Corrigendum to "Knot Floer homology detects fibred knots"

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Abstract

We correct a mistake on the citation of JSJ theory in [4]. Some arguments in [4] are also slightly modified accordingly.

An important step in [4] uses JSJ theory [2, 3] to deduce some topological information about the knot complement when the knot Floer homology is monic, see [4, Section 6]. The version of JSJ theory cited there is from [1]. However, as pointed out by Kronheimer, the definition of "product regions" in [1] is not the one we want. In this note, we will provide the necessary background on JSJ theory following [2]. Some arguments in [4] will then be modified.

Definition 1. An *n–manifold pair* is a pair (M, T) where M is an n–manifold and T is an $(n-1)$–manifold contained in ∂M. A 3–manifold pair (M, T) is *irreducible* if M is irreducible and T is incompressible. An irreducible 3–manifold pair (M, T) is *Haken* if each component of M contains an incompressible surface.

Definition 2. [2, Page 10] A compact 3–manifold pair $(\mathcal{S}, \mathcal{T})$ is called an *I–pair* if $\mathcal{S}$ is an I–bundle over a compact surface, and $\mathcal{T}$ is the corresponding ∂I–bundle. A compact 3–manifold pair $(\mathcal{S}, \mathcal{T})$ is called an *S^1–pair* if $\mathcal{S}$ is a Seifert fibred manifold and $\mathcal{T}$ is a union of Seifert fibres in some Seifert fibration. A *Seifert pair* is a compact 3–manifold pair $(\mathcal{S}, \mathcal{T})$, each component of which is an I–pair or an S^1–pair.

Definition 3. [2, Page 138] A *characteristic pair* for a compact, irreducible 3–manifold pair (M, T) is a perfectly-embedded Seifert pair $(\Sigma, \Phi) \subset (M, \mathrm{int}(T))$ such that if f is any essential, nondegenerate map of an arbitrary Seifert pair $(\mathcal{S}, \mathcal{T})$ into (M, T), f is homotopic, as a map of pairs, to a map f' such that $f'(\mathcal{S}) \subset \Sigma$ and $f'(\mathcal{T}) \subset \Phi$.

The definition of a perfectly-embedded pair can be found in [2, Page 4]. We note that the definition requires that $\Sigma \cap \partial M = \Phi$, so Σ is disjoint from $\partial M - T$.

The main result in JSJ theory is the following theorem.

Theorem 4 (Jaco–Shalen [2], Johannson [3]). *Every Haken 3–manifold pair (M, T) has a characteristic pair. This characteristic pair is unique up to ambient isotopy relative to $(\partial M - \mathrm{int}(T))$.*

Definition 5. Let (M, γ) be a sutured manifold. A 3–manifold pair $(P, Q) \subset (M, R(\gamma))$ is a *product pair* if $P = F \times [0,1], Q = F \times \{0,1\}$ for some compact surface F, and $F \times 0 \subset R_-(\gamma), F \times 1 \subset R_+(\gamma)$. We also require that $P \cap A = \emptyset$ or A for any annular component A of γ. A product pair is *gapless* if no component of its exterior is a product pair.

Definition 6. Suppose (M, γ) is a taut sutured manifold, (Σ, Φ) is the characteristic pair for $(M, R(\gamma))$. The *characteristic product pair* for M is the union of all components of (Σ, Φ) which are product pairs. A *maximal product pair* for M is a gapless product pair $(\mathcal{P}, \mathcal{Q})$ such that it contains the characteristic product pair, and if $(\mathcal{P}', \mathcal{Q}') \supset (\mathcal{P}, \mathcal{Q})$ is another gapless product pair, then there is an ambient isotopy relative to γ that takes $(\mathcal{P}', \mathcal{Q}')$ to $(\mathcal{P}, \mathcal{Q})$.

The existence of maximal product pairs follows from the definition, although the uniqueness is not guaranteed. The exterior of a maximal product pair is also a sutured manifold. By definition the exterior does not contain essential product annuli or essential product disks.

Now we are ready to modify the arguments in [4]. The next theorem is a reformulation of [4, Theorem 6.2]. The proof is not changed though.

Theorem 6.2′ *Suppose (M, γ) is an irreducible balanced sutured manifold, γ has only one component, and (M, γ) is vertically prime. Let $\mathcal{E}$ be the subgroup of $H_1(M)$ spanned by the first homologies of product annuli in M. If $\widehat{HFS}(M, \gamma) \cong \mathbb{Z}$, then $\mathcal{E} = H_1(M)$.* □

Corollary 7. *In the last theorem, suppose (Π, Ψ) is the characteristic product pair for M, then the map*

$$i_*\colon H_1(\Pi) \to H_1(M)$$

is surjective.

Proof. We recall that such an M is a homology product [4, Proposition 3.1].

Suppose (Σ, Φ) is the characteristic pair for $(M, R(\gamma))$, then any product annulus can be homotoped into (Σ, Φ) without crossing γ. Let $\Phi_+ = \big(\Phi \cap R_+(\gamma)\big) \subset \mathrm{int}(R_+(\gamma))$. Theorem 6.2′ implies that the map $H_1(\Phi_+) \to H_1(R_+(\gamma))$ is surjective, so $\partial \Phi_+$ consists of separating circles in $R_+(\gamma)$. If a component $(\mathcal{S}, \mathcal{T})$ of (Σ, Φ) is an S^1–pair, then $\mathcal{T} \cap R_+(\gamma)$ consists of annuli by definition. We conclude that each annulus is null-homologous in $H_1(R_+(\gamma))$.

Suppose a product annulus A contributes to $H_1(M)$ nontrivially, and it can be homotoped into a component (σ, φ) of (Σ, Φ). Given the result from the last paragraph, this (σ, φ) cannot be an S^1–pair. It is neither a twisted I–bundle since the two components of ∂A are contained in different components of $R(\gamma)$. So (σ, φ) must be a trivial I–bundle, and the two components of φ lie in different components of $R(\gamma)$. In other words, (σ, φ) is a product pair. Now our desired result follows from Theorem 6.2′. □

The following proof of the main theorem in [4] is only slightly changed. Basically we use "maximal product pair" here instead of the wrong notion "characteristic product region" in [4].

Proof of [4, Theorem 1.1]. Suppose (M,γ) is the sutured manifold obtained by cutting open $Y-\mathrm{int}(\mathrm{Nd}(K))$ along F, $(\mathcal{P},\mathcal{Q})$ is a maximal product pair for M. We need to show that M is a product. By [4, Proposition 3.1], M is a homology product. Moreover, by [4, Theorem 4.1], we can assume M is vertically prime.

If M is not a product, then $M-\mathcal{P}$ is nonempty. Thus there exist some product annuli in (M,γ), which split off $\mathcal{P}$ from M. Let (M',γ') be the remaining sutured manifold. By definition $(\mathcal{P},\mathcal{Q})$ contains the characteristic product pair for M. Corollary 7 then implies that the map $H_1(\mathcal{P})\to H_1(M)$ is surjective. So $R_\pm(\gamma')$ are planar surfaces, and $M'\cap\mathcal{P}$ consists of separating product annuli in M. Since we assume that M is vertically prime, M' must be connected. (See the first paragraph in the proof of [4, Theorem 5.1].) Moreover, M' is also vertically prime. By [4, Theorem 5.1], $\widehat{HFS}(M',\gamma')\cong\mathbb{Z}$.

We add some product 1–handles to M' to get a new sutured manifold (M'',γ'') with γ'' connected. By [4, Proposition 2.9], $\widehat{HFS}(M'',\gamma'')\cong\mathbb{Z}$. It is easy to see that M'' is also vertically prime. [4, Proposition 3.1] shows that M'' is a homology product.

Let H be one of the product 1–handles added to M' such that H connects two different components of γ'. By Theorem 6.2$'$, there is at least one product annulus A in M'', such that A cannot be homotoped to be disjoint from the cocore of H. Isotope A if necessary, we find that at least one component of $A\cap M'$ is an essential product disk in M', a contradiction to the assumption that $(\mathcal{P},\mathcal{Q})$ is a maximal product pair. □

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