpha-1)/2}\psi_{AC}^{(\alpha-1)/2}\psi_C^{(\alpha-1)/2}\right\}\\
&= \operatorname{Tr}_{BC}\left\{\psi_C^{(\alpha-1)/2}\psi_{AC}^{(\alpha-1)/2}\psi_D^{(\alpha-1)/2}\psi_{ABCD}\psi_D^{(\alpha-1)/2}\psi_{AC}^{(\alpha-1)/2}\psi_C^{(\alpha-1)/2}\right\} &(7.13)\\
&= \operatorname{Tr}_{BC}\left\{\psi_D^{(\alpha-1)/2}\psi_C^{(\alpha-1)/2}\psi_{AC}^{(\alpha-1)/2}\psi_{ABCD}\psi_{AC}^{(\alpha-1)/2}\psi_C^{(\alpha-1)/2}\psi_D^{(\alpha-1)/2}\right\} &(7.14)\\
&= \operatorname{Tr}_{BC}\left\{\psi_D^{(\alpha-1)/2}\psi_C^{(\alpha-1)/2}\psi_{BD}^{(\alpha-1)/2}\psi_{ABCD}\psi_{BD}^{(\alpha-1)/2}\psi_C^{(\alpha-1)/2}\psi_D^{(\alpha-1)/2}\right\} &(7.15)\\
&= \operatorname{Tr}_{BC}\left\{\psi_D^{(\alpha-1)/2}\psi_{BD}^{(\alpha-1)/2}\psi_C^{(\alpha-1)/2}\psi_{ABCD}\psi_C^{(\alpha-1)/2}\psi_{BD}^{(\alpha-1)/2}\psi_D^{(\alpha-1)/2}\right\} &(7.16)\\
&= \operatorname{Tr}_{BC}\left\{\psi_D^{(\alpha-1)/2}\psi_{BD}^{(\alpha-1)/2}\psi_{ABD}^{(\alpha-1)/2}\psi_{ABCD}\psi_{ABD}^{(\alpha-1)/2}\psi_{BD}^{(\alpha-1)/2}\psi_D^{(\alpha-1)/2}\right\} &(7.17)\\
&= \operatorname{Tr}_{B}\left\{\psi_D^{(\alpha-1)/2}\psi_{BD}^{(\alpha-1)/2}\psi_{ABD}^{(\alpha-1)/2}\psi_{ABD}\psi_{ABD}^{(\alpha-1)/2}\psi_{BD}^{(\alpha-1)/2}\psi_D^{(\alpha-1)/2}\right\} &(7.18)\\
&= \operatorname{Tr}_{B}\left\{\psi_D^{(\alpha-1)/2}\psi_{BD}^{(\alpha-1)/2}\psi_{ABD}^{\alpha}\psi_{BD}^{(\alpha-1)/2}\psi_D^{(\alpha-1)/2}\right\}. &(7.19)
\end{aligned}$$

In the above, we have applied (7.4) several times. ■

Theorem 33. *The following duality relation holds for all $\alpha \in (0,1) \cup (1,\infty)$ for a pure four-party state ψ_{ABCD}:*

$$\widetilde{I}_\alpha(A;B|C)_\psi = \widetilde{I}_\alpha(B;A|D)_\psi. \tag{7.20}$$

Proof. Our proof uses ideas similar to those in the proof of Ref. 50, Theorem 10. We start by considering the case $\alpha > 1$. We recall that it is possible to express the α-norm with its dual norm (see, e.g., Ref. 50, Lemma 12)

$$\begin{aligned}
&\inf_{\sigma_{BC}}\sup_{\omega_C}\left\|\psi_{ABC}^{1/2}\psi_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\sigma_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\psi_{AC}^{(1-\alpha)/2\alpha}\psi_{ABC}^{1/2}\right\|_\alpha =\\
&\qquad\inf_{\sigma_{BC}}\sup_{\omega_C}\sup_{\tau_{ABC}}\operatorname{Tr}\left\{\psi_{ABC}^{1/2}\psi_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\sigma_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\psi_{AC}^{(1-\alpha)/2\alpha}\psi_{ABC}^{1/2}\tau_{ABC}^{(\alpha-1)/\alpha}\right\}. \qquad (7.21)
\end{aligned}$$

So it suffices to prove the following relation:

$$\inf_{\sigma_{BC}} \sup_{\omega_C} \sup_{\tau_{ABC}} \operatorname{Tr}\left\{\psi_{ABC}^{1/2}\psi_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\sigma_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\psi_{AC}^{(1-\alpha)/2\alpha}\psi_{ABC}^{1/2}\tau_{ABC}^{(\alpha-1)/\alpha}\right\} = \inf_{\sigma_{AD}} \sup_{\tau_D} \sup_{\omega_{ABD}} \operatorname{Tr}\left\{\psi_{ABD}^{1/2}\psi_{BD}^{(1-\alpha)/2\alpha}\tau_D^{(\alpha-1)/2\alpha}\sigma_{AD}^{(1-\alpha)/\alpha}\tau_D^{(\alpha-1)/2\alpha}\psi_{AC}^{(1-\alpha)/2\alpha}\psi_{ABD}^{1/2}\omega_{ABD}^{(\alpha-1)/\alpha}\right\}, \quad (7.22)$$

because

$$\widetilde{I}_\alpha(B;A|D)_\psi = \inf_{\sigma_{AD}} \sup_{\tau_D} \sup_{\omega_{ABD}} \frac{\alpha}{\alpha-1}\log \operatorname{Tr}\left\{\psi_{ABD}^{1/2}\psi_{BD}^{(1-\alpha)/2\alpha}\tau_D^{(\alpha-1)/2\alpha}\sigma_{AD}^{(1-\alpha)/\alpha}\tau_D^{(\alpha-1)/2\alpha}\psi_{AC}^{(1-\alpha)/2\alpha}\psi_{ABD}^{1/2}\omega_{ABD}^{(\alpha-1)/\alpha}\right\}. \quad (7.23)$$

Indeed, we will prove that

$$\operatorname{Tr}\left\{\psi_{ABC}^{1/2}\psi_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\sigma_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\psi_{AC}^{(1-\alpha)/2\alpha}\psi_{ABC}^{1/2}\tau_{ABC}^{(\alpha-1)/\alpha}\right\} = \operatorname{Tr}\left\{\psi_{ABD}^{1/2}\psi_{BD}^{(1-\alpha)/2\alpha}(\tau_D^T)^{(\alpha-1)/2\alpha}(\sigma_{AD}^T)^{(1-\alpha)/\alpha}(\tau_D^T)^{(\alpha-1)/2\alpha}\psi_{BD}^{(1-\alpha)/2\alpha}\psi_{ABD}^{1/2}(\omega_{ABD}^T)^{(\alpha-1)/\alpha}\right\}, \quad (7.24)$$

from which one can conclude (7.22), which has the optimizations.

Proceeding, we observe that

$$\begin{aligned}
&\operatorname{Tr}\left\{\psi_{ABC}^{1/2}\psi_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\sigma_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\psi_{AC}^{(1-\alpha)/2\alpha}\psi_{ABC}^{1/2}\tau_{ABC}^{(\alpha-1)/\alpha}\right\}\\
&= \langle\Gamma|\,\psi_{ABC}^{1/2}\psi_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\sigma_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\psi_{AC}^{(1-\alpha)/2\alpha}\psi_{ABC}^{1/2}\tau_{ABC}^{(\alpha-1)/\alpha}|\Gamma\rangle_{ABC|D} && (7.25)\\
&= \langle\Gamma|\,\psi_{ABC}^{1/2}\psi_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\sigma_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\psi_{AC}^{(1-\alpha)/2\alpha}\psi_{ABC}^{1/2}(\tau_D^T)^{(\alpha-1)/\alpha}|\Gamma\rangle_{ABC|D} && (7.26)\\
&= \langle\psi|\,\psi_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}(\tau_D^T)^{(\alpha-1)/2\alpha}\sigma_{BC}^{(1-\alpha)/\alpha}(\tau_D^T)^{(\alpha-1)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\psi_{AC}^{(1-\alpha)/2\alpha}|\psi\rangle_{ABCD} && (7.27)\\
&= \langle\psi|\,\psi_{BD}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}(\tau_D^T)^{(\alpha-1)/2\alpha}\sigma_{BC}^{(1-\alpha)/\alpha}(\tau_D^T)^{(\alpha-1)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\psi_{BD}^{(1-\alpha)/2\alpha}|\psi\rangle_{ABCD} && (7.28)\\
&= \langle\psi|\,\omega_C^{(\alpha-1)/2\alpha}\psi_{BD}^{(1-\alpha)/2\alpha}(\tau_D^T)^{(\alpha-1)/2\alpha}\sigma_{BC}^{(1-\alpha)/\alpha}(\tau_D^T)^{(\alpha-1)/2\alpha}\psi_{BD}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}|\psi\rangle_{ABCD} && (7.29)\\
&= \langle\Gamma|\,\psi_{ABD}^{1/2}\omega_C^{(\alpha-1)/2\alpha}\psi_{BD}^{\frac{1-\alpha}{2\alpha}}(\tau_D^T)^{\frac{\alpha-1}{2\alpha}}\sigma_{BC}^{\frac{1-\alpha}{\alpha}}(\tau_D^T)^{\frac{\alpha-1}{2\alpha}}\psi_{BD}^{\frac{1-\alpha}{2\alpha}}\omega_C^{(\alpha-1)/2\alpha}\psi_{ABD}^{1/2}|\Gamma\rangle_{ABD|C} && (7.30)\\
&= \langle\Gamma|\,\omega_C^{(\alpha-1)/2\alpha}\psi_{ABD}^{1/2}\psi_{BD}^{\frac{1-\alpha}{2\alpha}}(\tau_D^T)^{\frac{\alpha-1}{2\alpha}}\sigma_{BC}^{\frac{1-\alpha}{\alpha}}(\tau_D^T)^{\frac{\alpha-1}{2\alpha}}\psi_{BD}^{\frac{1-\alpha}{2\alpha}}\psi_{ABD}^{1/2}\omega_C^{(\alpha-1)/2\alpha}|\Gamma\rangle_{ABD|C} && (7.31)\\
&= \langle\Gamma|\,(\omega_{ABD}^T)^{\frac{\alpha-1}{2\alpha}}\psi_{ABD}^{1/2}\psi_{BD}^{\frac{1-\alpha}{2\alpha}}(\tau_D^T)^{\frac{\alpha-1}{2\alpha}}\sigma_{BC}^{\frac{1-\alpha}{\alpha}}(\tau_D^T)^{\frac{\alpha-1}{2\alpha}}\psi_{BD}^{\frac{1-\alpha}{2\alpha}}\psi_{ABD}^{1/2}(\omega_{ABD}^T)^{\frac{\alpha-1}{2\alpha}}|\Gamma\rangle_{ABD|C}, && (7.32)
\end{aligned}$$

where we used standard transpose trick (7.3) for the maximally entangled vector $|\Gamma\rangle_{ABD|C}$ and the first identity from Lemma 31. For the vector

$$|\varphi\rangle_{ABCD} \equiv (\tau_D^T)^{(\alpha-1)/2\alpha}\psi_{BD}^{(1-\alpha)/2\alpha}\psi_{ABD}^{1/2}(\omega_{ABD}^T)^{(\alpha-1)/2\alpha}|\Gamma\rangle_{ABD|C}, \quad (7.33)$$

we get from the second identity in Lemma 31 that

$$\begin{aligned}
&\langle\Gamma|\,(\omega_{ABD}^T)^{\frac{\alpha-1}{2\alpha}}\psi_{ABD}^{1/2}\psi_{BD}^{\frac{1-\alpha}{2\alpha}}(\tau_D^T)^{\frac{\alpha-1}{2\alpha}}\sigma_{BC}^{\frac{1-\alpha}{\alpha}}(\tau_D^T)^{\frac{\alpha-1}{2\alpha}}\psi_{BD}^{\frac{1-\alpha}{2\alpha}}\psi_{ABD}^{1/2}(\omega_{ABD}^T)^{\frac{\alpha-1}{2\alpha}}|\Gamma\rangle_{ABD|C}\\
&= \langle\varphi|\,\sigma_{BC}^{\frac{1-\alpha}{\alpha}}|\varphi\rangle_{ABCD} && (7.34)\\
&= \langle\varphi|\,(\sigma_{AD}^T)^{\frac{1-\alpha}{\alpha}}|\varphi\rangle_{ABCD} && (7.35)\\
&= \langle\Gamma|\,(\omega_{ABD}^T)^{\frac{\alpha-1}{2\alpha}}\psi_{ABD}^{1/2}\psi_{BD}^{\frac{1-\alpha}{2\alpha}}(\tau_D^T)^{\frac{\alpha-1}{2\alpha}}(\sigma_{AD}^T)^{\frac{1-\alpha}{\alpha}}(\tau_D^T)^{\frac{\alpha-1}{2\alpha}}\psi_{BD}^{\frac{1-\alpha}{2\alpha}}\psi_{ABD}^{1/2}(\omega_{ABD}^T)^{\frac{\alpha-1}{2\alpha}}|\Gamma\rangle_{ABD|C} && (7.36)\\
&= \operatorname{Tr}\left\{(\omega_{ABD}^T)^{\frac{\alpha-1}{2\alpha}}\psi_{ABD}^{1/2}\psi_{BD}^{\frac{1-\alpha}{2\alpha}}(\tau_D^T)^{\frac{\alpha-1}{2\alpha}}(\sigma_{AD}^T)^{(1-\alpha)/\alpha}(\tau_D^T)^{\frac{\alpha-1}{2\alpha}}\psi_{BD}^{\frac{1-\alpha}{2\alpha}}\psi_{ABD}^{1/2}(\omega_{ABD}^T)^{\frac{\alpha-1}{2\alpha}}\right\} && (7.37)\\
&= \operatorname{Tr}\left\{\psi_{ABD}^{1/2}\psi_{BD}^{(1-\alpha)/2\alpha}(\tau_D^T)^{(\alpha-1)/2\alpha}(\sigma_{AD}^T)^{(1-\alpha)/\alpha}(\tau_D^T)^{(\alpha-1)/2\alpha}\psi_{BD}^{(1-\alpha)/2\alpha}\psi_{ABD}^{1/2}(\omega_{ABD}^T)^{(\alpha-1)/\alpha}\right\}. && (7.38)
\end{aligned}$$

For the case $\alpha \in (0,1)$ the proof is similar, where we also use Ref. 50, Lemma 12. We omit the details for this case. ■

VIII. MONOTONICITY IN α

From numerical evidence and proofs for some special cases, we think it is natural to put forward the following conjecture:

Conjecture 34. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})_{++}$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})_{++}$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)_{++}$. Then all of the Rényi core quantities Δ_α and $\widetilde{\Delta}_\alpha$ derived from (5.3), (5.7)-(5.11) and (6.1), (6.7)-(6.11), respectively, are monotone non-decreasing in α. That is, for $0 \leq \alpha \leq \beta$, the following inequalities hold

$$\Delta_\alpha\left(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}\right) \leq \Delta_\beta\left(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}\right), \tag{8.1}$$

$$\widetilde{\Delta}_\alpha\left(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}\right) \leq \widetilde{\Delta}_\beta\left(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}\right), \tag{8.2}$$

and similar inequalities hold for all orderings of the last three arguments of Δ_α and $\widetilde{\Delta}_\alpha$.

If Conjecture 34 is true, we could conclude that all non-sandwiched and sandwiched Rényi generalizations of the conditional mutual information are monotone non-decreasing in α for positive definite operators. Another implication of monotonicity in $\alpha \geq 1/2$ for $\widetilde{\Delta}_\alpha\left(\rho_{ABC}, \rho_{AC}, \rho_C, \rho_{BC}\right)$ would be that a tripartite quantum state ρ_{ABC} is a quantum Markov state if and only if

$$\widetilde{\Delta}_\alpha\left(\rho_{ABC}, \rho_{AC}, \rho_C, \rho_{BC}\right) = 0 \tag{8.3}$$

(with $\alpha \geq 1/2$). This would generalize the results from Ref. 28 to the case $\alpha \neq 1$.

Note that this conjecture does not follow straightforwardly from the following monotonicity

$$D_\alpha\left(\rho \| \sigma\right) \leq D_\beta\left(\rho \| \sigma\right), \tag{8.4}$$

$$\widetilde{D}_\alpha\left(\rho \| \sigma\right) \leq \widetilde{D}_\beta\left(\rho \| \sigma\right), \tag{8.5}$$

which holds for $0 \leq \alpha \leq \beta$.[65,50] However, for classical states ρ_{ABC}, the conjecture is clearly true for $\Delta_\alpha\left(\rho_{ABC}, \rho_{AC}, \rho_C, \rho_{BC}\right)$ and $\widetilde{\Delta}_\alpha\left(\rho_{ABC}, \rho_{AC}, \rho_C, \rho_{BC}\right)$ by appealing to the above known inequalities.

Observe that some of the conjectured inequalities are redundant. For example, if

$$\Delta_\alpha\left(\rho_{ABC}, \tau_{AC}, \theta_{BC}, \omega_C\right) \leq \Delta_\beta\left(\rho_{ABC}, \tau_{AC}, \theta_{BC}, \omega_C\right) \tag{8.6}$$

holds for all $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})_{++}$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})_{++}$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)_{++}$, then the following monotonicity holds as well

$$\Delta_\alpha\left(\rho_{ABC}, \theta_{BC}, \tau_{AC}, \omega_C\right) \leq \Delta_\beta\left(\rho_{ABC}, \theta_{BC}, \tau_{AC}, \omega_C\right), \tag{8.7}$$

due to a symmetry under the exchange of systems A and B. Similar statements apply to other pairs of inequalities, so that it suffices to prove only six of the 12 monotonicities discussed above in order to establish the other six. However, as we will see below, a single proof of the monotonicity for each kind of Rényi conditional mutual information (non-sandwiched and sandwiched) should suffice because we think one could easily generalize such a proof to the other cases.

A. Approaches for proving the conjecture

We briefly outline some approaches for proving the conjecture. One idea is to follow a proof technique from Ref. 65, Lemma 3 and Ref. 50, Theorem 7. If the derivative of $\Delta_\alpha\left(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}\right)$ and $\widetilde{\Delta}_\alpha\left(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}\right)$ with respect to α is non-negative, then we can conclude that these functions are monotone increasing with α. It is possible to prove that the derivatives are non-negative when α is in a neighborhood of one, by computing Taylor expansions of these functions. We explore this approach further in Appendix E.

B. Numerical evidence

To test the conjecture in (8.1) and its variations, we conducted several numerical experiments. First, we selected states ρ_{ABC}, τ_{AC}, ω_C, θ_{BC} at random,[16] with the dimensions of the local systems never exceeding six. We then computed the numerator in (E6) for values of γ ranging from -0.99

to 10 with a step size of 0.05 (so that $\alpha = \gamma + 1$ goes from 0.01 to 11). For each value of γ, we conducted 1000 numerical experiments. The result was that the numerator in (E6) was always non-negative. We then conducted the same set of experiments for the various operator orderings and always found the numerator to be non-negative.

To test the conjecture in (8.2) and its variations, we conducted similar numerical experiments. First, we selected states ρ_{ABC}, τ_{AC}, ω_C, θ_{BC}, μ_{ABC} at random,[16] with the dimensions of the local systems never exceeding six. We then computed the numerator in (E15) for values of γ ranging from -10 to 0.99 with a step size of 0.05 (so that $\alpha = 1/(1-\gamma)$ goes from ≈ 0.091 to ≈ 100). For each value of γ, we conducted 1000 numerical experiments. The result was that the numerator in (E15) was always non-negative. We then conducted the same set of experiments for the various operator orderings and always found the numerator to be non-negative.

C. Special cases of the conjecture

We can prove that the conjecture is true in a number of cases, due to the special form that the Rényi conditional mutual information takes in these cases. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$. We define the following quantities, which are the same as (2.3) and (2.5), respectively:

$$I_\alpha(A;B|C)_{\rho|\rho} \equiv \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\rho_{BC}^{1-\alpha}\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\right\}, \tag{8.8}$$

$$\widetilde{I}_\alpha(A;B|C)_{\rho|\rho} \equiv \frac{\alpha}{\alpha-1}\log \left\|\rho_{ABC}^{1/2}\rho_{AC}^{(1-\alpha)/2\alpha}\rho_C^{(\alpha-1)/2\alpha}\rho_{BC}^{(1-\alpha)/\alpha}\rho_C^{(\alpha-1)/2\alpha}\rho_{AC}^{(1-\alpha)/2\alpha}\rho_{ABC}^{1/2}\right\|_\alpha, \tag{8.9}$$

so that

$$I_0(A;B|C)_{\rho|\rho} = -\log \operatorname{Tr}\left\{\rho_{ABC}^{0}\rho_{AC}^{1/2}\rho_C^{-1/2}\rho_{BC}\rho_C^{-1/2}\rho_{AC}^{1/2}\right\}, \tag{8.10}$$

$$I_2(A;B|C)_{\rho|\rho} = \log \operatorname{Tr}\left\{\rho_{ABC}^{2}\left(\rho_{AC}^{1/2}\rho_C^{-1/2}\rho_{BC}\rho_C^{-1/2}\rho_{AC}^{1/2}\right)^{-1}\right\}. \tag{8.11}$$

Recall that the following inequality holds for all $\alpha \in (0,1)\cup(1,\infty)$:[18]

$$\widetilde{D}_\alpha(\rho\|\sigma) \le D_\alpha(\rho\|\sigma). \tag{8.12}$$

Using the monotonicity given in (8.5) and the above inequality, we can conclude that

$$I_0(A;B|C)_{\rho|\rho} \le I_2(A;B|C)_{\rho|\rho}, \tag{8.13}$$

$$I_{\min}(A;B|C)_{\rho|\rho} \le I_{\max}(A;B|C)_{\rho|\rho}, \tag{8.14}$$

$$I_{\min}(A;B|C)_{\rho|\rho} \le I_2(A;B|C)_{\rho|\rho}, \tag{8.15}$$

where $I_{\max}(A;B|C)_{\rho|\rho}$ and $I_{\min}(A;B|C)_{\rho|\rho}$ are defined in (6.56) and (6.57), respectively. However, we cannot relate to the (von Neumann entropy based) conditional mutual information because its representation in terms of the relative entropy does not feature the operator $\rho_{AC}^{1/2}\rho_C^{-1/2}\rho_{BC}\rho_C^{-1/2}\rho_{AC}^{1/2}$ as its second argument but instead has $\exp\{\log\rho_{BC} + \log\rho_{AC} - \log\rho_C\}$.

Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})_{++}$, $\omega_C \in \mathcal{S}(\mathcal{H}_C)_{++}$, and $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})_{++}$. Tomamichel has informed us that the inequality in (8.2) and its variations are true for $0 \le \alpha \le \beta$ and such that $1/\alpha + 1/\beta = 2$.[63] This is because in such a case, we have that $\alpha/(1-\alpha) = -\beta(1-\beta)$, so that

$$\left[\tau_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\theta_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\tau_{AC}^{(1-\alpha)/2\alpha}\right]^{\alpha/(1-\alpha)} = \left[\tau_{AC}^{(1-\beta)/2\beta}\omega_C^{(\beta-1)/2\beta}\theta_{BC}^{(1-\beta)/\beta}\omega_C^{(\beta-1)/2\beta}\tau_{AC}^{(1-\beta)/2\beta}\right]^{\beta/(1-\beta)}, \tag{8.16}$$

and similar equalities hold for the five other operator orderings. Since this is the case, the monotonicity follows directly from the ordinary monotonicity of the sandwiched Rényi relative entropy. By a similar line of reasoning, the inequality in (8.1) and its variations are true for $0 \le \alpha \le \beta$ and such that $\alpha + \beta = 2$. Similarly, in such a case, we have that $1-\alpha = -(1-\beta)$, so that

$$\left[\tau_{AC}^{(1-\alpha)/2}\omega_C^{(\alpha-1)/2}\theta_{BC}^{1-\alpha}\omega_C^{(\alpha-1)/2}\tau_{AC}^{(1-\alpha)/2}\right]^{1/(1-\alpha)} = \left[\tau_{AC}^{(1-\beta)/2}\omega_C^{(\beta-1)/2}\theta_{BC}^{1-\beta}\omega_C^{(\beta-1)/2}\tau_{AC}^{(1-\beta)/2}\right]^{1/(1-\beta)}, \tag{8.17}$$

and similar equalities hold for the five other operator orderings. Then the monotonicity again follows from the ordinary monotonicity of the Rényi relative entropy. The observations in (8.13) and (8.14) are then special cases of the above observations.

D. Implications for tripartite states with small conditional mutual information

It has been an open question since the work in Ref. 28 to characterize tripartite quantum states ρ_{ABC} with small conditional mutual information $I(A;B|C)_\rho$. That is, given that the various quantum Markov state conditions in (4.37) and (4.38)-(4.40) are equivalent to $I(A;B|C)_\rho$ being equal to zero, we would like to understand what happens when we perturb these various conditions. In this section, we pursue this direction and explicitly show how Conjecture 34 could be used to address this important question.

Several researchers have already considered what happens when perturbing the quantum Markov state condition in (4.37), but we include a discussion here for completeness. To begin with, we know that if there exists a quantum Markov state $\mu_{ABC} \in \mathcal{M}_{A-C-B}$ such that

$$\|\rho_{ABC} - \mu_{ABC}\|_1 \leq \varepsilon \tag{8.18}$$

then

$$I(A;B|C)_\mu = 0, \tag{8.19}$$

$$I(A;B|C)_\rho \leq 8\varepsilon \log \min\{d_A, d_B\} + 4h_2(\varepsilon), \tag{8.20}$$

where

$$h_2(x) \equiv -x\log x - (1-x)\log(1-x) \tag{8.21}$$

is the binary entropy, which obeys

$$\lim_{\varepsilon \searrow 0} h_2(\varepsilon) = 0. \tag{8.22}$$

The first line is by definition and the second follows from an application of the Alicki-Fannes inequality.[3] However, the example in Ref. 12 and the subsequent development in Ref. 22 exclude a particular converse of the above bound. That is, by Ref. 12, Lemma 6, there exists a sequence of states ρ^d_{ABC} such that

$$I(A;B|C)_{\rho^d} = 2\log((d+2)/d), \tag{8.23}$$

which goes to zero as $d \to \infty$. However, for this same sequence of states, the following constant lower bound is known:

$$\min_{\mu_{ABC} \in \mathcal{M}_{A-C-B}} D_0\left(\rho^d_{ABC} \| \mu_{ABC}\right) \geq \log\sqrt{4/3}, \tag{8.24}$$

by Ref. 22, Theorem 1. By employing monotonicity of the Rényi relative entropy with respect to the Rényi parameter, so that $D_{1/2} \geq D_0$, and the well-known relation $1 - \|\omega - \tau\|_1/2 \leq \mathrm{Tr}\{\sqrt{\omega}\sqrt{\tau}\}$ for $\omega, \tau \in \mathcal{S}(\mathcal{H})$ (see, e.g., Ref. 10, Eq. (22)), we can readily translate the bound in (8.24) to a constant lower bound on the trace distance of ρ^d_{ABC} to the set of quantum Markov states

$$\left\|\rho^d_{ABC} - \mathcal{M}_{A-C-B}\right\|_1 \equiv \min_{\mu_{ABC} \in \mathcal{M}_{A-C-B}} \left\|\rho^d_{ABC} - \mu_{ABC}\right\|_1 \geq 2\left(1 - (3/4)^{1/4}\right) \approx 0.139. \tag{8.25}$$

So (8.23) and (8.25) imply that a Pinsker-like bound of the form $I(A;B|C)_\rho \geq K\,\|\rho_{ABC} - \mathcal{M}_{A-C-B}\|_1^2$ cannot hold in general, with K a dimension-independent constant.

We now focus on a perturbation of the conditions in (4.38) and (4.39). It appears that these cases will be promising for applications if Conjecture 34 is true. The following proposition states that the conditional mutual information is small if it is possible to recover the system A from system C alone (or by symmetry, if one can get B from C alone). We note that (8.28) was proven independently in Ref. 23, Eq. (8).

Proposition 35. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$, $\mathcal{R}_{C\to AC}$ be a CPTP "recovery" map, and $\varepsilon \in [0,1]$. Suppose that it is possible to recover the system A from system C alone, in the following sense:

$$\|\rho_{ABC} - \omega_{ABC}\|_1 \leq \varepsilon, \tag{8.26}$$

where

$$\omega_{ABC} \equiv \mathcal{R}_{C\to AC}(\rho_{BC}). \tag{8.27}$$

Then the conditional mutual informations $I(A;B|C)_\rho$ and $I(A;B|C)_\omega$ obey the following bounds:

$$I(A;B|C)_\rho \leq 4\varepsilon \log d_B + 2h_2(\varepsilon), \tag{8.28}$$

$$I(A;B|C)_\omega \leq 4\varepsilon \log d_B + 2h_2(\varepsilon), \tag{8.29}$$

where d_B is the dimension of the B system and $h_2(\varepsilon)$ is defined in (8.21). By symmetry, a related bound holds if one can recover system B from system C alone.

Proof. Consider that

$$\begin{aligned} I(A;B|C)_\rho &= H(B|C)_\rho - H(B|AC)_\rho && (8.30)\\ &\leq H(B|AC)_\omega - H(B|AC)_\rho && (8.31)\\ &\leq H(B|AC)_\omega - H(B|AC)_\omega + 4\varepsilon \log d_B + 2h_2(\varepsilon) && (8.32)\\ &= 4\varepsilon \log d_B + 2h_2(\varepsilon). && (8.33) \end{aligned}$$

The first inequality follows because the conditional entropy is monotone increasing under quantum operations on the conditioning system (the map $\mathcal{R}_{C\to AC}$ is applied to the system C of state ρ_{ABC} to produce ω_{ABC} and the conditional entropy only increases under such processing). The second inequality is a result of (8.26) and the Alicki-Fannes inequality[3] (continuity of conditional entropy). Similarly, consider that

$$\begin{aligned} I(A;B|C)_\omega &= H(B|C)_\omega - H(B|AC)_\omega && (8.34)\\ &\leq H(B|C)_\rho - H(B|AC)_\omega + 4\varepsilon \log d_B + 2h_2(\varepsilon) && (8.35)\\ &\leq H(B|AC)_\omega - H(B|AC)_\omega + 4\varepsilon \log d_B + 2h_2(\varepsilon) && (8.36)\\ &= 4\varepsilon \log d_B + 2h_2(\varepsilon). && (8.37) \end{aligned}$$

The first inequality is from the fact that (8.26) implies that

$$\|\rho_{BC} - \omega_{BC}\|_1 \leq \varepsilon \tag{8.38}$$

and the Alicki-Fannes' inequality. The second is again from monotonicity of conditional entropy. ■

The implications of Conjecture 34 are nontrivial. For example, if it were true, then we could conclude a converse of Proposition 35, that if the conditional mutual information is small, then it is possible to recover the system A from system C alone (or by symmetry, that one can get B from C alone). That is, the following relation would hold for $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$:

$$\begin{aligned} I(A;B|C)_\rho &\geq I_{\min}(A;B|C)_{\rho|\rho} && (8.39)\\ &= -\log F\left(\rho_{ABC}, \rho_{AC}^{1/2}\rho_C^{-1/2}\rho_{BC}\rho_C^{-1/2}\rho_{AC}^{1/2}\right) && (8.40)\\ &= -\log F\left(\rho_{ABC}, \mathcal{R}^P_{C\to AC}(\rho_{BC})\right) && (8.41)\\ &\geq -\log\left[1 - \left(\frac{1}{2}\left\|\rho_{ABC} - \mathcal{R}^P_{C\to AC}(\rho_{BC})\right\|_1\right)^2\right] && (8.42)\\ &\geq \frac{1}{4}\left\|\rho_{ABC} - \mathcal{R}^P_{C\to AC}(\rho_{BC})\right\|_1^2, && (8.43) \end{aligned}$$

where $\mathcal{R}^P_{C\to AC}$ is Petz's transpose map discussed in Ref. 28

$$\mathcal{R}^P_{C\to AC}(\cdot) \equiv \rho_{AC}^{1/2}\rho_C^{-1/2}(\cdot)\rho_C^{-1/2}\rho_{AC}^{1/2}. \tag{8.44}$$

In the above, the first inequality would follow from Conjecture 34, the second is a result of well known relations between trace distance and fidelity,[25] and the last is a consequence of the inequality

$-\log(1-x) \geq x$, valid for $x \leq 1$. Thus, the truth of Conjecture 34 would establish the truth of an open conjecture from Ref. 37 (up to a constant). As pointed out in Ref. 37, this would then imply that for tripartite states ρ_{ABC} with conditional mutual information $I(A;B|C)_\rho$ small (i.e., states that fulfill strong subadditivity with near equality), Petz's transpose map for the partial trace over A is good for recovering ρ_{ABC} from ρ_{BC}. Hence, even though ρ_{ABC} does not have to be close to a quantum Markov state if $I(A;B|C)_\rho$ is small (as discussed above), A would still be nearly independent of B from the perspective of C in the sense that ρ_{ABC} could be approximately recovered from ρ_{BC} alone. This would give an operationally useful characterization of states that fulfill strong subadditivity with near equality and would be helpful for answering some open questions concerning squashed entanglement, as discussed in Ref. 73.

For the quantum Markov state condition in (4.40), for simplicity, we consider instead the "relative entropy distance" between ρ_{ABC} and ς_{ABC}, where

$$\varsigma_{ABC} \equiv \exp\{\log \rho_{AC} + \log \rho_{BC} - \log \rho_C\}. \tag{8.45}$$

So if

$$D(\rho_{ABC} \| \varsigma_{ABC}) \leq \varepsilon, \tag{8.46}$$

then we can conclude that

$$I(A;B|C)_\rho = D(\rho_{ABC} \| \varsigma_{ABC}) \leq \varepsilon. \tag{8.47}$$

If desired, one can also obtain an ε-dependent upper bound on $I(A;B|C)_{\varsigma'}$, where $\varsigma'_{ABC} \equiv \varsigma_{ABC}/\operatorname{Tr}\{\varsigma_{ABC}\}$, which vanishes in the limit as ε goes to zero. This can be accomplished by employing the bound in Corollary 4 and by bounding $\operatorname{Tr}\{\varsigma_{ABC}\}$ from below by $1 - \|\rho_{ABC} - \varsigma_{ABC}\|_1$. The bound in Corollary 4 also serves as a converse of these bounds: if the conditional mutual information is small, then the trace distance between ρ_{ABC} and ς_{ABC} is small. However, it is not clear that a perturbation of the quantum Markov state condition in (4.40) will be as useful in applications as a perturbation of (4.38) and (4.39) would be, mainly because the map $\rho_{ABC} \to \exp\{\log \rho_{AC} + \log \rho_{BC} - \log \rho_C\}$ is non-linear (as discussed in Ref. 35).

IX. DISCUSSION

This paper has defined several Rényi generalizations of the CQMI quantities that satisfy properties that should find use in applications. Namely, we showed that these generalizations are non-negative and are monotone under local quantum operations on one of the systems A or B. An important open question is to prove that they are monotone under local quantum operations on both systems. Some of the Rényi generalizations satisfy a generalization of the duality relation $I(A;B|C) = I(A;B|D)$, which holds for a four-party pure state ψ_{ABCD}. We conjecture that these Rényi generalizations of the CQMI are monotone non-decreasing in the Rényi parameter α, and we have proved that this conjecture is true when α is in a neighborhood of one and in some other special cases. The truth of this conjecture in general would have implications in condensed matter physics, as detailed in Ref. 37, and quantum communication complexity, as mentioned in Ref. 68.

Based on the fact that the conditional mutual information can be written as

$$I(A;B|C)_\rho = D(\rho_{ABC} \| \exp\{\log \rho_{AC} + \log \rho_{BC} - \log \rho_C\}), \tag{9.1}$$

one could consider another Rényi generalization of the conditional mutual information, such as

$$D_\alpha(\rho_{ABC} \| \exp\{\log \rho_{AC} + \log \rho_{BC} - \log \rho_C\}), \tag{9.2}$$

or with the sandwiched variant. However, it is unclear to us whether (9.2) is monotone under local operations, which we have argued is an important property for a Rényi generalization of conditional mutual information.

There are many directions to consider going forward from this paper. First, one could improve many of the results here on a technical level. It would be interesting to understand in depth the limits in (4.5), (5.6), and (6.6) in order to establish the most general support conditions for the Δ, Δ_α, and $\widetilde{\Delta}_\alpha$ quantities, respectively, as has been done for the quantum and Rényi relative entropies, as

recalled in (3.2), (3.4), and (3.6). Next, if one could establish uniform convergence of the Δ_α and $\widetilde{\Delta}_\alpha$ quantities as α goes to one, then we could conclude that the optimized versions of these quantities converge to the conditional mutual information in this limit. One might also attempt to extend Theorem 11, Theorem 20, and Lemma 23 to hold for positive semi-definite density operators.

As far as applications are concerned, one could explore a Rényi squashed entanglement and determine if several properties hold which are analogous to the squashed entanglement.[13] Such a quantity might be helpful in strengthening Ref. 13, Proposition 10, so that the squashed entanglement could be interpreted as a strong converse upper bound on distillable entanglement. More generally, it might be helpful in strengthening the main result of Ref. 61, so that the upper bound established on the two-way assisted quantum capacity could be interpreted as a strong converse rate. The quantities defined here might be useful in the context of one-shot information theory, for example, to establish a one-shot state redistribution protocol as an extension of the main result of Ref. 19. Preliminary results on Rényi squashed entanglement and discord are discussed in our follow-up paper.[56] One could also explore applications of the Rényi conditional mutual informations in the context of condensed matter physics or high energy physics, as the Rényi entropy has been employed extensively in these contexts.[9]

Finally, these potential applications in information theory and physics should help in singling out some of our many possible definitions for Rényi conditional mutual information.

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APPENDIX A: SIBSON IDENTITY FOR THE RÉNYI CONDITIONAL MUTUAL INFORMATION

The Rényi conditional mutual information in Definition 7 has an explicit form, much like other Rényi information quantities.[41,57,26,64] We prove this in two steps, first by proving the following Sibson identity.[58]

Lemma 36. The following quantum Sibson identity holds when $\operatorname{supp}(\rho_{ABC}) \subseteq \operatorname{supp}(\sigma_{BC})$ *and for* $\alpha \in (0,1) \cup (1,\infty)$*:*

$$\Delta_\alpha(\rho_{ABC}, \rho_{AC}, \rho_C, \sigma_{BC}) = \Delta_\alpha(\rho_{ABC}, \rho_{AC}, \rho_C, \sigma^*_{BC}) + D_\alpha(\sigma^*_{BC} \| \sigma_{BC}), \tag{A1}$$

with the state σ^*_{BC} *having the form*

$$\sigma^*_{BC} \equiv \frac{\left(\operatorname{Tr}_A\left\{\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\right\}\right)^{1/\alpha}}{\operatorname{Tr}\left\{\left(\operatorname{Tr}_A\left\{\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\right\}\right)^{1/\alpha}\right\}}. \tag{A2}$$

Proof. The relation for σ^*_{BC} implies that

$$\left[\sigma^*_{BC}\operatorname{Tr}\left\{\left(\operatorname{Tr}_A\left\{\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\right\}\right)^{1/\alpha}\right\}\right]^{\alpha} = \operatorname{Tr}_A\left\{\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\right\}. \tag{A3}$$

Then consider that

$$\Delta_\alpha\left(\rho_{ABC},\rho_{AC},\rho_C,\sigma_{BC}\right)$$
$$= \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\sigma_{BC}^{1-\alpha}\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\right\} \tag{A4}$$
$$= \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\sigma_{BC}^{1-\alpha}\right\} \tag{A5}$$
$$= \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\operatorname{Tr}_A\left\{\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\right\}\sigma_{BC}^{1-\alpha}\right\} \tag{A6}$$
$$= \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\left[\sigma_{BC}^{*}\right]^{\alpha}\sigma_{BC}^{1-\alpha}\right\} + \frac{\alpha}{\alpha-1}\log \operatorname{Tr}\left\{\left(\operatorname{Tr}_A\left\{\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\right\}\right)^{1/\alpha}\right\}. \tag{A7}$$

Now consider expanding the following:

$$\Delta_\alpha\left(\rho_{ABC},\rho_{AC},\rho_C,\sigma_{BC}^{*}\right)$$
$$= \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\left[\sigma_{BC}^{*}\right]^{1-\alpha}\right\} \tag{A8}$$
$$= \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\operatorname{Tr}_A\left\{\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\right\}\left[\sigma_{BC}^{*}\right]^{1-\alpha}\right\} \tag{A9}$$
$$= \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\left[\operatorname{Tr}_A\left\{\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\right\}\right]^{1/\alpha}\right\} \tag{A10}$$
$$+\log \operatorname{Tr}\left\{\left(\operatorname{Tr}_A\left\{\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\right\}\right)^{1/\alpha}\right\} \tag{A11}$$
$$= \frac{\alpha}{\alpha-1}\log \operatorname{Tr}\left\{\left(\operatorname{Tr}_A\left\{\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\right\}\right)^{1/\alpha}\right\}. \tag{A12}$$

Putting everything together, we can conclude the statement of the lemma. ■

Corollary 37. The Rényi conditional mutual information has the following explicit form for $\alpha \in (0,1)\cup(1,\infty)$:

$$I_\alpha(A;B|C)_\rho = \frac{\alpha}{\alpha-1}\log \operatorname{Tr}\left\{\left(\rho_C^{(\alpha-1)/2}\operatorname{Tr}_A\left\{\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\right\}\rho_C^{(\alpha-1)/2}\right)^{1/\alpha}\right\}. \tag{A13}$$

The infimum in $I_\alpha(A;B|C)_\rho$ is achieved uniquely by the state in (A2), so that it can be replaced by a minimum.

Proof. This follows from the previous lemma:

$$I_\alpha(A;B|C)_\rho = \inf_{\sigma_{BC}} \Delta_\alpha\left(\rho_{ABC},\rho_{AC},\rho_C,\sigma_{BC}\right) \tag{A14}$$
$$= \inf_{\sigma_{BC}}\left[\Delta_\alpha\left(\rho_{ABC},\rho_{AC},\rho_C,\sigma_{BC}^{*}\right) + D_\alpha\left(\sigma_{BC}^{*}\|\sigma_{BC}\right)\right] \tag{A15}$$
$$= \Delta_\alpha\left(\rho_{ABC},\rho_{AC},\rho_C,\sigma_{BC}^{*}\right) \tag{A16}$$
$$= \frac{\alpha}{\alpha-1}\log \operatorname{Tr}\left\{\left(\operatorname{Tr}_A\left\{\rho_C^{(\alpha-1)/2}\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\rho_C^{(\alpha-1)/2}\right\}\right)^{1/\alpha}\right\}. \tag{A17}$$

■

Other Sibson identities hold for other variations of the Rényi conditional mutual information (whenever the innermost operator is optimized over and the others are the marginals of ρ_{ABC}). The proof for this is the same as given above.

APPENDIX B: CONVERGENCE OF THE RÉNYI CONDITIONAL MUTUAL INFORMATION

Before giving a proof of Theorem 11, we first establish the following lemma, which is a slight extension of Ref. 50, Proposition 15.

Lemma 38. Let $Z(\alpha) \in \mathcal{B}(\mathcal{H})_{++}$ be an operator-valued function and let $f(\alpha)$ be a function, both continuously differentiable in α for all $\alpha \in (0,\infty)$. Then the derivative $\frac{d}{d\alpha}\operatorname{Tr}\{Z(\alpha)^{f(\alpha)}\}$ exists

and is equal to

$$\frac{d}{d\alpha}\mathrm{Tr}\left\{Z(\alpha)^{f(\alpha)}\right\} = \left(\frac{d}{d\alpha}f(\alpha)\right)\mathrm{Tr}\left\{Z(\alpha)^{f(\alpha)}\log Z(\alpha)\right\} + f(\alpha)\,\mathrm{Tr}\left\{Z(\alpha)^{f(\alpha)-1}\frac{d}{d\alpha}Z(\alpha)\right\}. \tag{B1}$$

Proof. We proceed as in Ref. 51, Theorem 2.7 or Ref. 50, Proposition 15. Consider that

$$\begin{aligned} &Z(\alpha+h)^{f(\alpha+h)} - Z(\alpha)^{f(\alpha)} \\ &= \int_0^1 ds \frac{d}{ds}\left[Z(\alpha+h)^{sf(\alpha+h)}Z(\alpha)^{(1-s)f(\alpha)}\right] \qquad \text{(B2)} \\ &= \int_0^1 ds\, Z(\alpha+h)^{sf(\alpha+h)}\left[\log Z(\alpha+h)^{f(\alpha+h)} - \log Z(\alpha)^{f(\alpha)}\right] Z(\alpha)^{(1-s)f(\alpha)}. \qquad \text{(B3)} \end{aligned}$$

Taking the trace, we get

$$\begin{aligned} &\mathrm{Tr}\left\{Z(\alpha+h)^{f(\alpha+h)}\right\} - \mathrm{Tr}\left\{Z(\alpha)^{f(\alpha)}\right\} \\ &\quad = f(\alpha+h)\int_0^1 ds\,\mathrm{Tr}\left\{Z(\alpha)^{(1-s)f(\alpha)}Z(\alpha+h)^{sf(\alpha+h)}\left[\log Z(\alpha+h) - \log Z(\alpha)\right]\right\} \\ &\qquad (f(\alpha+h) - f(\alpha))\int_0^1 ds\,\mathrm{Tr}\left\{Z(\alpha)^{(1-s)f(\alpha)}Z(\alpha+h)^{sf(\alpha+h)}\log Z(\alpha)\right\}. \end{aligned} \tag{B4}$$

Dividing by h and taking the limit as $h \to 0$, we find

$$\begin{aligned} &\lim_{h\to 0}\frac{1}{h}\left[\mathrm{Tr}\left\{Z(\alpha+h)^{f(\alpha+h)}\right\} - \mathrm{Tr}\left\{Z(\alpha)^{f(\alpha)}\right\}\right] \\ &\quad = f(\alpha)\int_0^1 ds\,\mathrm{Tr}\left\{Z(\alpha)^{(1-s)f(\alpha)}Z(\alpha)^{sf(\alpha)}\lim_{h\to 0}\frac{1}{h}\left[\log Z(\alpha+h) - \log Z(\alpha)\right]\right\} \\ &\qquad + \lim_{h\to 0}\frac{f(\alpha+h) - f(\alpha)}{h}\int_0^1 ds\,\mathrm{Tr}\left\{Z(\alpha)^{(1-s)f(\alpha)}Z(\alpha)^{sf(\alpha)}\log Z(\alpha)\right\}, \end{aligned} \tag{B5}$$

which is equal to

$$f(\alpha)\,\mathrm{Tr}\left\{Z(\alpha)^{f(\alpha)}\frac{d}{d\alpha}\left[\log Z(\alpha)\right]\right\} + \left(\frac{d}{d\alpha}f(\alpha)\right)\mathrm{Tr}\left\{Z(\alpha)^{f(\alpha)}\log Z(\alpha)\right\}. \tag{B6}$$

Carrying out the same arguments as in Ref. 51, Theorem 2.7 or Ref. 50, Proposition 15 in order to compute $\frac{d}{d\alpha}\left[\log Z(\alpha)\right]$, we recover the formula in the statement of the lemma. ∎

We now provide a proof of Theorem 11. The idea is similar to that in the proof of Theorem 9. To this end, we again invoke L'Hôpital's rule. We begin by defining

$$G(\alpha) \equiv \rho_C^{(\alpha-1)/2}\mathrm{Tr}_A\left\{\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\right\}\rho_C^{(\alpha-1)/2}, \tag{B7}$$

which implies that

$$I_\alpha(A;B|C)_\rho = \frac{1}{1-\frac{1}{\alpha}}\log\mathrm{Tr}\left\{G(\alpha)^{1/\alpha}\right\}. \tag{B8}$$

Applying Lemma 38 to $G(\alpha)$ and the function $1/\alpha$, we find that

$$\frac{d}{d\alpha}\mathrm{Tr}\left\{G(\alpha)^{1/\alpha}\right\} = -\frac{1}{\alpha^2}\mathrm{Tr}\left\{G(\alpha)^{1/\alpha}\log G(\alpha)\right\} + \frac{1}{\alpha}\mathrm{Tr}\left\{G(\alpha)^{(1-\alpha)/\alpha}\frac{d}{d\alpha}G(\alpha)\right\}. \tag{B9}$$

Also, we have that

$$\begin{aligned}\frac{d}{d\alpha}G(\alpha) &= \frac{d}{d\alpha}\left[\rho_C^{(\alpha-1)/2}\mathrm{Tr}_A\left\{\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\right\}\rho_C^{(\alpha-1)/2}\right]\\ &= \frac{1}{2}(\log\rho_C)\,\rho_C^{(\alpha-1)/2}\mathrm{Tr}_A\left\{\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\right\}\rho_C^{(\alpha-1)/2}\\ &\quad-\frac{1}{2}\rho_C^{(\alpha-1)/2}\mathrm{Tr}_A\left\{(\log\rho_{AC})\,\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\right\}\rho_C^{(\alpha-1)/2}\\ &\quad+\rho_C^{(\alpha-1)/2}\mathrm{Tr}_A\left\{\rho_{AC}^{(1-\alpha)/2}(\log\rho_{ABC})\,\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\right\}\rho_C^{(\alpha-1)/2}\\ &\quad-\frac{1}{2}\rho_C^{(\alpha-1)/2}\mathrm{Tr}_A\left\{\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}(\log\rho_{AC})\,\rho_{AC}^{(1-\alpha)/2}\right\}\rho_C^{(\alpha-1)/2}\\ &\quad+\frac{1}{2}\rho_C^{(\alpha-1)/2}\mathrm{Tr}_A\left\{\rho_{AC}^{(1-\alpha)/2}\rho_{ABC}^{\alpha}\rho_{AC}^{(1-\alpha)/2}\right\}(\log\rho_C)\,\rho_C^{(\alpha-1)/2}. \qquad \text{(B10)}\end{aligned}$$

Applying L'Hôpital's rule gives

$$\lim_{\alpha\to1} I_\alpha(A;B|C)_\rho = \lim_{\alpha\to1}\frac{-\mathrm{Tr}\left\{G(\alpha)^{1/\alpha}\log G(\alpha)\right\}+\alpha\mathrm{Tr}\left\{G(\alpha)^{(1-\alpha)/\alpha}\frac{d}{d\alpha}G(\alpha)\right\}}{\mathrm{Tr}\left\{G(\alpha)^{1/\alpha}\right\}}. \qquad \text{(B11)}$$

Consider that

$$\lim_{\alpha\to1} G(\alpha)^{(1-\alpha)/\alpha} = \left[\rho_C^0\mathrm{Tr}_A\left\{\rho_{AC}^0\rho_{ABC}\rho_{AC}^0\right\}\rho_C^0\right]^0 \qquad \text{(B12)}$$

$$= \rho_{BC}^0. \qquad \text{(B13)}$$

Evaluating the limits above one at a time and using that $\mathrm{supp}(\rho_{ABC})\subseteq\mathrm{supp}(\rho_{AC})\subseteq\mathrm{supp}(\rho_C)$ (see, e.g., Ref. 54, Lemma B.4.1), we find that

$$\lim_{\alpha\to1}\frac{1}{\mathrm{Tr}\left\{G(\alpha)^{1/\alpha}\right\}} = \frac{1}{\mathrm{Tr}\left\{\rho_C^0\mathrm{Tr}_A\left\{\rho_{AC}^0\rho_{ABC}\rho_{AC}^0\right\}\rho_C^0\right\}} \qquad \text{(B14)}$$

$$= 1, \qquad \text{(B15)}$$

$$\lim_{\alpha\to1}-\mathrm{Tr}\left\{G(\alpha)^{1/\alpha}\log G(\alpha)\right\} = -\mathrm{Tr}\left\{\left[\rho_C^0\mathrm{Tr}_A\left\{\rho_{AC}^0\rho_{ABC}\rho_{AC}^0\right\}\rho_C^0\right]\log\left[\rho_C^0\mathrm{Tr}_A\left\{\rho_{AC}^0\rho_{ABC}\rho_{AC}^0\right\}\rho_C^0\right]\right\} \qquad \text{(B16)}$$

$$= -\mathrm{Tr}\left\{\rho_{BC}\log\rho_{BC}\right\}, \qquad \text{(B17)}$$

$$\begin{aligned}\lim_{\alpha\to1}\frac{d}{d\alpha}G(\alpha) &= \frac{1}{2}(\log\rho_C)\,\rho_C^0\mathrm{Tr}_A\left\{\rho_{AC}^0\rho_{ABC}\rho_{AC}^0\right\}\rho_C^0-\frac{1}{2}\rho_C^0\mathrm{Tr}_A\left\{(\log\rho_{AC})\,\rho_{AC}^0\rho_{ABC}\rho_{AC}^0\right\}\rho_C^0\\ &\quad+\rho_C^0\mathrm{Tr}_A\left\{\rho_{AC}^0(\log\rho_{ABC})\,\rho_{ABC}\rho_{AC}^0\right\}\rho_C^0-\frac{1}{2}\rho_C^0\mathrm{Tr}_A\left\{\rho_{AC}^0\rho_{ABC}(\log\rho_{AC})\,\rho_{AC}^0\right\}\rho_C^0\\ &\quad+\frac{1}{2}\rho_C^0\mathrm{Tr}_A\left\{\rho_{AC}^0\rho_{ABC}\rho_{AC}^0\right\}(\log\rho_C)\,\rho_C^0. \qquad \text{(B18)}\end{aligned}$$

Putting all of this together, we can see that the limit in (B11) evaluates to

$$\lim_{\alpha\to1} I_\alpha(A;B|C)_\rho = \Delta(\rho_{ABC},\rho_{AC},\rho_C,\rho_{BC}) \qquad \text{(B19)}$$

$$= I(A;B|C)_\rho. \qquad \text{(B20)}$$

APPENDIX C: CONVERGENCE OF THE $\widetilde{\Delta}_\alpha$ QUANTITIES

This section presents a proof of Theorem 20. We will consider L'Hôpital's rule in order to evaluate the limit of $\widetilde{\Delta}_\alpha$ as $\alpha\to1$, due to the presence of the denominator term $\alpha-1$ in $\widetilde{\Delta}_\alpha$. Consider that

$$\widetilde{Q}_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) = \mathrm{Tr}\left\{\left[Z_{ABC}(\alpha)\right]^\alpha\right\}, \qquad \text{(C1)}$$

where

$$Z_{ABC}(\alpha) \equiv \rho_{ABC}^{1/2}\tau_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\theta_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\tau_{AC}^{(1-\alpha)/2\alpha}\rho_{ABC}^{1/2}. \qquad \text{(C2)}$$

We begin by computing

$$\begin{aligned}\frac{d}{d\alpha}Z_{ABC}(\alpha) = \left(-\frac{1}{\alpha^2}\right)\Bigg[&\frac{1}{2}\rho_{ABC}^{1/2}(\log\tau_{AC})\,\tau_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\theta_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\tau_{AC}^{(1-\alpha)/2\alpha}\rho_{ABC}^{1/2}\\ &-\frac{1}{2}\rho_{ABC}^{1/2}\tau_{AC}^{(1-\alpha)/2\alpha}(\log\omega_C)\,\omega_C^{(\alpha-1)/2\alpha}\theta_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\tau_{AC}^{(1-\alpha)/2\alpha}\rho_{ABC}^{1/2}\\ &+\rho_{ABC}^{1/2}\tau_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}(\log\theta_{BC})\,\theta_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\tau_{AC}^{(1-\alpha)/2\alpha}\rho_{ABC}^{1/2}\\ &-\frac{1}{2}\rho_{ABC}^{1/2}\tau_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\theta_{BC}^{(1-\alpha)/\alpha}(\log\omega_C)\,\omega_C^{(\alpha-1)/2\alpha}\tau_{AC}^{(1-\alpha)/2\alpha}\rho_{ABC}^{1/2}\\ &+\frac{1}{2}\rho_{ABC}^{1/2}\tau_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\theta_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}(\log\tau_{AC})\,\tau_{AC}^{(1-\alpha)/2\alpha}\rho_{ABC}^{1/2}\Bigg].\end{aligned} \tag{C3}$$

Applying Lemma 38 to $Z_{ABC}(\alpha)$ and the function α, we find that

$$\frac{d}{d\alpha}\mathrm{Tr}\left\{[Z_{ABC}(\alpha)]^\alpha\right\} = \mathrm{Tr}\left\{[Z_{ABC}(\alpha)]^\alpha\log Z_{ABC}(\alpha)\right\} + \alpha\,\mathrm{Tr}\left\{[Z_{ABC}(\alpha)]^{\alpha-1}\frac{d}{d\alpha}Z_{ABC}(\alpha)\right\}, \tag{C4}$$

and

$$\lim_{\alpha\to1}\alpha[Z_{ABC}(\alpha)]^{\alpha-1} = \left[\rho_{ABC}^{1/2}\tau_{AC}^0\omega_C^0\theta_{BC}^0\omega_C^0\tau_{AC}^0\rho_{ABC}^{1/2}\right]^0 \tag{C5}$$

$$\equiv [Z_{ABC}(1)]^0, \tag{C6}$$

we find that

$$\begin{aligned}\lim_{\alpha\to1}\frac{d}{d\alpha}\widetilde{Q}_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC})&\\ = \mathrm{Tr}\left\{\rho_{ABC}^{1/2}\tau_{AC}^0\omega_C^0\theta_{BC}^0\omega_C^0\tau_{AC}^0\rho_{ABC}^{1/2}\log\rho_{ABC}^{1/2}\tau_{AC}^0\omega_C^0\theta_{BC}^0\omega_C^0\tau_{AC}^0\rho_{ABC}^{1/2}\right\}&\\ -\frac{1}{2}\mathrm{Tr}\left\{[Z_{ABC}(1)]^0\rho_{ABC}^{1/2}(\log\tau_{AC})\,\tau_{AC}^0\omega_C^0\theta_{BC}^0\omega_C^0\tau_{AC}^0\rho_{ABC}^{1/2}\right\}&\\ +\frac{1}{2}\mathrm{Tr}\left\{[Z_{ABC}(1)]^0\rho_{ABC}^{1/2}\tau_{AC}^0(\log\omega_C)\,\omega_C^0\theta_{BC}^0\omega_C^0\tau_{AC}^0\rho_{ABC}^{1/2}\right\}&\\ -\mathrm{Tr}\left\{[Z_{ABC}(1)]^0\rho_{ABC}^{1/2}\tau_{AC}^0\omega_C^0(\log\theta_{BC})\,\theta_{BC}^0\omega_C^0\tau_{AC}^0\rho_{ABC}^{1/2}\right\}&\\ +\frac{1}{2}\mathrm{Tr}\left\{[Z_{ABC}(1)]^0\rho_{ABC}^{1/2}\tau_{AC}^0\omega_C^0\theta_{BC}^0(\log\omega_C)\,\omega_C^0\tau_{AC}^0\rho_{ABC}^{1/2}\right\}&\\ -\frac{1}{2}\mathrm{Tr}\left\{[Z_{ABC}(1)]^0\rho_{ABC}^{1/2}\tau_{AC}^0\omega_C^0\theta_{BC}^0\omega_C^0(\log\tau_{AC})\,\tau_{AC}^0\rho_{ABC}^{1/2}\right\}.&\end{aligned} \tag{C7}$$

Since we assume that $\mathrm{supp}(\rho_{ABC})$ is contained in each of $\mathrm{supp}(\tau_{AC})$, $\mathrm{supp}(\omega_C)$, and $\mathrm{supp}(\theta_{BC})$, we can see that

$$\lim_{\alpha\to1}\frac{d}{d\alpha}\widetilde{Q}_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) = \Delta(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}), \tag{C8}$$

$$\lim_{\alpha\to1}\widetilde{Q}_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) = 1, \tag{C9}$$

by applying the relations $\rho_{ABC} = \rho_{ABC}^0\rho_{ABC}\rho_{ABC}^0$, $\rho_{ABC}^0\tau_{AC}^0 = \rho_{ABC}^0$, $\rho_{ABC}^0\theta_{BC}^0 = \rho_{ABC}^0$, $\rho_{ABC}^0\omega_C^0 = \rho_{ABC}^0$, $[Z_{ABC}(1)]^0 = \rho_{ABC}^0$, and their Hermitian conjugates. Applying L'Hôpital's rule, we find that

$$\lim_{\alpha\to1}\widetilde{\Delta}_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}) = \lim_{\alpha\to1}\frac{\frac{d}{d\alpha}\widetilde{Q}_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC})}{\widetilde{Q}_\alpha(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC})} \tag{C10}$$

$$= \Delta(\rho_{ABC},\tau_{AC},\omega_C,\theta_{BC}). \tag{C11}$$

Essentially the same proof establishes the limiting relation for the other $\widetilde{\Delta}_\alpha$ quantities defined from (6.7) to (6.11).

APPENDIX D: CONVERGENCE TO $\Delta_{\max}$

This section gives a proof of Proposition 29. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})_{++}$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})_{++}$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)_{++}$. We prove that

$$\lim_{\alpha\to\infty} \widetilde{\Delta}_\alpha\left(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}\right) = \Delta_{\max}\left(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}\right). \tag{D1}$$

The method of proof is the same as that for Ref. 50, Theorem 5. By the reverse triangle inequality for the α norm, we have that

$$\begin{aligned}\Big| \left\|\rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\theta_{BC}^{\frac{1-\alpha}{\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\rho_{ABC}^{\frac{1}{2}}\right\|_\alpha - \left\|\rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\omega_C^{\frac{1}{2}}\theta_{BC}^{-1}\omega_C^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\rho_{ABC}^{\frac{1}{2}}\right\|_\alpha \Big| \\ \leq \left\|\rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\theta_{BC}^{\frac{1-\alpha}{\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\rho_{ABC}^{\frac{1}{2}} - \rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\omega_C^{\frac{1}{2}}\theta_{BC}^{-1}\omega_C^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\rho_{ABC}^{\frac{1}{2}}\right\|_\alpha.\end{aligned} \tag{D2}$$

Then

$$\begin{aligned}&\lim_{\alpha\to\infty} \widetilde{\Delta}_\alpha\left(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}\right) \\ &= \lim_{\alpha\to\infty} \frac{\alpha}{\alpha-1}\log\left\|\rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\theta_{BC}^{\frac{1-\alpha}{\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\rho_{ABC}^{\frac{1}{2}}\right\|_\alpha \qquad \text{(D3)} \\ &\leq \log\left(\begin{array}{c}\lim_{\alpha\to\infty}\left\|\rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\omega_C^{\frac{1}{2}}\theta_{BC}^{-1}\omega_C^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\rho_{ABC}^{\frac{1}{2}}\right\|_\alpha + \\ \lim_{\alpha\to\infty}\left\|\rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\theta_{BC}^{\frac{1-\alpha}{\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\rho_{ABC}^{\frac{1}{2}} - \rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\omega_C^{\frac{1}{2}}\theta_{BC}^{-1}\omega_C^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\rho_{ABC}^{\frac{1}{2}}\right\|_\alpha\end{array}\right) \qquad \text{(D4)} \\ &\leq \log\left(\begin{array}{c}\lim_{\alpha\to\infty}\left\|\rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\omega_C^{\frac{1}{2}}\theta_{BC}^{-1}\omega_C^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\rho_{ABC}^{\frac{1}{2}}\right\|_\alpha + \dim\left(\mathcal{H}_{ABC}\right)\times \\ \lim_{\alpha\to\infty}\left\|\rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\theta_{BC}^{\frac{1-\alpha}{\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\rho_{ABC}^{\frac{1}{2}} - \rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\omega_C^{\frac{1}{2}}\theta_{BC}^{-1}\omega_C^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\rho_{ABC}^{\frac{1}{2}}\right\|_\infty\end{array}\right) \qquad \text{(D5)} \\ &= \log\left\|\rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\omega_C^{\frac{1}{2}}\theta_{BC}^{-1}\omega_C^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\rho_{ABC}^{\frac{1}{2}}\right\|_\infty \qquad \text{(D6)} \\ &= \Delta_{\max}\left(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}\right) \qquad \text{(D7)}\end{aligned}$$

and

$$\begin{aligned}&\lim_{\alpha\to\infty} \widetilde{\Delta}_\alpha\left(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}\right) \\ &= \lim_{\alpha\to\infty} \frac{\alpha}{\alpha-1}\log\left\|\rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\theta_{BC}^{\frac{1-\alpha}{\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\rho_{ABC}^{\frac{1}{2}}\right\|_\alpha \qquad \text{(D8)} \\ &\geq \log\left(\begin{array}{c}\lim_{\alpha\to\infty}\left\|\rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\omega_C^{\frac{1}{2}}\theta_{BC}^{-1}\omega_C^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\rho_{ABC}^{\frac{1}{2}}\right\|_\alpha - \\ \lim_{\alpha\to\infty}\left\|\rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\theta_{BC}^{\frac{1-\alpha}{\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\rho_{ABC}^{\frac{1}{2}} - \rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\omega_C^{\frac{1}{2}}\theta_{BC}^{-1}\omega_C^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\rho_{ABC}^{\frac{1}{2}}\right\|_\alpha\end{array}\right) \qquad \text{(D9)} \\ &\geq \log\left(\begin{array}{c}\lim_{\alpha\to\infty}\left\|\rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\omega_C^{\frac{1}{2}}\theta_{BC}^{-1}\omega_C^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\rho_{ABC}^{\frac{1}{2}}\right\|_\alpha - \dim\left(\mathcal{H}_{ABC}\right)\times \\ \lim_{\alpha\to\infty}\left\|\rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\theta_{BC}^{\frac{1-\alpha}{\alpha}}\omega_C^{\frac{\alpha-1}{2\alpha}}\tau_{AC}^{\frac{1-\alpha}{2\alpha}}\rho_{ABC}^{\frac{1}{2}} - \rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\omega_C^{\frac{1}{2}}\theta_{BC}^{-1}\omega_C^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\rho_{ABC}^{\frac{1}{2}}\right\|_\infty\end{array}\right) \qquad \text{(D10)} \\ &= \log\left\|\rho_{ABC}^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\omega_C^{\frac{1}{2}}\theta_{BC}^{-1}\omega_C^{\frac{1}{2}}\tau_{AC}^{-\frac{1}{2}}\rho_{ABC}^{\frac{1}{2}}\right\|_\infty \qquad \text{(D11)} \\ &= \Delta_{\max}\left(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}\right). \qquad \text{(D12)}\end{aligned}$$

APPENDIX E: APPROACHES FOR PROVING CONJECTURE 34 AND PROOF FOR A SPECIAL CASE

This section gives more details regarding the approach outlined in Sec. VIII A for proving Conjecture 34. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$, $\tau_{AC} \in \mathcal{S}(\mathcal{H}_{AC})_{++}$, $\theta_{BC} \in \mathcal{S}(\mathcal{H}_{BC})_{++}$, and $\omega_C \in \mathcal{S}(\mathcal{H}_C)_{++}$. We begin by introducing a variable

$$\gamma = \alpha - 1, \tag{E1}$$

and with

$$Y(\gamma) \equiv \rho_{ABC}^{1+\gamma}\tau_{AC}^{\frac{-\gamma}{2}}\omega_C^{\frac{\gamma}{2}}\theta_{BC}^{-\gamma}\omega_C^{\frac{\gamma}{2}}\tau_{AC}^{\frac{-\gamma}{2}}, \tag{E2}$$

it follows that $\Delta_\alpha(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC})$ is equal to

$$\frac{1}{\alpha-1}\log \mathrm{Tr}\left\{\rho_{ABC}^{\alpha}\tau_{AC}^{\frac{1-\alpha}{2}}\omega_C^{\frac{\alpha-1}{2}}\theta_{BC}^{1-\alpha}\omega_C^{\frac{\alpha-1}{2}}\tau_{AC}^{\frac{1-\alpha}{2}}\right\} = \frac{1}{\gamma}\log \mathrm{Tr}\left\{Y(\gamma)\right\}. \tag{E3}$$

Since $d\gamma/d\alpha = 1$,

$$\frac{d}{d\alpha}\left[\frac{1}{\alpha-1}\log \mathrm{Tr}\left\{\rho_{ABC}^{\alpha}\tau_{AC}^{\frac{1-\alpha}{2}}\omega_C^{\frac{\alpha-1}{2}}\theta_{BC}^{1-\alpha}\omega_C^{\frac{\alpha-1}{2}}\tau_{AC}^{\frac{1-\alpha}{2}}\right\}\right] = \frac{d}{d\gamma}\left[\frac{1}{\gamma}\log \mathrm{Tr}\left\{Y(\gamma)\right\}\right]. \tag{E4}$$

We can then explicitly compute the derivative

$$\frac{d}{d\gamma}\left[\frac{1}{\gamma}\log \mathrm{Tr}\left\{Y(\gamma)\right\}\right] = -\frac{1}{\gamma^2}\log \mathrm{Tr}\left\{Y(\gamma)\right\} + \frac{\mathrm{Tr}\left\{\frac{d}{d\gamma}Y(\gamma)\right\}}{\gamma \mathrm{Tr}\left\{Y(\gamma)\right\}} \tag{E5}$$

$$= \frac{\gamma \mathrm{Tr}\left\{\frac{d}{d\gamma}Y(\gamma)\right\} - \mathrm{Tr}\left\{Y(\gamma)\right\}\log \mathrm{Tr}\left\{Y(\gamma)\right\}}{\gamma^2 \mathrm{Tr}\left\{Y(\gamma)\right\}}. \tag{E6}$$

So

$$\begin{aligned}\gamma\frac{d}{d\gamma}Y(\gamma) = {}& \log \rho_{ABC}^{\gamma}Y(\gamma) + \rho_{ABC}^{1+\gamma}\left[\log \tau_{AC}^{-\gamma/2}\right]\tau_{AC}^{\frac{-\gamma}{2}}\omega_C^{\frac{\gamma}{2}}\theta_{BC}^{-\gamma}\omega_C^{\frac{\gamma}{2}}\tau_{AC}^{\frac{-\gamma}{2}} \\ &+ \rho_{ABC}^{1+\gamma}\tau_{AC}^{\frac{-\gamma}{2}}\left[\log \omega_C^{\gamma/2}\right]\omega_C^{\frac{\gamma}{2}}\theta_{BC}^{-\gamma}\omega_C^{\frac{\gamma}{2}}\tau_{AC}^{\frac{-\gamma}{2}} + \rho_{ABC}^{1+\gamma}\tau_{AC}^{\frac{-\gamma}{2}}\omega_C^{\frac{\gamma}{2}}\left[\log \theta_{BC}^{-\gamma}\right]\theta_{BC}^{-\gamma}\omega_C^{\frac{\gamma}{2}}\tau_{AC}^{\frac{-\gamma}{2}} \\ &+ \rho_{ABC}^{1+\gamma}\tau_{AC}^{\frac{-\gamma}{2}}\omega_C^{\frac{\gamma}{2}}\theta_{BC}^{-\gamma}\omega_C^{\frac{\gamma}{2}}\left[\log \omega_C^{\gamma/2}\right]\tau_{AC}^{\frac{-\gamma}{2}} + Y(\gamma)\log \tau_{AC}^{-\gamma/2}.\end{aligned} \tag{E7}$$

If it is true that the numerator in (E6) is non-negative for all ρ_{ABC}, then we can conclude the monotonicity in α.

A potential path for proving the conjecture for the sandwiched version is to follow a similar approach developed by Tomamichel *et al.* (see the proof of Ref. 50, Theorem 7). Since we can write

$$\widetilde{\Delta}_\alpha(\rho_{ABC}, \tau_{AC}, \omega_C, \theta_{BC}) = \max_{\gamma_{ABC}} \widetilde{D}_\alpha(\rho, \tau, \omega, \theta, \gamma), \tag{E8}$$

where

$$\widetilde{D}_\alpha(\rho, \tau, \omega, \theta, \mu) \equiv \frac{\alpha}{\alpha-1}\log \mathrm{Tr}\left\{\rho_{ABC}^{1/2}\tau_{AC}^{(1-\alpha)/2\alpha}\omega_C^{(\alpha-1)/2\alpha}\theta_{BC}^{(1-\alpha)/\alpha}\omega_C^{(\alpha-1)/2\alpha}\tau_{AC}^{(1-\alpha)/2\alpha}\rho_{ABC}^{1/2}\mu_{ABC}^{(\alpha-1)/\alpha}\right\}, \tag{E9}$$

it suffices to prove that $\widetilde{D}_\alpha(\rho, \tau, \omega, \theta, \mu)$ is monotone in α. For this purpose, the idea is similar to the above (i.e., try to show that the derivative of $\widetilde{D}_\alpha(\rho, \tau, \omega, \theta, \mu)$ with respect to α is non-negative). To this end, now let

$$\gamma = \frac{\alpha-1}{\alpha}, \tag{E10}$$

and with

$$Z(\gamma) \equiv \rho_{ABC}^{1/2}\tau_{AC}^{\frac{-\gamma}{2}}\omega_C^{\frac{\gamma}{2}}\theta_{BC}^{-\gamma}\omega_C^{\frac{\gamma}{2}}\tau_{AC}^{\frac{-\gamma}{2}}\rho_{ABC}^{1/2}\mu_{ABC}^{\gamma}, \tag{E11}$$

it follows that (E9) is equal to

$$\widetilde{D}_\alpha(\rho, \tau, \omega, \theta, \mu) = \frac{1}{\gamma}\log \mathrm{Tr}\left\{Z(\gamma)\right\}. \tag{E12}$$

Then since $d\gamma/d\alpha = 1/\alpha^2$,

$$\frac{d}{d\alpha}\left[\widetilde{D}_\alpha(\rho, \tau, \omega, \theta, \mu)\right] = \frac{1}{\alpha^2}\frac{d}{d\gamma}\left[\frac{1}{\gamma}\log \mathrm{Tr}\left\{Z(\gamma)\right\}\right]. \tag{E13}$$

Computing the derivative then results in

$$\frac{d}{d\gamma}\left[\frac{1}{\gamma}\log \mathrm{Tr}\left\{Z\left(\gamma\right)\right\}\right] = -\frac{1}{\gamma^2}\log \mathrm{Tr}\left\{Z\left(\gamma\right)\right\} + \frac{\mathrm{Tr}\left\{\frac{d}{d\gamma}Z\left(\gamma\right)\right\}}{\gamma \mathrm{Tr}\left\{Z\left(\gamma\right)\right\}} \tag{E14}$$

$$= \frac{\gamma \mathrm{Tr}\left\{\frac{d}{d\gamma}Z\left(\gamma\right)\right\} - \mathrm{Tr}\left\{Z\left(\gamma\right)\right\}\log \mathrm{Tr}\left\{Z\left(\gamma\right)\right\}}{\gamma^2 \mathrm{Tr}\left\{Z\left(\gamma\right)\right\}}. \tag{E15}$$

The calculation of the derivative $\gamma \mathrm{Tr}\left\{\frac{d}{d\gamma}Z\left(\gamma\right)\right\}$ is very similar to what we have shown above. So, in order to prove the conjecture, it suffices to prove that the numerator of the last line above is non-negative.

If the above approach is successful, one could take essentially the same approach to prove all of the other conjectured monotonicities detailed in Conjecture 34.

1. Proof of Conjecture 34 for α in a neighborhood of one

We can prove that the numerator of (E6) is non-negative for γ in a neighborhood of zero. To this end, consider a Taylor expansion of $Y\left(\gamma\right)$ in (E2) around γ equal to zero (so around α equal to one). Indeed, consider that

$$X^{1+\gamma} = X + \gamma X \log X + \frac{\gamma^2}{2} X \log^2 X + O\left(\gamma^3\right), \tag{E16}$$

$$X^{\gamma} = I + \gamma \log X + \frac{\gamma^2}{2}\log^2 X + O\left(\gamma^3\right). \tag{E17}$$

For our case, we make the following substitutions into $\mathrm{Tr}\{Y\left(\gamma\right)\}$:

$$\rho_{ABC}^{1+\gamma} = \rho_{ABC} + \gamma \rho_{ABC}\log \rho_{ABC} + \frac{\gamma^2}{2}\rho_{ABC}\log^2 \rho_{ABC} + O\left(\gamma^3\right), \tag{E18}$$

$$\theta_{BC}^{\frac{-\gamma}{2}} = I - \frac{\gamma}{2}\log \theta_{BC} + \frac{\gamma^2}{8}\log^2 \theta_{BC} + O\left(\gamma^3\right), \tag{E19}$$

$$\omega_{C}^{\frac{\gamma}{2}} = I + \frac{\gamma}{2}\log \omega_{C} + \frac{\gamma^2}{8}\log^2 \omega_{C} + O\left(\gamma^3\right), \tag{E20}$$

$$\tau_{AC}^{-\gamma} = I - \gamma \log \tau_{AC} + \frac{\gamma^2}{2}\log^2 \tau_{AC} + O\left(\gamma^3\right). \tag{E21}$$

After a rather tedious calculation, we find that

$$\mathrm{Tr}\left\{Y\left(\gamma\right)\right\} = \mathrm{Tr}\left\{\rho_{ABC}\right\} + \gamma\Delta\left(\rho,\tau,\omega,\theta\right) + \frac{\gamma^2}{2}\left[V\left(\rho,\tau,\omega,\theta\right) + \left[\Delta\left(\rho,\tau,\omega,\theta\right)\right]^2\right] + O\left(\gamma^3\right), \tag{E22}$$

where $V\left(\rho,\tau,\omega,\theta\right)$ is a quantity for which it seems natural to call the *tripartite information variance*

$$V\left(\rho,\tau,\omega,\theta\right) \equiv \mathrm{Tr}\left\{\rho_{ABC}[\log \rho_{ABC} - \log \tau_{AC} - \log \theta_{BC} + \log \omega_{C} - \Delta\left(\rho,\tau,\omega,\theta\right)]^2\right\}. \tag{E23}$$

A special case of this is a quantity which we can call the *conditional mutual information variance* of ρ_{ABC}

$$V(A;B|C)_{\rho} \equiv \mathrm{Tr}\left\{\rho_{ABC}\left[\log \rho_{ABC} - \log \rho_{AC} - \log \rho_{BC} + \log \rho_{C} - I(A;B|C)_{\rho}\right]^2\right\}. \tag{E24}$$

The mutual information variance defined in Ref. 66 is a special case of the above quantity when C is trivial. For any Hermitian operator H, we have that

$$\left\langle H^2\right\rangle_{\rho} - \left\langle H\right\rangle_{\rho}^2 \geq 0. \tag{E25}$$

So taking $H \equiv \log \rho_{ABC} - \log \tau_{AC} - \log \theta_{BC} + \log \omega_{C}$, we conclude that $V\left(\rho,\tau,\omega,\theta\right) \geq 0$, an observation central to our development here. We will make the abbreviations $\Delta \equiv \Delta\left(\rho,\tau,\omega,\theta\right)$ and $V \equiv V\left(\rho,\tau,\omega,\theta\right)$ from here forward, so that

$$\mathrm{Tr}\left\{Y\left(\gamma\right)\right\} = 1 + \gamma\Delta + \frac{\gamma^2}{2}\left[V + \Delta^2\right] + O\left(\gamma^3\right). \tag{E26}$$

So this implies that

$$\gamma \mathrm{Tr}\left\{\frac{d}{d\gamma}Y(\gamma)\right\} = \gamma\Delta + \gamma^2\left[V + \Delta^2\right] + O\left(\gamma^3\right), \tag{E27}$$

$$\mathrm{Tr}\left\{Y(\gamma)\right\}\log \mathrm{Tr}\left\{Y(\gamma)\right\} = \left[1 + \gamma\Delta + \frac{\gamma^2}{2}\left[V + \Delta^2\right] + O\left(\gamma^3\right)\right]\log\left[1 + \gamma\Delta + \frac{\gamma^2}{2}\left[V + \Delta^2\right] + O\left(\gamma^3\right)\right]. \tag{E28}$$

Then for small γ, we have the following Taylor expansion for the logarithm:

$$\log\left[1 + \gamma\Delta + \frac{\gamma^2}{2}\left[V + \Delta^2\right] + O\left(\gamma^3\right)\right] = \gamma\Delta + \frac{\gamma^2}{2}\left[V + \Delta^2\right] - \frac{\gamma^2\Delta^2}{2} + O\left(\gamma^3\right) \tag{E29}$$

$$= \gamma\Delta + \frac{\gamma^2}{2}V + O\left(\gamma^3\right), \tag{E30}$$

which gives

$$\begin{aligned}\mathrm{Tr}\left\{Y(\gamma)\right\}\log \mathrm{Tr}\left\{Y(\gamma)\right\} &= \left[1 + \gamma\Delta + \frac{\gamma^2}{2}\left[V + \Delta^2\right] + O\left(\gamma^3\right)\right]\left[\gamma\Delta + \frac{\gamma^2}{2}V + O\left(\gamma^3\right)\right] \\ &= \gamma\Delta + \frac{\gamma^2}{2}V + \gamma^2\Delta^2 + O\left(\gamma^3\right).\end{aligned} \tag{E31}$$

Finally, we can say that

$$\begin{aligned}\gamma \mathrm{Tr}\left\{\frac{d}{d\gamma}Y(\gamma)\right\} - \mathrm{Tr}\left\{Y(\gamma)\right\}\log \mathrm{Tr}\left\{Y(\gamma)\right\} &= \gamma\Delta + \gamma^2\left[V + \Delta^2\right] - \left[\gamma\Delta + \frac{\gamma^2}{2}V + \gamma^2\Delta^2\right] + O\left(\gamma^3\right) \\ &= \frac{\gamma^2}{2}V + O\left(\gamma^3\right).\end{aligned} \tag{E32}$$

If $V > 0$, we can conclude that as long as γ is very near to zero, all terms $O\left(\gamma^3\right)$ are negligible in comparison to $\frac{\gamma^2}{2}V$, and the monotonicity holds in such a regime. A development similar to the above, one establishes the other variations of (8.1) for γ in a neighborhood of zero. (Note that this argument does not work if $V = 0$.)

A similar kind of development shows that the conjecture in (8.2) and its variations hold for γ in a neighborhood of zero. We only sketch the main idea since it is similar to the previous development. We first observe that we can rewrite $\mathrm{Tr}\{Z(\gamma)\}$ in the following way:

$$\mathrm{Tr}\left\{Z(\gamma)\right\} = \langle\varphi|\,\tau_{AC}^{\frac{-\gamma}{2}}\omega_C^{\frac{\gamma}{2}}\theta_{BC}^{-\gamma}\omega_C^{\frac{\gamma}{2}}\tau_{AC}^{\frac{-\gamma}{2}}\rho_{ABC}^{1/2}\otimes\left(\mu_{A'B'C'}^T\right)^{\gamma}|\varphi\rangle, \tag{E33}$$

where $A'B'C'$ are some systems isomorphic to ABC and

$$|\varphi\rangle_{ABC,A'B'C'} \equiv \rho_{ABC}^{1/2}\otimes I_{A'B'C'}|\Gamma\rangle_{ABC,A'B'C}, \tag{E34}$$

with $|\Gamma\rangle$ the maximally entangled vector. Then a Taylor expansion about $\gamma = 0$ (another tedious calculation) gives that

$$\mathrm{Tr}\left\{Z(\gamma)\right\} = \mathrm{Tr}\left\{\rho_{ABC}\right\} + \gamma\langle\varphi|\,H_{ABC,A'B'C'}\,|\varphi\rangle + \frac{\gamma^2}{2}\langle\varphi|\,H_{ABC,A'B'C'}^2\,|\varphi\rangle + O\left(\gamma^3\right), \tag{E35}$$

where

$$H_{ABC,A'B'C'} \equiv \log\omega_C - \log\tau_{AC} - \log\theta_{BC} + \log\mu_{A'B'C'}^T. \tag{E36}$$

Then we know that

$$\langle\varphi|\,H_{ABC,A'B'C'}^2\,|\varphi\rangle - \left[\langle\varphi|\,H_{ABC,A'B'C'}\,|\varphi\rangle\right]^2 \geq 0. \tag{E37}$$

From here, we can show that the numerator of (E15) is non-negative for small γ by following the same development as in (E26)-(E32) (substitute $\langle\varphi|\,H_{ABC,A'B'C'}\,|\varphi\rangle$ for Δ and the LHS in (E37) for V). The development for the other variations of (8.2) is similar.

APPENDIX F: DIMENSION BOUNDS AND OTHER INEQUALITIES

For the bounds in this appendix, we make the following definitions:

$$I''_\alpha(A;B|C)_\rho \equiv \inf_{\sigma_{ABC}} \Delta_\alpha\left(\rho_{ABC},\sigma_{AC},\sigma_C,\sigma_{BC}\right), \tag{F1}$$

$$\widetilde{I}''_\alpha(A;B|C)_\rho \equiv \inf_{\sigma_{ABC}} \widetilde{\Delta}_\alpha\left(\rho_{ABC},\sigma_{AC},\sigma_C,\sigma_{BC}\right), \tag{F2}$$

where the optimizations are over σ_{ABC} such that $\operatorname{supp}(\rho_{ABC}) \subseteq \operatorname{supp}(\sigma_{ABC})$.

Proposition 39. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$. The following dimension bound holds for $\alpha \in [0,1) \cup (1,2]$:

$$I''_\alpha(A;B|C)_\rho \leq 2\min\left\{\log d_A, \log d_B\right\}, \tag{F3}$$

and the following one holds for $\alpha \in (1/2,1) \cup (1,\infty)$:

$$\widetilde{I}''_\alpha(A;B|C)_\rho \leq 2\min\left\{\log d_A, \log d_B\right\}. \tag{F4}$$

Proof. We first prove that the following dimension bounds hold

$$I''_\alpha(A;B|C)_\rho \leq \log d_A - H_\alpha(A|BC)_\rho, \tag{F5}$$

$$I''_\alpha(A;B|C)_\rho \leq \log d_B - H_\alpha(B|AC)_\rho, \tag{F6}$$

$$\widetilde{I}''_\alpha(A;B|C)_\rho \leq \log d_A - \widetilde{H}_\alpha(A|BC)_\rho, \tag{F7}$$

$$\widetilde{I}''_\alpha(A;B|C)_\rho \leq \log d_B - \widetilde{H}_\alpha(B|AC)_\rho. \tag{F8}$$

The inequality in (F6) follows from

$$\begin{aligned}
I''_\alpha(A;B|C)_\rho &= \inf_{\sigma_{ABC}} \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\rho^\alpha_{ABC}\sigma^{(1-\alpha)/2}_{AC}\sigma^{(\alpha-1)/2}_{C}\sigma^{1-\alpha}_{BC}\sigma^{(\alpha-1)/2}_{C}\sigma^{(1-\alpha)/2}_{AC}\right\} && \text{(F9)}\\
&\leq \inf_{\sigma_{AC}} \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\rho^\alpha_{ABC}\sigma^{(1-\alpha)/2}_{AC}\sigma^{(\alpha-1)/2}_{C}(\pi_B\otimes\sigma_C)^{1-\alpha}\sigma^{(\alpha-1)/2}_{C}\sigma^{(1-\alpha)/2}_{AC}\right\} && \text{(F10)}\\
&= \inf_{\sigma_{AC}} \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\rho^\alpha_{ABC}\pi^{1-\alpha}_B\sigma^{1-\alpha}_{AC}\right\} && \text{(F11)}\\
&= \log d_B - \left(-\min_{\sigma_{AC}} \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\rho^\alpha_{ABC}\sigma^{1-\alpha}_{AC}\right\}\right) && \text{(F12)}\\
&= \log d_B - H_\alpha(B|AC)_\rho, && \text{(F13)}
\end{aligned}$$

where $\pi_B = I_B/d_B$ and $H_\alpha(B|AC)_\rho \equiv -\inf_{\sigma_{AC}} D_\alpha(\rho_{ABC}\|I_B\otimes\sigma_{AC})$ as in (1.10). The bound in (F5) follows similarly by choosing $\sigma_{ABC} = \pi_A \otimes \sigma_{BC}$. The proofs for the sandwiched Rényi CMIs follow similarly, except we end up with the sandwiched Rényi conditional entropy in the upper bound.

To prove (F3), we use the duality relation proved in Ref. 65, Lemma 6. From (F5), we know that

$$\begin{aligned}
I''_\alpha(A;B|C)_\rho &\leq \log d_A + \inf_{\sigma_{BC}} \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\rho^\alpha_{ABC}\sigma^{1-\alpha}_{BC}\right\} && \text{(F14)}\\
&\leq \log d_A + \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\rho^\alpha_{ABC}\rho^{1-\alpha}_{BC}\right\} && \text{(F15)}\\
&\equiv \log d_A - H_\alpha(A|BC)_{\rho|\rho} && \text{(F16)}\\
&= \log d_A + H_\beta(A|D)_{\rho|\rho} && \text{(F17)}\\
&\leq \log d_A + H_\beta(A)_\rho && \text{(F18)}\\
&\leq 2\log d_A, && \text{(F19)}
\end{aligned}$$

where $H_\alpha(A|BC)_{\rho|\rho} \equiv -D_\alpha(\rho_{ABC}\|I_A\otimes\rho_{BC})$. The second equality follows from the duality from Ref. 65, Lemma 6, i.e.,

$$H_\alpha(A|BC)_{\rho|\rho} = -H_\beta(A|D)_{\rho|\rho}, \tag{F20}$$

where ρ_{ABCD} is a purification of ρ_{ABC} and β is chosen so that $\alpha + \beta = 2$. The third inequality follows from data processing and the last from a dimension bound on the Rényi entropy.

The inequality (F4) follows from the duality of the sandwiched conditional Rényi entropy Ref. 50, Theorem 10:

$$\widetilde{H}_\alpha(A|BC)_\rho = -\widetilde{H}_\beta(A|D)_\rho, \tag{F21}$$

where $\widetilde{H}_\alpha(A|BC)_\rho \equiv -\inf_{\sigma_{BC}} \widetilde{D}_\alpha(\rho_{ABC}\|I_A \otimes \sigma_{BC})$, ρ_{ABCD} is a purification of ρ_{ABC} and β is chosen so that $\frac{1}{\alpha} + \frac{1}{\beta} = 2$. So this means that

$$\begin{aligned}
\widetilde{I}''_\alpha(A;B|C)_\rho &\leq \log d_A - \widetilde{H}_\alpha(A|BC)_\rho && \text{(F22)}\\
&= \log d_A + \widetilde{H}_\beta(A|D)_\rho && \text{(F23)}\\
&\leq \log d_A + \widetilde{H}_\beta(A)_\rho && \text{(F24)}\\
&\leq 2\log d_A, && \text{(F25)}
\end{aligned}$$

where the second inequality follows from data processing and the last is a universal bound on the Rényi entropy. ■

Proposition 40. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$. The following bounds hold for $\alpha \in [0,1) \cup (1,\infty)$:

$$I''_\alpha(A;B|C)_\rho \leq I_\alpha(A;BC)_\rho, \tag{F26}$$

$$I''_\alpha(A;B|C)_\rho \leq I_\alpha(B;AC)_\rho, \tag{F27}$$

and the following hold for $\alpha \in (0,1) \cup (1,\infty)$:

$$\widetilde{I}''_\alpha(A;B|C)_\rho \leq \widetilde{I}_\alpha(A;BC)_\rho, \tag{F28}$$

$$\widetilde{I}''_\alpha(A;B|C)_\rho \leq \widetilde{I}_\alpha(B;AC)_\rho. \tag{F29}$$

Proof. A proof for the first inequality follows from

$$\begin{aligned}
I''_\alpha(A;B|C)_\rho &= \inf_{\sigma_{ABC}} \frac{1}{\alpha-1} \log \operatorname{Tr}\left\{\rho_{ABC}^\alpha \sigma_{AC}^{(1-\alpha)/2} \sigma_C^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \sigma_C^{(\alpha-1)/2} \sigma_{AC}^{(1-\alpha)/2}\right\} && \text{(F30)}\\
&\leq \inf_{\sigma_{BC}} \frac{1}{\alpha-1} \log \operatorname{Tr}\left\{\rho_{ABC}^\alpha (\rho_A \otimes \sigma_C)^{(1-\alpha)/2} \sigma_C^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \sigma_C^{(\alpha-1)/2} (\rho_A \otimes \sigma_C)^{(1-\alpha)/2}\right\} && \text{(F31)}\\
&= \inf_{\sigma_{BC}} \frac{1}{\alpha-1} \log \operatorname{Tr}\left\{\rho_{ABC}^\alpha \left(\rho_A^{1-\alpha} \otimes \sigma_{BC}^{1-\alpha}\right)\right\} && \text{(F32)}\\
&\equiv I_\alpha(A;BC)_\rho, && \text{(F33)}
\end{aligned}$$

as defined in (1.11). A proof for the second inequality follows similarly by choosing $\sigma_{ABC} = \rho_B \otimes \sigma_{AC}$. Proofs for the last two inequalities are similar, except the sandwiched Rényi mutual information is defined for a bipartite state ρ_{AB} as

$$\widetilde{I}_\alpha(A;B)_\rho \equiv \inf_{\sigma_B} \frac{1}{\alpha-1} \log \operatorname{Tr}\left\{\left[(\rho_A \otimes \sigma_B)^{\frac{1-\alpha}{2\alpha}} \rho_{AB} (\rho_A \otimes \sigma_B)^{\frac{1-\alpha}{2\alpha}}\right]^\alpha\right\}. \tag{F34}$$

■

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