Embedding infinite cyclic covers of knot spaces into 3-space*

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Abstract

We say a knot k in the 3-sphere $\mathbb{S}^3$ has *Property IE* if the infinite cyclic cover of the knot exterior embeds into $\mathbb{S}^3$. Clearly all fibred knots have Property IE.

There are infinitely many non-fibred knots with Property IE and infinitely many non-fibred knots without property IE. Both kinds of examples are established here for the first time. Indeed we show that if a genus 1 non-fibred knot has Property IE, then its Alexander polynomial $\Delta_k(t)$ must be either 1 or $2t^2 - 5t + 2$, and we give two infinite families of non-fibred genus 1 knots with Property IE and having $\Delta_k(t) = 1$ and $2t^2 - 5t + 2$ respectively.

Hence among genus one non-fibred knots, no alternating knot has Property IE, and there is only one knot with Property IE up to ten crossings.

We also give an obstruction to embedding infinite cyclic covers of a compact 3-manifold into any compact 3-manifold.

Keywords: Embedding; Non-fiber knots; Infinite cyclic coverings.

MSC: 57M10, 57M25, 57N30

§1. Introduction

In this paper all surfaces and 3-manifolds are orientable, and all surfaces in 3-manifolds are proper, embedded and two-sided. Suppose S (resp. P) is a surface (resp. 3-manifold) in a 3-manifold M, we use $M \setminus S$ (resp. $M \setminus P$) to denote the manifold obtained by cutting M along S (resp. removing int$P$, the interior of P, from M).

Suppose S is a connected non-separating surface in M. Then $X = M \setminus S$ has two copies of S in ∂X, denoted by $S^+ \sqcup S^-$. Taking countably many copies of X: $\{X_i\}_{i=-\infty}^{+\infty}$, and identifying S^+_{i-1} with S^-_i for all i, we get an infinite cyclic cover of M, denoted by $\widetilde{M}_S$.

Let k be a knot in $\mathbb{S}^3$, $E(k)$ be the exterior of k, S be a Seifert surface of k. Then $E(k)$ has a unique infinite cyclic cover, simply denoted by $\widetilde{E}(k)$. If k is a fibred knot with fiber S, then $\widetilde{E}(k)$ is homeomorphic to $S \times \mathbb{R}$ which clearly embeds into $\mathbb{S}^3$. The paper will address the following

Question 1. *Suppose k is a non-fibred knot, when does $\widetilde{E}(k)$ embed into $\mathbb{S}^3$?*

The third named author was introduced to Question 1 during conversations with Professor Robert D. Edwards in the spring of 1984, and Edwards attributed Question 1 to Professor J. Stallings.

It is natural to ask the following more general and flexible

Question 2. *When does an infinite cyclic cover of a compact 3-manifold embed into a compact 3-manifold?*

Definition 1.1 We say a knot k in $\mathbb{S}^3$ has *Property IE*, if the infinite cyclic cover $\widetilde{E}(k)$ embeds into $\mathbb{S}^3$. We say a knot k in $\mathbb{S}^3$ has *Property DIE*, if $(\widetilde{E}(k), \tau) \subset (\mathbb{S}^3, f)$, that is, the deck transformation τ of $\widetilde{E}(k)$ embeds into a dynamical system f on $\mathbb{S}^3$. (We say a dynamical system g on a space P *embeds* into a dynamical system f on a space Y, denoted by $(P, g) \subset (Y, f)$, if there is an embedding $P \subset Y$ such that $f|P = g$.)

The organization of this paper goes as below.

§2 and §3 are the main parts of the paper. All knots involved in §2 and §3 are of genus 1 and non-fibred. It is well known that the only genus 1 fibred knots are 3_1 and 4_1 in the knot table.

In §2, we give a partial positive answer to Questions 1 and 2. In §2.1, beginning with a discrete dynamical system f on $\mathbb{S}^3$ (or a compact 3-manifold Y), we construct a compact 3-manifold M (closed or with torus boundary) such that $(\widetilde{M_S}, \tau) \subset (\mathbb{S}^3, f)$ or $\subset (Y, f)$, where τ is the deck transformation on the infinite cyclic cover $\widetilde{M_S}$. In §2.2 we prove that the simplest non-trivial example provided by construction in §2.1 is $E(9_{46})$, the exterior of the 46-th knot of nine crossings in the knot table, see [R] or [BZ], therefore provide the first known positive example to Question 1. A subtle point in the verification is to choose a right projection of 9_{46}, which significantly simplifies the process. But a key point is to choose 9_{46} among all knots in $\mathbb{S}^3$ to compare with. In §2.3, we give a sufficient condition for the 3-manifolds constructed in §2.1 to be complements of knots in $\mathbb{S}^3$, and then we prove that there are infinitely many non-fibred genus 1 knots having Property DIE by invoking Thurston and Soma's results on Gromov volume of 3-manifolds.

In §3, we give a partial negative answer to Question 1. By invoking Freedman-Freedman's version of Kneser-Haken finiteness theorem and results of Gabai (and Novikov) on foliation and on surgery, we prove that if a genus 1 non-fibred knot k has Property IE, then $E(k)$ is constructed as in §2.1, and hence k has Property DIE. It follows that the Alexander polynomial of such knots must be 1 or $2t^2 - 5t + 2$, and the Alexander invariant is also restricted. So "most" genus 1 non-fibred knots do not have Property IE. In particular, among all non-fibred genus 1 knots, no alternating knots have Property IE, and up to crossing numbers ≤ 10 only 9_{46} has Property IE. On the other hand, two infinite families of genus 1 non-fibred knots with Property IE constructed in §2.3 have $\Delta_k(t) = 1$ and $\Delta_k(t) = 2t^2 - 5t + 2$ respectively.

§4 is a remark about Property IE on connected sum, which provides knots of any given genus g (non-prime when $g > 1$), some of them have Property IE and some do not.

§5 gives a homological obstruction to embedding infinite cyclic covers of a compact 3-manifold into any compact 3-manifold (Theorem 5.1), therefore gives a partial negative answer to Question 2.

Comments.

1. If we replace the term "unknotted solid torus" by "unknotted handlebody of genus g for any $g > 1$", constructions in §2.1 can be used to study Property DIE of knots with higher genera, although the arguments become more complicated. The knots having Property DIE provide interesting dynamics in $\mathbb{S}^3$.

2. Theorem 5.1 as well as the constructions in §2.1 still holds for closed n-manifold and connected non-separating bicollared properly embedded $(n-1)$-submanifold S in M.

3. For knot k in S^3, the homological obstruction in Theorem 5.1 vanishes for

$E(k)$ (read Remark 2). We wonder if Question 2 has positive answer when we restrict to $E(k)$ for knots k in $\mathbb{S}^3$.

4. Two references [JNW] and [CL] were not cited in our proofs. But [JNW] suggested us the construction in §2.1 and [CL] inspired us to prove Lemma 3.1.

Acknowledgements. We are grateful to Dr. Hao Zheng for drawing the pictures, to Professor William Browder for a helpful conversation with the second author, to Professor Robert D. Edwards for bringing the third author to this simply stated intuitive question, to Professor David Gabai for a comment on alternating knots.

§2. Infinitely many genus 1 non-fibred knots have Property DIE

§2.1. A construction of compact 3-manifolds having infinite cyclic covers in $\mathbb{S}^3$ or in a compact 3-manifold

Step 1. We first consider a rather general case. Let Y be a closed 3-manifold, and $P \subset Y$ be a submanifold of dimension three with connected and non-empty ∂P. Suppose that there is a homeomorphism

$$f : Y \to Y \qquad \text{such that} \qquad f(P) \subset \mathrm{int} P.$$

Let $X = P \setminus f(P)$. Then $\partial X = \partial P \cup \partial f(P)$. Let $M = X/f$ be the closed 3-manifold obtained from X by identifying ∂P and $\partial f(P)$ via f, and $S \subset M$ be the image of ∂P and $\partial f(P)$ after identification. Then S is a connected non-separating surface in M. Clearly the infinite cyclic cover $\widetilde{M}_S$ is identified with $\cup_{k=-\infty}^{+\infty} f^k(X) \subset Y$ and $f| \cup_{k=-\infty}^{+\infty} f^k(X)$ gives the deck transformation τ. Hence $(\widetilde{M}_S, \tau) \subset (Y, f)$.

We say the construction above is *non-trivial*, if X is not homeomorphic to $\partial P \times [0, 1]$.

Step 2. Continue from Step 1. Let $Y = \mathbb{S}^3$ and let P be an unknotted solid torus P in $\mathbb{S}^3$, and let P' be a solid torus in $\mathrm{int} P$, such that P' is still unknotted in $\mathbb{S}^3$. Since both P and P' are unknotted in $\mathbb{S}^3$, there is a homeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$ such that $f(P) = P'$. Then $X = P \setminus f(P)$ is an example of Step 1.

Step 3. Continue from Step 2. Let Γ be a proper arc in X with one end in ∂P and the other in $\partial f(P)$. Let $N(\Gamma)$ be the regular neighborhood of Γ in X. Up to isotopy we may assume $f(\partial P \cap N(\Gamma)) = \partial f(P) \cap N(\Gamma)$. Let $X^* = X \setminus N(\Gamma)$. Then X^* is obtained from X by digging a tunnel from ∂P to $\partial f(P)$. Let $M^* = X^*/f$, $S^* = M^* \cap S$, where M^* is obtained from M by removing a solid torus. Clearly the infinite cyclic cover $\widetilde{M}^*_{S^*}$ is identified with $\cup_{k=-\infty}^{+\infty} f^k(X^*) \subset \mathbb{S}^3$, and $f|\cup_{k=-\infty}^{+\infty} f^k(X^*)$ gives the deck transformation τ. We summarize the discussion above as

Proposition 2.1 *M^* is a compact 3-manifold with torus boundary, and $(\widetilde{M}^*_S, \tau) \subset (\mathbb{S}^3, f)$. In particular if M^* is homeomorphic to $E(k)$ for a knot $k \subset S^3$, then k has Property DIE.*

§2.2. The knot 9_{46} has Property DIE

A simplest non-trivial construction in Proposition 2.1 is indicated in Figure 1, where P' is a 2-braid in P and the tunnel is "unknotted". In this subsection, all notions in Step 3 of §2.1 refer to Figure 1.

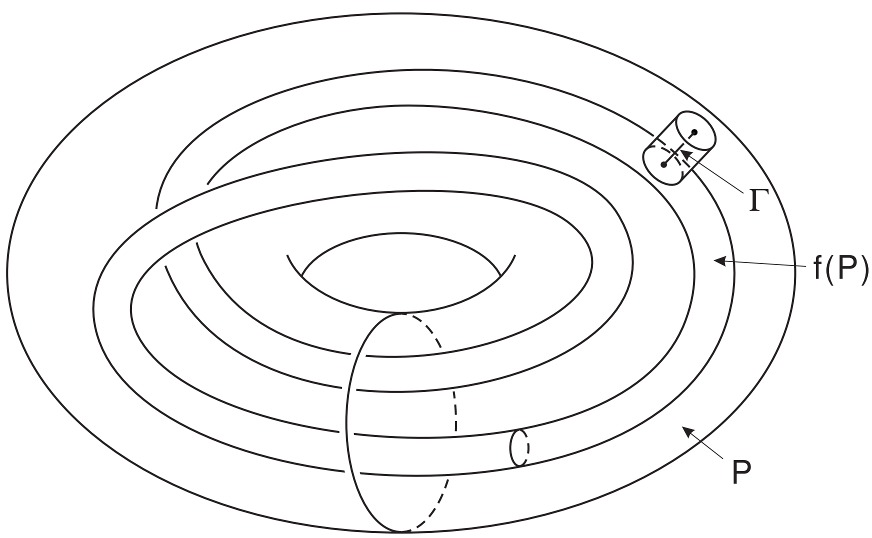

Figure 1

We will verify that M^* is homeomorphic to $E = E(9_{46})$, the exterior of knot $9_{46} \subset \mathbb{S}^3$ in the knot table. Our verification consists of three steps:

Step 1. Compute $\pi_1(M^*)$ and $\pi_1(\partial M^*) \subset \pi_1(M^*)$. Cutting M^* open along S^*, we get back to X^*, which is already presented in Figure 1. Its boundary $\partial X^* = S^*_- \cup \text{annulus} \cup S^*_+$, where S^*_- and S^*_+ are 1-punctured tori on the inner boundary $\partial f(P)$ and the outer boundary ∂P respectively. The annulus is the boundary of the tunnel.

Choose meridian μ_+ and longitude λ_+ on S^*_+ such that μ_+ bounds a disc in P and λ_+ bounds a disc in $\mathbb{S}^3 \setminus P$. Similarly choose meridian μ_- and longitude λ_- on S^*_- such that μ_- bounds a disc in $f(P)$ and λ_- bounds a disc in $\mathbb{S}^3 \setminus f(P)$, where $\mu_\pm$ and $\lambda_\pm$ are as indicated in Figure 2.

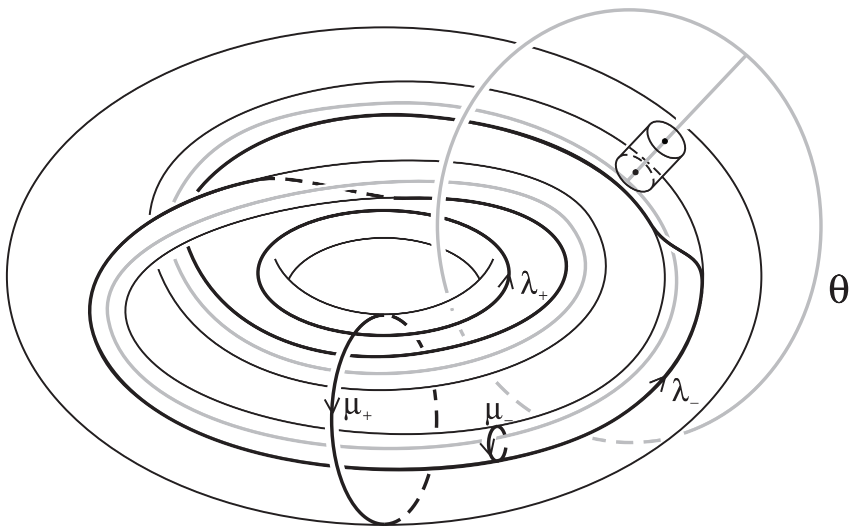

Figure 2

Since f is a homeomorphism on $\mathbb{S}^3$ which sends the unknotted solid torus P to $f(P)$, we must have $f(\lambda_+) = \lambda_-$, and $f(\mu_+) = \mu_-$. Now $M^* = X^*/f$ as in Step 3 of §2.1.

Note that in Figure 2, X^* is the complement of a graph Θ (shown in gray in Figure 2) in $\mathbb{S}^3$, where Θ consists of the centerline of $f(P)$, the centerline of $\mathbb{S}^3 \setminus P$, joined by the centerline γ of the tunnel.

If we ignore the image of X^* in Figure 2, but with Θ, $\lambda_\pm$ and $\mu_\pm$ remaining, then we have the Figure 3 below. Let B^3 be a 3-ball containing the arc γ in Θ, as indicated in Figure 3. It is an observation that the complement of Θ is homeomorphic to the complement of two unknotted arcs in the 3-ball $\mathbb{S}^3 \setminus B^3$. Hence X^* is a handlebody of genus 2.

Two generators a, b of $\pi_1(X^*)$ are indicated in Figure 3, where we use the Wirtinger presentation [R], the base point in X^* being above the page. Representing $\lambda_\pm$ and $\mu_\pm$ in terms of a, b, we have $\lambda_- = abab^{-1}$, $\mu_- = b$; $\lambda_+ = a$, $\mu_+ = baba^{-1}$. By HNN extension, we have

$$\pi_1(M^*) =< a, b, t |\ tat^{-1} = abab^{-1},\ tbaba^{-1}t^{-1} = b >,$$

and $\pi_1(\partial M^*) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by t and $[\lambda_-, \mu_-] = [abab^{-1}, b]$.

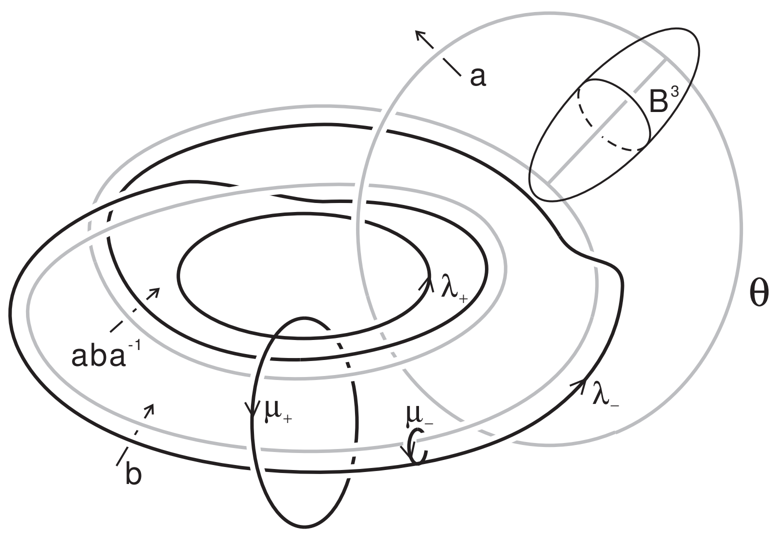

Figure 3

Step 2. Compute $\pi_1(E)$ and $\pi_1(\partial E) \subset \pi_1(E)$. We choose the projection of 9_{46} provided in [R, p. 211] rather than in the knot table of [R], as Figure 4 below. The Seifert surface T of 9_{46} in Figure 4 is the 1-punctured torus presented as a plumbing of two unknotted and untwisted bands $B(\alpha)$ and $B(\beta)$ with oriented centerlines α and β respectively. $\pi_1(T)$ is generated by α and β.

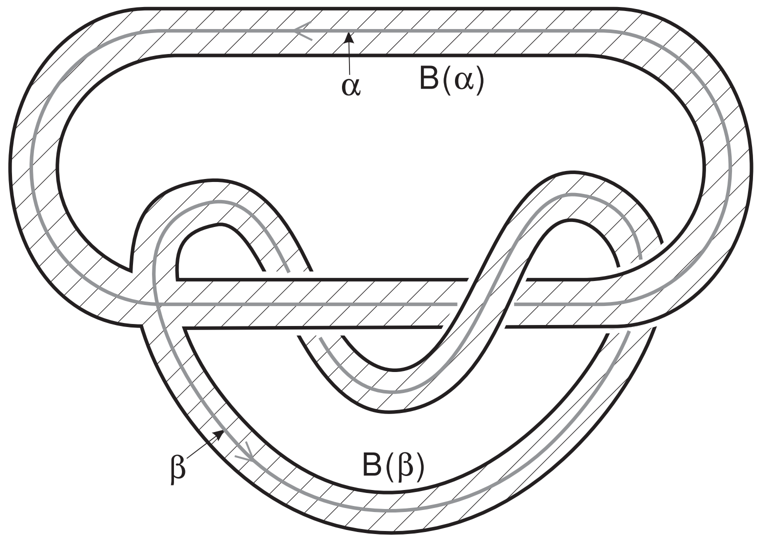

Figure 4

Cutting E open along T, we get a compact 3-manifold Q, which is the complement of T in $\mathbb{S}^3$, therefore Q is also homeomorphic to the complement of the one point union of the two circles $\alpha \cup \beta$. By a handle sliding argument (see [R, p. 95]) one can check that Q is also a handlebody of genus 2. Two generators c, d of $\pi_1(Q)$ are indicated in Figure 5.

First pushing α and β off T towards the minus side of T, we get two generators α_-, β_- of $\pi_1(T_-)$ in $\pi_1(Q)$; and then pushing α and β off T towards the plus side of T, we get two generators α_+, β_+ of $\pi_1(T_+)$ in $\pi_1(Q)$, all shown in Figure 5. It can be easily computed that $\alpha_- = cdcd^{-1}, \beta_- = d, \alpha_+ = c, \beta_+ = dcdc^{-1}$. So

$$\pi_1(E) = < c, d, s |\ scs^{-1} = cdcd^{-1},\ sdcdc^{-1}s^{-1} = d >,$$

and $\pi_1(\partial E) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by s and $[\alpha_-, \beta_-] = [cdcd^{-1}, d]$.

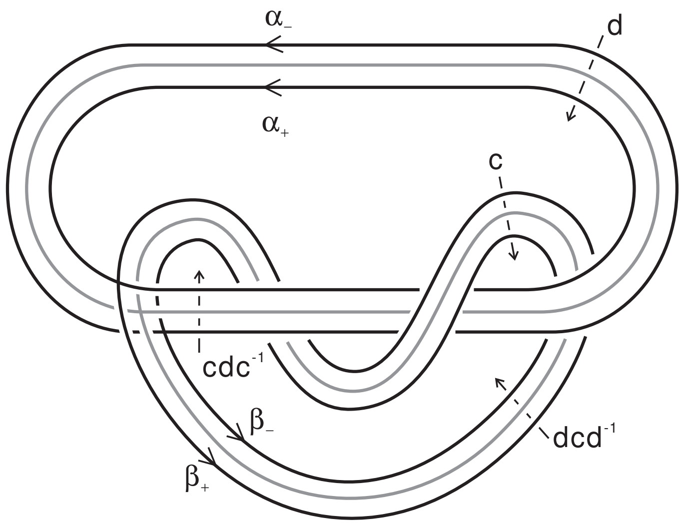

Figure 5

Step 3. Now we have an isomorphism

$$\phi : \pi_1(M^*) \to \pi_1(E), \text{ such that } a \mapsto c, b \mapsto d, t \mapsto s,$$

which maps $\pi_1(\partial M^*)$ isomorphically onto $\pi_1(\partial E)$.

Both M^*, E are $\mathbb{P}^2$-irreducible, sufficiently large manifolds, so Waldhausen's theorem [H, Theorem 13.6] (or [BZ, p. 308 B7] more directly), implies that M^* is homeomorphic to E. We finished the verification.

§2.3. Infinitely many genus 1 knots have Property DIE

Let P, P', X^* and X^*/f be as given in Step 3 of §2.1.

Proposition 2.2

(1) If a meridian disk D of P meets the core of P' in exactly 2 points tranversely, then X^*/f is the complement of a genus 1 knot in a homotopy 3-sphere.

(2) Furthermore if X^* is homeomorphic to a handlebody of genus 2, then $X^*/f = E(k)$ for some genus 1 knot $k \subset \mathbb{S}^3$.

(3) There are infinitely many genus 1 knots $k \subset \mathbb{S}^3$ such that $E(k)$ are obtained by the construction in §2.1.

Proof. Figure 6 indicates that there are infinitely many embeddings $P' \subset P$, such that both the conditions in Proposition 2.2 (1) and (2) are satisfied. The verification of X^* to be the handlebody of genus 2 is the same as we did in Figure 3 in §2.2. (Note that if we choose the tunnel jointing ∂P and $\partial P'$ to be knotted, then the condition in (2) is not satisfied in general.)

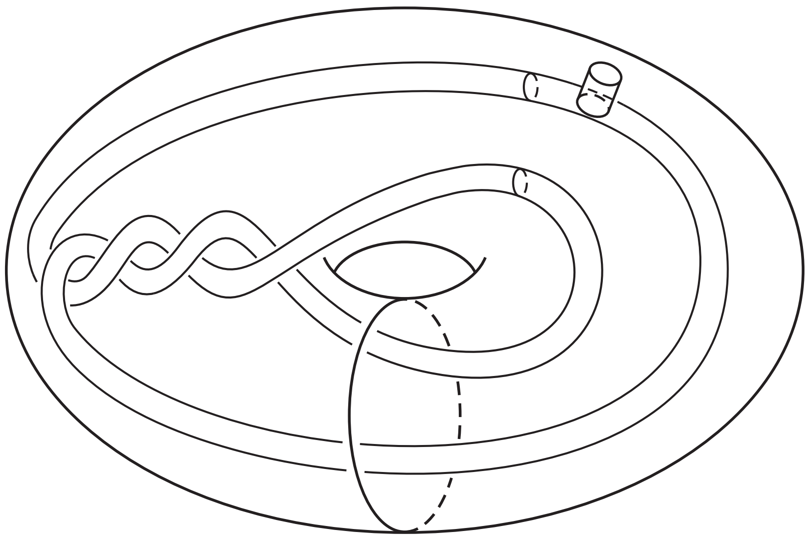

Figure 6

(1) We will find a presentation for $\pi_1(M^*)$ as in Step 1 of §2.2 (but the process is simpler since we need less precise information about the presentation.) First from Figure 6 we get Figure 7, (as what we did from Figure 1 to Figure 3 in Step 1 of §2.2,) where a, b, b' are elements in $G = \pi_1(X^*)$. Then as in Step 1 of §2.2 we can compute $\pi_1(X^*/f)$ via HNN extension as

$$\pi_1(X^*/f) =< G, t | tat^{-1} = c, tbb't^{-1} = b >,$$

where c is the element in G representing λ_-, and t is represented by a loop γ in $\partial(X^*/f) = T^2$. Note $\mu_+ = bb'$ because the meridian disk intersects P' twice.

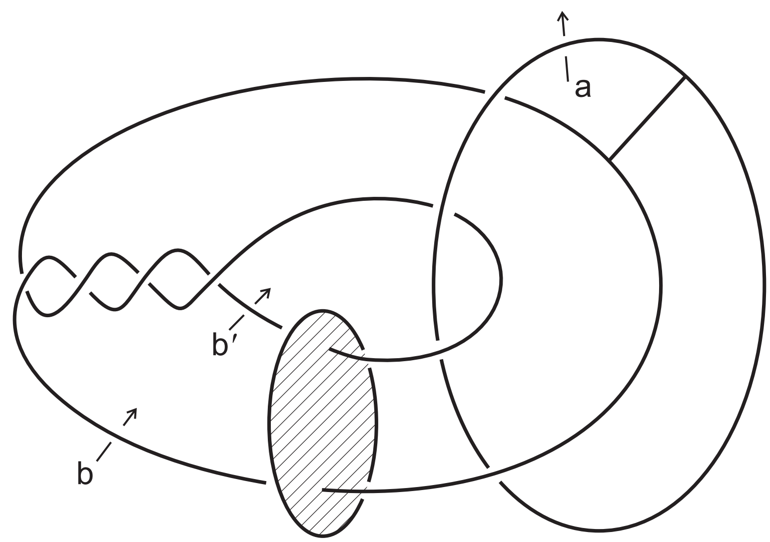

Figure 7

A Dehn filling along γ will kill t and provide a new manifold $M_1 = (X^*/f)(\gamma)$ with

$$\pi_1(M_1) =< G|a = c, bb' = b >=< G|a = c, b' = 1 > .$$

If we add a 2-handle to X^* along the loop representing b', the new manifold is obviously a solid torus. So $< G\,|b' = 1 >\cong \mathbb{Z}$. Thus $\pi_1(M_1)$ is a quotient group of $\mathbb{Z}$. A computation in homology will show that $\mathrm{H}_1(M_1;\mathbb{Z}) = 0$, hence $\pi_1(M_1) = 1$. Thus X^*/f is the complement of a knot k in the homotopy 3-sphere M_1.

(2) Furthermore suppose X^* is homeomorphic to a handlebody of genus 2. Note $X^*/f = X^* \cup_f N(\partial P \setminus N(\Gamma))$, and $M_1 = (X^*/f)(\gamma)$ can be viewed as a quotient of $X^* \cup_f N(\partial P \setminus N(\Gamma))$ by identifying the annulus $\partial(X^*/f) \cap N(\partial P \setminus N(\Gamma))$ with the annulus $\partial(X^*/f) \cap X^*$. Hence M_1 has a Heegaard splitting $X^* \cup_h N(\partial P \setminus N(\Gamma))$ of genus 2, where h is determined by f and γ. By Theorem 1 of [BH] M_1 is a 2-fold cyclic covering of S^3, branched over a 3-bridge link. It follows that M_1 is homeomorphic to $\mathbb{S}^3$ by Thurston's orbifold theorem (see [BP]), and hence $X^*/f = E(k)$ for a knot k in $\mathbb{S}^3$.

(3) We refine our notations related to Figure 6: Denote P', X^* and f by P'_n, X^*_n and f_n, if the crossing number of the core of $P' \subset P$ in Figure 6 is n, $n \in \mathbb{Z}$. Then

we have $X_n^*/f_n^* = E(k_n)$ for some knot $k_n \subset \mathbb{S}^3$ according to (2). If there are only finitely many different homeomorphism types for $E(k_n)$, then there are only finitely many $E(k_n, 0)$, the zero surgery manifold on k_n. It follows that the Gromov volumes $\{V(E(k_n, 0))\}$ take only finitely many values. Note that $E(k_n, 0)$ is homeomorphic to $(P \setminus P_n')/f_n$, and $E(k_n, 0) \setminus S_n = P \setminus P_n'$. Since ∂P is incompressible in $P \setminus P_n'$, $V(E(k_n, 0)) = V(P \setminus P_n')$ by a theorem of Soma [S, Theorem 1], it follows that $\{V(P \setminus P_n')\}$ take only finitely many values.

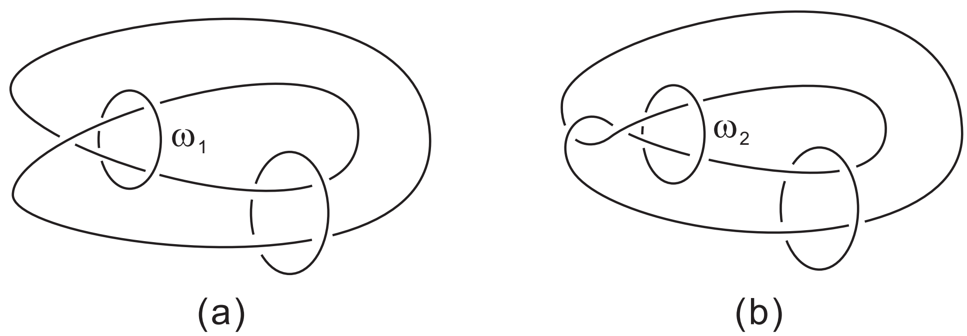

Figure 8

Consider two 3-component links L_1 and L_2 with marked components ω_1 and ω_2 respectively, indicated in Figure 8, (a) and (b). Note ω_i is unknotted in $\mathbb{S}^3$, the standard arguments (see [R, Chap. 9]) show that

$$P \setminus P_{2n+1}' = E(L_1)(\omega_1, 1/n), \qquad P \setminus P_{2n+2}' = E(L_2)(\omega_2, 1/n),$$

where $E(L_i)(\omega_i, 1/n)$ is the $1/n$-Dehn filling along ω_i. It is also known that both L_1 and L_2 are hyperbolic links. (This fact can be checked by SnapPea [W].) According to Thurston's theory about Gromov volume on 3-manifolds (see [Th, Chapters 5 and 6]), we have

(i) $V(E(L_i)(w_i, 1/n)) < V(E(L_i))$,

(ii) $\lim_{n\to\infty} V(E(L_i)(w_i, 1/n)) = V(E(L_i))$.

It follows that $\{V(P \setminus P_n')\}$ take infinitely many values, a contradiction. $\square$

Remark 1. All knots constructed in Proposition 2.2 bound the gunus 1 surface. All knots k_n in Propositon 2.2 (3) are non-fibred, see the end of §3, and also $k_1 = 9_{46}$.

§3. "Most" Genus 1 knots do not have Property IE

Let k be a non-fibred knot of genus 1 in $\mathbb{S}^3$. Recall the notations $E(k)$, S, $X = E(k) \setminus S, \widetilde{E}(k)$ defined in the beginning of the paper. Suppose also S is of genus 1.

Let S_n ($n = \dots, -2, -1, 0, 1, 2, \dots$) denote the copies of S in $\widetilde{E}(k)$. For integers $m < n$, let $X_{[m,n]}$ denote the sub-manifold of $\widetilde{E}(k)$ between S_m and S_n, and $A_{[m,n]}$ denote the annulus bounded by $\partial S_m \sqcup \partial S_n$ on $\partial X_{[m,n]}$. Assume $\widetilde{E}(k)$ is already embedded in $\mathbb{S}^3$, and $Y_{]m,n[} = \mathbb{S}^3 \setminus X_{[m,n]}$. We always use X_n to denote $X_{[n,n+1]}$ for simplicity. The readers should be aware that the subscript n here has different meaning from the n in the last section.

Lemma 3.1 *For any integer $N > 0$, ∂S_0 bounds a disk D in $Y_{]-N,N[}$.*

Proof. Consider the separating surfaces $S_n^* = A_{[N,n]} \cup S_n$ ($n = N+1, N+2, \dots$) in $Y_{]-N,N[}$. They are mutually non-parallel, since k is non-fibred. Since each S_n^* has the first Betti number 2, Freedman-Freedman's version of Kneser-Haken finiteness theorem [FF] implies that they must be compressible in $Y_{]-N,N[}$ when n is sufficiently large. Suppose D is a compressing disk of S_n^*. If ∂D is parallel to ∂S_n^* on S_n^*, then the lemma is proved, since ∂S_n^* is parallel to ∂S_0 on $\partial Y_{]-N,N[}$. If ∂D is not parallel to ∂S_n^*, surger S_n^* along ∂D, we still get a disk D' in $Y_{]-N,N[}$, with $\partial D' = \partial S_n^*$, since S_n^* is a 1-punctured torus. $\square$

Now fix an N sufficiently large, we can thicken $D \cup A_{[-N,N]}$ in $Y_{]-N,N[}$ to get a 2-handle $D \times I$, which is attached to $X_{[-N,N]}$ along the annulus $A_{[-N,N]}$. Let $D_{-N}, \dots, D_N$ be a collection of $D \times \{t\}$'s in the 2-handle, so that $\partial D_i = \partial S_i$, $i = -N, \dots, N$. From now on, all subscripts in this section are bounded by N, as is understood.

Let $\widehat{S}_i$ denote the torus $S_i \cup D_i$. Let $\widehat{X}_i$ be the manifold bounded by $\widehat{S}_i$ and $\widehat{S}_{i+1}$ in $\mathbb{S}^3$, and more generally, $\widehat{X}_{[m,n]}$ be the manifold bounded by $\widehat{S}_m$ and $\widehat{S}_n$ in $\mathbb{S}^3$.

Lemma 3.2 *$\widehat{X}_i$ is irreducible and ∂-irreducible. Moreover $\widehat{X}_i$ is not a product.*

Proof. Since S is a minimal genus Seifert surface of k, $E(k)$ admits a taut foliation $\mathcal{F}$ such that S is a leaf of $\mathcal{F}$ and $\mathcal{F}|\partial E(k)$ is foliated by circles by [G1, Theorem 3.1]. Then $\mathcal{F}$ can be extended to a taut foliation $\widehat{\mathcal{F}}$ on $E(k,0)$, the zero surgery manifold on k, such that $\widehat{S}$ is a leaf of $\widehat{\mathcal{F}}$, where $\widehat{S}$ is obtained by capping disc on S. Moreover since $E(k)$ is not fibred, $E(k,0)$ is not fibred by [G1, Corollary 8.19], in particular $E(k,0) \neq S^2 \times S^1$. By Novikov's theorem [N], each leaf of the taut foliation $\widehat{\mathcal{F}}$ is π_1-injective in $E(k,0)$ and π_2(E(k,0)) = 0. Then $E(k,0)$ is irreducible by the sphere theroem [H, Chap. 3], and furthermore $\widehat{S}$ is incompressible. It follows that $E(k,0) \setminus \widehat{S}$ is irreducible, ∂-irreducible, and is not a product.

Since each $\widehat{X}_i$ is homeomorphic to $E(k,0) \setminus \widehat{S}$, Lemma 3.2 is proved. □

Each $\widehat{S}_i$ separates $\mathbb{S}^3$ into 2 components. We say the component containing $\widehat{X}_i$ *lies on the plus side of* $\widehat{S}_i$, the component containing $\widehat{X}_{i-1}$ *lies on the minus side of* $\widehat{S}_i$. $\widehat{S}_i$ bounds a solid torus on the plus side or the minus side, since every torus in $\mathbb{S}^3$ bounds a solid torus. In fact, we can prove the stronger

Proposition 3.3 *Each $\widehat{S}_i$ bounds solid tori on both sides.*

Proof. Without loss of generality, we can assume $\widehat{S}_0$ bounds a solid torus P_0 on the minus side. Our argument proceeds in the following steps.

Step 1. For each $n < 0$, $\widehat{S}_n$ bounds a solid torus P_n on the minus side.

Otherwise, assume some $\widehat{S}_n$ does not bounds a solid torus on the minus side, then $\widehat{S}_n$ bounds a solid torus P_+ on the positive side. Hence $\widehat{S}_n$ cuts P_0 into 2 parts: $\widehat{X}_{[n,0]}$ and $P_0 \setminus \widehat{X}_{[n,0]} = \mathbb{S}^3 \setminus P_+$. By Lemma 3.2, $\widehat{S}_n$ is incompressible in $\widehat{X}_{[n,0]}$; $\widehat{S}_n$ is also incompressible in $\mathbb{S}^3 \setminus P_+$ since P_+ is knotted. So $P_0 = (\mathbb{S}^3 \setminus P_+) \cup_{\widehat{S}_n} \widehat{X}_{[n,0]}$ cannot have $\pi_1 = \mathbb{Z}$.

By Step 1, we have a nested sequence of solid tori

$$... \subset P_{n-1} \subset P_n \subset P_{n+1} \subset ... \subset P_0.$$

We assume that these tori adapt the orientation of $\mathbb{S}^3$. Let μ_n, $\lambda_n \subset \widehat{S}_n$ be a oriented meridian-longitude system of P_n, $n < 0$, so that

(1) the algebraic intersection number of μ_n and λ_n is 1,

(2) the linking number of λ_n and μ_{n+1}, which is defined as the winding number of P_n in P_{n+1}, is ≥ 0.

Step 2. Suppose P_n has winding number w_n in P_{n+1}, $n<0$. Then all w_n are equal, denoted by w.

Clearly $P_n \setminus P_{n-1}$ is homeomorphic to the complement in $\mathbb{S}^3$ of a 2-component link with linking number w_{n-1}, so $H_1(P_n \setminus P_{n-1}; \mathbb{Z})$ has a basis λ_n, μ_{n-1}, and $H_1(P_n \setminus P_{n-1}, \widehat{S}_n; \mathbb{Z})$ is isomorphic to $Z_{w_{n-1}}$.

Note that the deck translation $\tau : \widetilde{E}(k) \to \widetilde{E}(k)$, which sends X_i to X_{i+1}, induces a homeomorphism $\widehat{\tau} : \widehat{X}_{n-1} = P_n \setminus P_{n-1} \to \widehat{X}_n = P_{n+1} \setminus P_n$ with $\widehat{\tau}_n(\widehat{S}_n) = \widehat{S}_{n+1}$ for each $n<0$. It follows that $w_n = w_{n-1}$.

Step 3. We claim that $\widehat{\tau}$ sends μ_{n-1} to μ_n for $n \leq -1$. There are 2 cases:

Case 1. $w = 0, 1$. Now P_n can not be a braid in P_{n+1}, otherwise $w = 1$ and $\widehat{X}_n$ is a product $T^2 \times I$, contradicts to Lemma 3.2. Then the results in [G2] imply that only trivial surgery on P_n yields solid torus. Since the Dehn surgery on the knot P_n in the solid torus P_{n+1} along $\widehat{\tau}(\mu_{n-1})$ again yields solid torus, $\widehat{\tau}(\mu_{n-1}) = \mu_n$ for $n \leq -1$.

Case 2. $w \geq 2$. Fix $n < 0$. Now λ_i, μ_i is a basis of $\mathrm{H}_1(\widehat{S}_i; Z)$, $i \leq 0$.

$$\widehat{\tau}_*(\lambda_n) = p\lambda_{n+1} + q\mu_{n+1}, \qquad \widehat{\tau}_*(\mu_n) = r\lambda_{n+1} + s\mu_{n+1}, \qquad ps - qr = 1.$$

For each integer $m > 0$, since μ_n is a w^m multiple in $\mathrm{H}_1(\widehat{X}_{[n-m,n]}; \mathbb{Z})$, $\widehat{\tau}_*(\mu_n)$ is also a w^m multiple in $\mathrm{H}_1(\widehat{X}_{[n-m+1,n+1]}; \mathbb{Z})$. Since μ_{n+1} is already a w^m multiple in $\mathrm{H}_1(\widehat{X}_{[n-m+1,n+1]}; \mathbb{Z})$, $r\lambda_{n+1}$ is also a w^m multiple.

Since $\{\lambda_{n+1}, \mu_{n-m+1}\}$ is a basis of $\mathrm{H}_1(\widehat{X}_{[n-m+1,n+1]})$ for $m > 0$, r should be a w^m multiple. Since r is a given integer, letting m be sufficiently large, we must have $r = 0$. Then $p = s = \pm 1$, i.e., $\widehat{\tau}_*(\mu_n) = \pm\mu_{n+1}$, the conclusion holds.

Step 4. When $n > 0$, $\widehat{S}_n$ bounds a solid torus on the minus side.

There is a properly embedded planar surface G in $\widehat{X}_{-2}$, $G \cap \widehat{S}_{-1} = \mu_{-1}$, $G \cap \widehat{S}_{-2}$

consists of parallel copies of μ_{-2}. By Step 3, $\widehat{\tau}(G)$ is a planar surface in $\widehat{X}_{-1}$, $\widehat{\tau}(G)\cap\widehat{S}_{-1}$ consists of parallel copies of μ_{-1}. $\widehat{\tau}(G)\cap\widehat{S}_0$ bounds a disk on the minus side of $\widehat{S}_0$, since each copy of μ_{-1} bounds a disk in P_{-1}. So $\widehat{\tau}(\mu_{-1})=\widehat{\tau}(G)\cap\widehat{S}_0=\mu_0$. Let $\mu_n=\widehat{\tau}^n(\mu_0)$ for $n>0$, the same argument as above shows that μ_n bounds a disk on the minus side of $\widehat{S}_n$, by induction.

Step 5. All $\widehat{S}_n$ bounds solid tori on both sides, $n\in\mathbb{N}$.

By Lemma 3.2, $\widehat{S}_n$ and $\widehat{S}_m$ are not parallel for $m\neq n$. By Haken's finiteness theorem, $\widehat{S}_n$ is compressible in $\mathbb{S}^3\setminus P_0$ when n is sufficiently large. The compressing disk can not lie on the minus side, since $\widehat{X}_{[0,n]}$ is ∂-irreducible by Lemma 3.2. So $\widehat{S}_n$ bounds a solid torus on the plus side when n is sufficiently large. Now proceed from Step 1 to Step 4, but reverse the direction, to get our conclusion. □

Theorem 3.4 *Suppose k is a non-fibred knot of genus 1 in $\mathbb{S}^3$. If k have Property IE, then k has Property DIE. Indeed, $E(k)$ can be obtained by the construction in §2.1.*

Moreover, the winding number w involved is either 0 or 2. Correspondingly, the Alexander invariant of k is either 0 or $\mathbb{Z}[t,t^{-1}]/(2t-1)\oplus\mathbb{Z}[t,t^{-1}]/(t-2)$, and the Alexander polynomial of k is either 1 or $2t^2-5t+2$.

Proof. Suppose $\widetilde{E}(k)$ is embedded into $\mathbb{S}^3$. We keep the notations in the proof of Proposition 3.3. First, extend $\tau|X_0:(X_0,S_0)\to(X_1,S_1)$ to a homeomorphism $\widehat{\tau}_1:(\widehat{X}_0,\widehat{S}_0)\to(\widehat{X}_1,\widehat{S}_1)$ as in the proof of Proposition 3.3.

According to Proposition 3.3, each $\widehat{S}_n$ bounds a solid torus P_n^- on the minus side, and a solid torus P_n^+ on the plus side. Suppose $\mu_n^-,\mu_n^+\subset S_n\subset\widehat{S}_n$ are meridians of P_n^-,P_n^+ respectively. By Step 3 (and its counterpart in Step 5) of Proposition 3.3,

$$\widehat{\tau}(\mu_n^-)=\mu_{n+1}^-,\qquad \widehat{\tau}(\mu_n^+)=\mu_{n+1}^+. \tag{3.1}$$

Hence we can further extend $\widehat{\tau}_1$ to $\widehat{\tau}_2:P_0^+\to P_1^+$, and finally we extend $\widehat{\tau}_2$ to $f:\mathbb{S}^3\to\mathbb{S}^3$ since both P_0^+ and P_1^+ are unknotted. Now we can reconstruct $E(k)$ from f as in §2.1, so k has Property DIE. We have finished the proof of the first part of Theorem 3.4.

By Step 2 (and its counterpart in Step 5) of Proposition 3.3, the winding number of P_n^- in P_{n+1}^- is a constant w^-, and the winding number of P_{n+1}^+ in P_n^+ is a constant w^+. It is easy to see that both w^- and w^+ are the linking number between μ_{n+1}^- and μ_n^+ (see the paragraph after Step 1 in the proof of Proposition 3.3), we have $w^- = w^+ = w$. Since $\widehat{\tau}| : \widehat{S}_n \to \widehat{S}_{n+1}$ is orientation preserving, by (3.1) we have

$$\widehat{\tau}_*^{-1}([\mu_n^+]) = \pm w[\mu_n^+], \qquad \widehat{\tau}_*([\mu_n^-]) = \pm w[\mu_n^-], \tag{3.2}$$

Note that $X_n \hookrightarrow \widehat{X}_n$ induces an isomorphism on 1-dimensional homology. Then by (3.2) the Alexander invariant of k has presentation [R, Chap 7]

$$\mathrm{H}_1(\widetilde{E}(k); \mathbb{Z}[t, t^{-1}]) =< \mu_n^+, \mu_n^-, t |\, t^{-1}([\mu_n^+]) = \pm w[\mu_n^+],\, t([\mu_n^-]) = \pm w[\mu_n^-] >,$$

and the Alexander matrix of k is

$$\begin{pmatrix} wt \mp 1 & 0 \\ 0 & t \mp w \end{pmatrix}.$$

Since $\Delta_k(1) = \pm 1$, w can only be 0 or 2, and the corresponding Alexander polynomials are 1 or $2t^2 - 5t + 2$ respectively, and the Alexander invariant of k are either 0 or $\mathbb{Z}[t, t^{-1}]/(2t-1) \oplus \mathbb{Z}[t, t^{-1}]/(t-2)$. We have finished the proof of Theorem 3.4. $\square$

Corollary 3.5 *Among all genus* 1 *non-fibred knots in* $\mathbb{S}^3$*,*

(1) up to ten crossings, 9_{46} *is the only one that has Property* IE*,*

(2) no alternating knot has Property IE*.*

Proof. (1) For knots with ≤ 10 crossings, no non-fibred knot has Alexander polynomial 1, and only 6_1 and 9_{46} have Alexander polynomial $2t^2 - 5t + 2$, see the tables in [BZ] and in [R]. But their Alexander invariants are not isomorphic (see [R, p.211]), so 6_1 does not have Property IE. Then by §2.2 (1) follows.

(2) If a genus 1 non-fibred knot k has Property IE, then $\Delta_k(-1) = 1 \quad \text{or} \quad 9$. Now suppose k is alternating, by a theorem of R.H. Crowell, (see [BZ, Proposition 13.30]) $\Delta_k(-1)$ is not smaller than the crossing number of k, and 9_{46} is not alternating. Hence (2) follows from (1). $\square$

Recall the two infinite families of knots k_{2n} and k_{2n+1} with Property IE, as well as the notion P'_n, defined in the proof of Proposition 2.2 (3). Since the winding number of P'_{2n} is 0 and the winding number of P'_{2n+1} is 2, according to the calculation in the proof of Theorem 3.4 we have $\Delta_{k_{2n}}(t) = 1$ and $\Delta_{k_{2n+1}}(t) = 2t^2 - 5t + 2$.

Corollary 3.6 *Among non-fibered genu 1 knots, both the subsets defined by $\Delta_k(t) = 1$ and by $\Delta_k(t) = 2t^2 - 5t + 2$ have infinitely many elements with Property IE.* $\square$

§4. A remark on connected sums

Lemma 4.1 *Suppose k_1 and k_2 are two knots in $\mathbb{S}^3$.*

(1) If $k_1 \# k_2$ has Property IE, then both k_1 and k_2 have Property IE.

(2) If k_1 has Property IE and k_2 is fibred, then $k_1 \# k_2$ has Property IE.

Note that there are fibred knots of any genus (just consider the connected sum of genus 1 fibred knots), and that $k_1 \# k_2$ is fibred if and only if both k_1 and k_2 are fibred (follows from the definitions of connected sum, fibred knot, and Stallings' fibration Theorem [H, Theorem 11.1]). Then by the main results in §2, §3 and Lemma 4.1 we have the following

Corollary 4.2 *Among non-fibred knots of genus g for any given integer $g > 0$, both the subsets defined by having Property IE and not having Property IE have infinitely many elements.* $\square$

Proof of Lemma 4.1. Denote $E(k_i)$ by E_i. Let $N_i = N(\mu_i)$ be the regular neighborhood of the meridian $\mu_i \subset \partial E_i$ in E_i. Let $E_i^* = E_i \setminus N_i$, and $A_i = E_i^* \cap N_i$. Then E_i^* is homeomorphic to E_i and A_i is an annulus. By definition of the connected sum, we have $E(k_1 \# k_2) = E_1^* \cup_h E_2^*$, where h is a homeomorphism identifying A_1 and A_2.

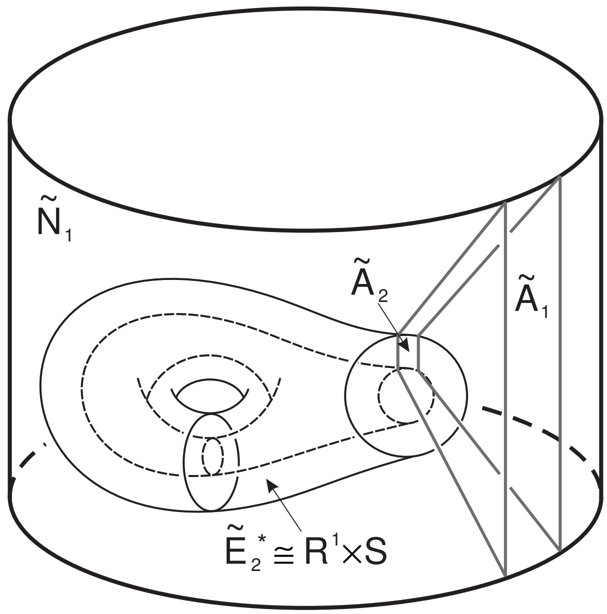

Figure 9

Let $p_i : \widetilde{E}_i \to E_i$ be the infinite cyclic covering, and let $\widetilde{E}_i^*, \widetilde{N}_i, \widetilde{A}_i$ be the preimage of E_i^*, N_i, A_i under p_i. Clearly the restriction of

$$p_i : (\widetilde{E}_i, \widetilde{E}_i^*, \widetilde{N}_i, \widetilde{A}_i) \to (E_i, E_i^*, N_i, A_i)$$

is the infinite cyclic covering on each of the four corresponding pairs. Moreover $\widetilde{E}_i^*, \widetilde{N}_i, \widetilde{A}_i$ are homeomorphic to $\widetilde{E}_i, R^1 \times D^2, R^1 \times I$ respectively and $\widetilde{E}(k_1 \# k_2) = \widetilde{E}_1^* \cup_{\widetilde{h}} \widetilde{E}_2^*$, where $\widetilde{h}$ is a homeomorphism identifying $\widetilde{A}_1$ with $\widetilde{A}_2$. Hence (1) follows.

We are going to prove (2). Now $\widetilde{E}_2^* = R^1 \times S$ for a once punctured surface S with $\widetilde{A}_2 = R^1 \times I$ properly embedded in $R^1 \times \partial S$.

Since there is an embedding $e : \widetilde{E}_2^* = R^1 \times S \to \widetilde{N}_1$ such that e sends $\widetilde{A}_2$ to $\widetilde{A}_1 \subset \partial \widetilde{N}_1$ homeomorphically, and $e(\widetilde{E}_2^*) \cap \widetilde{N}_1 = \widetilde{A}_1$ (see Figure 9), $\widetilde{E}(k_1 \# k_2) = \widetilde{E}_1 \cup_{\widetilde{h}} \widetilde{E}_2$ can be embedded into $E_1^* \cup \widetilde{N}_1 = \widetilde{E}_1$. Hence (2) follows. $\square$

§5. A partial negative answer to Question 2

In this section we use the notations in the first two paragraphs of §1. We will use $\mathrm{H}_i(\cdot)$ to denote $\mathrm{H}_i(\cdot\,;\mathbb{Q})$. Recall the following standard fact: Let

$$\cdots \to A \to B \to C \to \cdots$$

be an exact sequence of vector spaces. Then

$$\dim A + \dim C \geq \dim B. \tag{$*$}$$

Theorem 5.1 *Suppose M is a compact 3-manifold, S is a connected non-separating 2-sided proper surface in M. Let $X = M \setminus S$.*

(1) In the case $\partial M \neq \emptyset$, if $[S \cap T] \neq 0 \in \mathrm{H}_1(\partial M; \mathbb{Z})$ for each boundary component T of M and $\beta_1(X) > \beta_1(S) - \chi(\partial M)$, then $\widetilde{M}_S$ cannot be embedded into any compact 3-manifold.

(2) In the case $\partial M = \emptyset$, if $\beta_1(X) > \beta_1(S)$, then $\widetilde{M}_S$ cannot be embedded into any compact 3-manifold.

Proof. Suppose $\partial M \neq \emptyset$, $\widetilde{M} = \cup_{k=-\infty}^{+\infty} X_k$ can be embedded into a compact 3-manifold Y. We may assume $\partial Y = \emptyset$. Denote $\cup_{k=1}^{m} X_k$ by P_m.

We need first estimate $\beta_1(P_m)$. From $P_m = P_{m-1} \cup X_m$ and $S_m = P_{m-1} \cap X_m$, we have the Mayer-Vietoris sequence:

$$\cdots \to \mathrm{H}_1(S_m) \to \mathrm{H}_1(P_{m-1}) \oplus \mathrm{H}_1(X_m) \to \mathrm{H}_1(P_m) \to \cdots .$$

By $(*)$, we have the inequality:

$$\beta_1(P_m) \geq \beta_1(P_{m-1}) + \beta_1(X) - \beta_1(S).$$

Hence we easily deduce:

$$\beta_1(P_m) \geq m\beta_1(X) - (m-1)\beta_1(S). \tag{1}$$

We need then estimate $\beta_1(\partial P_m)$.

Cutting ∂M open along ∂S, we get a surface T'. ∂P_m is the union of $S_1^- \sqcup S_m^+$ and m copies of T'. Note that the cutting and gluing of surfaces are all along circles, which have Euler characteristic 0. So

$$\begin{aligned}
\chi(\partial P_m) &= \chi(S_1^- \sqcup S_m^+) + m\chi(T') \\
&= 2\chi(S) + m\chi(\partial M) \\
&= 2(1-\beta_1(S)) + m\chi(\partial M)
\end{aligned}$$

Then one can verify that

$$\beta_1(\partial P_m) = 2\beta_0(\partial P_m) - \chi(\partial P_m) = 2\beta_0(\partial P_m) + 2(\beta_1(S)-1) - m\chi(\partial M). \qquad (2)$$

Lemma 5.2 $\beta_0(\partial P_m) \le 2\beta_0(S \cap \partial M)$ for any m.

Proof. The bottom and the top of P_m are $S_1^- \sqcup S_m^+$, which consists of $2\beta_0(S \cap \partial M)$ boundary components. If for some m, $\beta_0(\partial P_m) > 2\beta_0(S \cap \partial M)$, then some component F of ∂P_m does not meet the top and the bottom of P_m. It follows that $F \subset P_m \subset \widetilde{M}_S$ provides a component of $\partial \widetilde{M}_S$, therefore $p(F)$ is a component of ∂M, where $p : \widetilde{M}_S \to M$ is the infinite cyclic covering map. Since the deck transformation group of the covering $p : \widetilde{M}_S \to M$ is the infinite cyclic group which contains no non-trivial finite subgroup, it follows that $p : F \to p(F)$ is a homeomorphism. Now $S \cap p(F) = \cup_{i=2}^m p(S_i \cap F)$.

Since S_i separates P_m, S_i separates F. Since F is closed, $S_i \cap F$ is homologically trivial in F. Hence $p(S_i \cap F)$ is homologically trivial in $p(F)$, and then $[S \cap p(F)] = 0$, contradicting the assumption in Theorem 5.1 (1). $\square$

By using (*) to various homology sequences, we have

$$\begin{aligned}
\beta_1(Y) &\ge \beta_1(Y, Y \setminus P_m) - \beta_0(Y \setminus P_m) && \text{by } (*) \\
&= \beta_1(P_m, \partial P_m) - \beta_0(Y \setminus P_m) && \text{by excision} \\
&\ge \beta_1(P_m, \partial P_m) - \beta_0(\partial P_m) && \text{since } \beta_0(Y \setminus P_m) \le \beta_0(\partial P_m) \\
&\ge \beta_1(P_m) - \beta_1(\partial P_m) - \beta_0(\partial P_m) && \text{by } (*) \\
&\ge m(\beta_1(X) - \beta_1(S) + \chi(\partial M)) + C && \text{by (1), (2) and Lemma 5.2}
\end{aligned}$$

where $C = 2 - \beta_1(S) - 6\beta_0(S \cap \partial M)$ is independent of m.

It follows that if $\beta_1(X) > \beta_1(S) - \chi(\partial M)$, $\beta_1(Y)$ would be arbitrarily large when m gets large. We reach a contradiction, since $\beta_1(Y)$ should be finite for a compact manifold Y. Theorem 5.1 (1) is proved.

A similar and more direct argument proves Theorem 5.1 (2) $\square$

Remark 2. Consider the connected sum $M = P \# E(k)$, where P is a homology 3-sphere with $\pi_1(P) \neq 1$ and k is a knot in $\mathbb{S}^3$. Let $S \subset M$ be a Seifert surface of $E(k)$, and $X = M \setminus S$. Then $\beta_1(X) \leq \beta_1(S)$ and $\chi(\partial M) = 0$. So the inequality in Theorem 5.1 (1) is not met. There is an essential 2-sphere S^2 in the connected sum, and $p^{-1}(S^2)$ is an infinite family of essential 2-spheres in $\widetilde{M}_S$, where $p : \widetilde{M}_S \to M$ is the infinite cyclic covering. Then $\widetilde{M}_S$ can not stay in a compact 3-manifold.

Otherwise suppose $\widetilde{M}_S \subset Y$ for a compact 3-manifold Y. Let $\cup_{i=1}^n S_i^2$ be n components in $p^{-1}(S^2)$ for any given n. Then clearly each component of $Y \setminus \cup_{i=1}^n S_i^2$ contains a copy of the 1-punctured homology 3-sphere P^* with $\pi_1(P^*) \neq 1$. Since P^* is not a subset of a punctured 3-sphere, no component of $Y \setminus \cup_{i=1}^n S_i^2$ is a punctured 3-sphere, which contradicts the Kneser finiteness theorem [H, Lemma 3.14].

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