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A polynomial time algorithm for the ground state of one-dimensional gapped local Hamiltonians

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Low entanglement approximations of the ground state

Definition 1. *Given a vector* $|v\rangle \in \mathcal{H}$*, by a Schmidt decomposition across the* $(i, i+1)$ *cut we shall mean a decomposition* $|v\rangle = \sum_{j=1}^{D} \lambda_j |a_j\rangle|b_j\rangle$ *with* $\{|a_j\rangle\}$ *(respectively* $\{|b_j\rangle\}$ *) a family of orthonormal vectors of* $\mathcal{H}_{[1,i]}$ *(respectively* $\mathcal{H}_{[i+1,n]}$*) and with* $\lambda_j \geq \lambda_{j+1} > 0$ *for all* $1 \leq j \leq D$*. The vectors* $|a_j\rangle$ *will be called the* left Schmidt vectors *across that cut, and the vectors* $|b_j\rangle$ *the* right Schmidt vectors*;* D *is the* Schmidt rank *across the cut.*

The following lemma follows from the 1D area law [1]. Although we will only need a polynomial bound on the Schmidt rank, we state the lemma using the best known parameters [2, Section 7].

Lemma 5. *For any constant* $c > 0$ *there is a constant* $C \geq 1$ *such that for every* n *there is a vector* $|v\rangle$ *with Schmidt rank bounded by* $exp(C(\ln n)^{3/4}\varepsilon^{-1/4})$ *across every cut such that* $|\langle\Gamma|v\rangle| \geq 1 - n^{-c}$.

The operation of *trimming* a state across a cut — removing Schmidt vectors associated with the smallest Schmidt coefficients — will be used repeatedly by our algorithm.

Definition 2. *Given a state* $|v\rangle \in \mathcal{H}$ *with Schmidt decomposition* $|v\rangle = \sum_j \lambda_j |a_j\rangle|b_j\rangle$ *across the* $(i, i+1)$ *cut and an integer* D*, define* $\mathrm{trim}_D^i|v\rangle := \sum_{j=1}^{D} \lambda_j |a_j\rangle|b_j\rangle$.[1]

The following well-known lemma states that among all vectors with Schmidt rank D across a certain cut i, $\mathrm{trim}_D^i|v\rangle$ provides the closest approximation to $|v\rangle$.

Lemma 6 (Eckart-Young theorem)**.** *Let* $|v\rangle \in \mathcal{H}$ *have Schmidt decomposition* $|v\rangle = \sum_i \lambda_i |a_i\rangle|v_i\rangle$ *across the* $(i, i+1)$ *cut. Then for any integer* D *the vector* $|v'\rangle = \mathrm{trim}_D^i|v\rangle / \|\mathrm{trim}_D^i|v\rangle\|$ *is such that* $\langle v'|v\rangle \geq |\langle w|v\rangle|$ *for any unit* $|w\rangle$ *of Schmidt rank at most* D *across the i-th cut.*

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[1] We note an ambiguity in the definition of $\mathrm{trim}_D^i|v\rangle$ in the case of degeneracies among the Schmidt decomposition. In our analysis it will never matter which eigenvectors associated with the same eigenvalue are kept.

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We will require the existence of close approximations to the ground state that have *constant* Schmidt rank across a given cut (and polynomial across the others).

Lemma 7 ([1, 2]). *For any cut $(i,i+1)$ and any constant δ, there exists a constant $B_\delta = exp(O(1/\varepsilon \log^3 d \log 1/\delta))$ such that the state $|v\rangle := \mathrm{trim}^i_{B_\delta}|\Gamma\rangle / \|\mathrm{trim}^i_{B_\delta}|\Gamma\rangle\|$ has the property that $|\langle\Gamma|v\rangle| \geq 1-\delta$.*

Lemma 8. *Let $\delta > 0$ be such that $\delta(1+1/\varepsilon) \leq \frac{1}{2}$ and $|w\rangle$ a unit vector with energy no larger than $\varepsilon_0 + \delta$. Then $|v\rangle := \mathrm{trim}_{B_\delta}|w\rangle / \|\mathrm{trim}_{B_\delta}|w\rangle\|$ has energy no larger than $\varepsilon_0 + 6\sqrt{\delta}$.*

Proof. By Lemma 14, $|\langle\Gamma|w\rangle| \geq 1-\delta/\varepsilon$. Let $|u\rangle := \mathrm{trim}^i_{B_\delta}|\Gamma\rangle / \|\mathrm{trim}^i_{B_\delta}|\Gamma\rangle\|$. Since by Lemma 7, $|\langle\Gamma|u\rangle| \geq 1-\delta$, Lemma 15 implies $|\langle w|u\rangle| \geq 1-\delta(1+1/\varepsilon) = 1-\delta'$. The Eckart-Young theorem (see Lemma 6) therefore implies that $\|\mathrm{trim}^i_{B_\delta}|w\rangle\|^2 \geq 1-\delta'$. Set $|w_0\rangle = \mathrm{trim}^i_{B_\delta}|w\rangle$, $|w_1\rangle = |w\rangle - |w_0\rangle$. We have

$$\langle w|H|w\rangle = \langle w_0|H-H_i|w_0\rangle + \langle w_1|H-H_i|w_1\rangle + \langle w|H_i|w\rangle,$$

and it follows that

$$\langle w_0|H|w_0\rangle \leq \varepsilon_0 + \delta - \langle w_1|H-H_i|w_1\rangle + \langle w_0|H_i|w_0\rangle - \langle w|H_i|w\rangle.$$

Using the fact that $|\langle w_0|H_i|w_0\rangle - \langle w|H_i|w\rangle| = |\langle w_0|H_i|w_0 - w\rangle + \langle w_0 - w|H_i|w\rangle| \leq 2\||w\rangle - |w_0\rangle\| \leq 2\sqrt{\delta'}$ along with the lower bound of $\langle w_1|H-H_i|w_1\rangle \geq (\varepsilon_0 - 1)\||w_1\rangle\|^2 \geq (\varepsilon_0-1)\delta'$ we have:

$$\langle w_0|H|w_0\rangle \leq \varepsilon_0(1-\delta') + \delta + 2\sqrt{\delta}.$$

This implies:

$$\langle v|H|v\rangle \leq \varepsilon_0 + \frac{\delta + 2\sqrt{\delta}}{1-\delta'} \leq \varepsilon_0 + 6\sqrt{\delta},$$

if $\delta' \leq 1/2$.

□

Corollary 1. *For any cut $(i,i+1)$ and any constant δ, there exists a constant B_δ such that there exists a state $|v\rangle$ with Schmidt rank B_δ that has energy at most $\varepsilon_0 + 6\sqrt{\delta}$ as well as the property that $\langle\Gamma|v\rangle \geq 1-\delta$.*

Proof. Setting $|v\rangle = \mathrm{trim}^i_{B_\delta}|\Gamma\rangle / \|\mathrm{trim}^i_{B_\delta}|\Gamma\rangle\|$, the result follows from Lemma 7 and Lemma 8. □

1. Extension

Proof of Lemma 1 from Methods. By definition $|S_i^{(1)}| = d|S_{i-1}|$. Clearly, the bond dimensions of vectors in $S_i^{(1)}$ are no larger than that of vectors in S_{i-1}. Given a witness $|v\rangle$ for S_{i-1} with Schmidt decomposition across the $(i-1,i)$ cut $|v\rangle = \sum_j \lambda_j |s_j\rangle|t_j\rangle$, decompose the first qudit of $|t_j\rangle$ on the computational basis as $|t_j\rangle = \sum_{k=0}^{d-1} |k\rangle|t_{jk}\rangle$. Then clearly $|v\rangle = \sum_{j,k} \lambda_j |s_j\rangle|k\rangle|t_{jk}\rangle$ is also a witness for $S_i^{(1)}$. □

2. Size trimming

Boundary contractions

The following claim (specifically part 3) shows how the boundary contraction can be used to decompose the problem of finding an approximate ground state into independent "left" and "right" subproblems:

Lemma 9 (Gluing)**.** *Given a density matrix σ on the space $\mathcal{H}_L \otimes \mathbb{C}^B$ and a state $|v\rangle = \sum_{j=1}^{B} \lambda_j |a_j\rangle|b_j\rangle$ on $\mathcal{H}_L \otimes \mathcal{H}_R$ the density matrix $\sigma' := U_v \sigma U_v^*$ on $\mathcal{H}_L \otimes \mathcal{H}_R$ satisfies the following properties:*

1. $\mathrm{tr}_{\mathcal{H}_R}(\sigma') = \mathrm{tr}_{\mathbb{C}^B}(\sigma)$,
2. $\| \mathrm{tr}_{[1,\cdots,i-1]}(\sigma') - \mathrm{tr}_{[1,\cdots,i-1]}(|v\rangle\langle v|)\|_1 = \| \mathrm{tr}_{[1,\ldots,i-1]}(\sigma) - \mathrm{cont}(v)\|_1$,
3. $\mathrm{tr}(\sigma' H) \le \mathrm{tr}(\sigma H_L) + \mathrm{tr}(|v\rangle\langle v|(H_R + H_i)) + n\| \mathrm{tr}_{[1,\ldots,i-1]}(\sigma) - \mathrm{cont}(v)\|_1$.

Proof. 1. Clear, since U_v is unitary.

2. We have

$$\begin{aligned}\| \mathrm{tr}_{[1,\cdots,i-1]}(\sigma') - \mathrm{tr}_{[1,\cdots,i-1]}(|v\rangle\langle v|)\|_1 &= \| \mathrm{tr}_{[1,\cdots,i-1]}(\sigma' - |v\rangle\langle v|)\|_1 \\ &= \| \mathrm{tr}_{[1,\cdots,i-1]}(U_v^*\sigma' U_v - U_v^*|v\rangle\langle v|)U_v\|_1 \\ &= \| \mathrm{tr}_{[1,\ldots,i-1]}(\sigma) - \mathrm{cont}(v)\|_1.\end{aligned}$$

3. Write $\mathrm{tr}(\sigma' H) = \mathrm{tr}(\sigma' H_L) + \mathrm{tr}(\sigma'(H_i + H_R))$. By the first item, $\mathrm{tr}(\sigma' H_L) = \mathrm{tr}(\sigma H_L)$. By the second item,

$$\begin{aligned}\mathrm{tr}(\sigma'(H_i + H_R)) - \mathrm{tr}(|v\rangle\langle v|(H_i + H_R)) &= (\mathrm{tr}_{[1,\ldots,i-1]}\,\sigma - \mathrm{cont}(v))(H_i + H_R)) \\ &\le n\| \mathrm{tr}_{[1,\ldots,i-1]}(\sigma) - \mathrm{cont}(v)\|_1.\end{aligned}$$

□

We show that the set $S_i^{(2)}$ defined in the second step of the algorithm is a $(i, p_1(n), q(n)b, 1/12)$-viable set, for some polynomial $q(n)$. The key observation is that, conditioned on the existence of a state $|w\rangle$ in $\mathrm{Span}\{S_i^{(1)}\} \otimes \mathcal{H}_R$ having both low energy and low bond dimension, the solution σ of the size trimming convex program (cf. (2) in *Methods*) for an X sufficiently close to the boundary contraction of $|w\rangle$ allows for the easy computation of the left Schmidt vectors of a good approximation to the ground state. This is shown in the following lemma; the subsequent Lemma 11 establishes the existence of $|w\rangle$.

Lemma 10. *Suppose there exists a state $|w\rangle$ in Span$\{S_i^{(1)}\} \otimes \mathcal{H}_R$ of bond dimension B_{c_ε} having energy at most $\varepsilon_0 + 6\sqrt{c_\varepsilon}$. Let X be the element of the net $\mathcal{N}$ that is closest to* $\mathrm{cont}(w)$ *and let σ be the solution to the size trimming convex program. Let $|u\rangle = \sum_j |u_j\rangle|j\rangle$ be the leading eigenvector of σ. Then there exist orthonormal vectors $\{|b_j\rangle \in \mathcal{H}_R\}$ such that $|u'\rangle := \sum_j |u_j\rangle|b_j\rangle$ has energy at most $\varepsilon_0 + \varepsilon/12$ and $|\langle u'|\Gamma\rangle| \ge 1 - 1/12$.*

Proof of Lemma 10. Apply Lemma 9 to σ and $|w\rangle$ to conclude that the energy of $\sigma' = U_w \sigma U_w^*$ can be upper bounded as follows

$$\begin{aligned}\mathrm{tr}(\sigma' H) &\le \mathrm{tr}(\sigma H_L) + \mathrm{tr}(|w\rangle\langle w|(H_R + H_i)) + n\| \mathrm{tr}_{[1,\ldots,i-1]}(\sigma) - \mathrm{cont}(w)\|_1 \\ &\le \varepsilon_0 + 6\sqrt{c_\varepsilon} + c_\varepsilon \\ &\le \varepsilon_0 + 7\sqrt{c_\varepsilon}, \end{aligned} \tag{1}$$

where we used the optimality of σ to bound $\mathrm{tr}(\sigma H_L) \le \mathrm{tr}(|w\rangle\langle w|H_L)$; indeed, $\mathrm{ls}(w)$ itself is a feasible solution to the size trimming convex program. Let $|v_j\rangle$ be the eigenvectors of σ', with corresponding eigenvalues $\lambda_1 \ge \cdots \ge \lambda_{B_{c_\varepsilon}}$. From (1) we get that $\sum_{j\in J} \lambda_j^2 \ge 1/2$ where $J = \{j : \mathrm{tr}(H|v_j\rangle\langle v_j|) \le \varepsilon_0 + 14\sqrt{c_\varepsilon}\}$.

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But since by definition $14\sqrt{c_\varepsilon} < \varepsilon/12 < \varepsilon/2$, any $|v_j\rangle$ with energy less than $\varepsilon_0 + 14\sqrt{c_\varepsilon}$ must satisfy $|\langle v_j|\Gamma\rangle|^2 > 1/2$. Thus there can only be one such $|v_j\rangle = |v_1\rangle$, and $\lambda_1^2 > 1/2$. Letting $|u\rangle := U_w^*|v_1\rangle$, $|u\rangle$ is the leading eigenvector of σ and has energy at most $\varepsilon_0 + \varepsilon/12$. Applying Lemma 14 to $|u'\rangle := |v_1\rangle$ establishes $|\langle u'|\Gamma\rangle| \geq 1 - 1/12$. □

In order to apply the previous lemma, we need to establish its hypothesis: that there exists a vector $|w\rangle$ with small bond dimension and low energy that lies in $\text{Span}\{S_i^{(1)}\} \otimes \mathcal{H}_R$.

Lemma 11. *There exists $|w\rangle$ in $\text{Span}\{S_i^{(1)}\} \otimes \mathcal{H}_R$ with bond dimension B_{c_ε} and energy bounded by $\varepsilon_0 + 6\sqrt{c_\varepsilon}$.*

Proof. Let $|v\rangle$ be the witness for $S_i^{(1)}$ being a $(i, c_\varepsilon/n)$-viable set. Since $|v\rangle$ is (c_ε/n)-close to $|\Gamma\rangle$ and H has norm at most n, its energy $|v\rangle$ is upper bounded by $\varepsilon_0 + c_\varepsilon$. Applying Lemma 8 to $|v\rangle$ we get a state $|w\rangle := \text{trim}_{B_{c_\varepsilon}}|v\rangle / \|\text{trim}_{B_{c_\varepsilon}}|v\rangle\|$ with energy bounded by $\varepsilon_0 + 6\sqrt{c_\varepsilon}$; moreover the left Schmidt vectors of $|w\rangle$ still lie in $\text{Span}\{S_i^{(1)}\}$. □

We note that since the set $S_i^{(1)}$ contains vectors specified using polynomial-size MPS, for any X a polynomial-size representation for the optimal solution σ to the size trimming convex program, (2) from *Methods*, can be computed efficiently. For this, we first compute an orthonormal basis $\{|f_k\rangle\}$ for $\text{Span}\{S_i^{(1)}\}$. Vectors in this basis van be represented as linear combinations of vectors in $S_i^{(1)}$. The variables of the convex program will be the polynomially many coefficients of σ on the $|f_k\rangle\langle f_\ell|$; to express the objective function as a function of these variables it suffices to compute each $\langle f_k|H_j|f_\ell\rangle$, which can be done efficiently by expanding the $|f_k\rangle$ on the vectors of $S_i^{(1)}$ and evaluating the resulting expression by using the MPS representations of the latter. The constraints can also be expressed as convex functions of the variables by pre-computing all $\langle f_k|H_j|f_\ell\rangle$. The remaining steps rely on the singular value decomposition which can be performed efficiently as well.

Proof of Lemma 2 from Methods. Together, Lemmas 10 and 11 establish that the vector $|u'\rangle$ from Lemma 10 is a witness for $S_i^{(2)}$ with error $\frac{1}{12}$. For every element of the net $\mathcal{N}$, step 2. of the algorithm generates at most B_{c_ε} vectors to be added to $S_i^{(2)}$ yielding a bound on the cardinality of $S_i^{(2)}$ of $p_1(n) = B_{c_\varepsilon}|\mathcal{N}|$ with $|\mathcal{N}| = (Bd/\eta)^{O((Bd)^2)}$. Finally since $S_i^{(2)} \subset \text{Span}\{S_i^{(1)}\}$ it is clear that each vector in $S_i^{(2)}$ has an MPS description with bond dimension bounded by the product of $|S_i^{(1)}|$ and the maximal bond dimension of the elements of $S_i^{(1)}$, i.e. dsb. □

3. Bond Trimming

Lemma 12. *Given a unit vector $|v\rangle$ with D non-zero Schmidt coefficients across the $(i, i+1)$ cut, for any $|u\rangle$ it holds that*

$$\left|\langle \text{trim}^i_{D/\varepsilon}(u)|v\rangle\right| \geq |\langle u|v\rangle| - \varepsilon.$$

Proof. Denote by $\lambda_1 \geq \lambda_2 \geq \dots$ the Schmidt coefficients of $|u\rangle$. We proceed by contradiction: assume $|w\rangle = |u\rangle - |\text{trim}_{D/\varepsilon}(u)\rangle$ has the property that $\langle w|v\rangle > \varepsilon$. Since $|v\rangle$ has only D non-zero Schmidt coefficients, by the Eckart-Young theorem (Lemma 6) this last condition implies that $\sum_{i=1}^{D} |\lambda_{\lceil D/\varepsilon\rceil + i}|^2 > \varepsilon$.

Using that the Schmidt coefficients are decreasing, we get

$$\sum_{j=1}^{\lceil D/\varepsilon \rceil} \lambda_j^2 \geq \left\lceil \frac{1}{\varepsilon} \right\rceil \sum_{j=1}^{D} \lambda_{\lceil D/\varepsilon \rceil + j}^2 > 1,$$

a contradiction. □

We also show that trimming a state across a given bond does not increase the Schmidt rank across any of the other bonds.

Lemma 13. *For any integer m the Schmidt rank of $|\mathrm{trim}_m^i(u)\rangle$ is no larger than the Schmidt rank of $|u\rangle$ across any cut $(j, j+1)$.*

Proof. Without loss of generality, assume $j \leq i$. Writing the Schmidt decomposition across cut $(i, i+1)$ as $|u\rangle = \sum \lambda_i |\alpha_i\rangle|\beta_i\rangle$ notice that

$$|\mathrm{trim}_m^i(u)\rangle = \Big(\mathrm{Id} \otimes \sum_{k=1}^{m} |\beta_k\rangle\langle\beta_k|\Big)|u\rangle.$$

Since this operator only acts strictly to the right of the $(j, j+1)$ cut, it cannot increase the Schmidt rank across that cut. □

Proof of Lemma 3 from Methods. Based on the analysis of the previous step of the algorithm, from Lemma 12 we know that there exists a witness $|u'\rangle$ for $S_i^{(2)}$ such that $|\langle u'|\Gamma\rangle| \geq 1 - 1/12$; furthermore $|u\rangle = \mathrm{ls}(u')$ is a member of the set $S_i^{(1)}$. Using Lemma 15 we get that $|\langle v|u'\rangle| \geq 1 - 5/24$. Applying Lemmas 12 and 13 to $|u'\rangle$ yields that the successive trimming of $|u'\rangle$ for each of the bonds $1, \ldots, i$ to Schmidt rank $p_2(n)$ results in a state $|w\rangle$ with

$$|\langle v|w\rangle| \geq |\langle v|u'\rangle| - n/(48n) \geq 1 - 11/48.$$

Applying Lemma 15 once more, $|\langle\Gamma|w\rangle| \geq 1 - 2(11/48 + 1/48) = 1/2$. Finally, observe that the left Schmidt vectors of $|w\rangle$ are identical to the left Schmidt vectors of the state obtained from applying the successive trimming procedure to $|u\rangle = \mathrm{ls}(u')$ instead. □

4. Error reduction

The sampling AGSP

We first state the following, which is implied by the variant of the Matrix-valued Chernoff bound [3, Theorem 1.6].

Theorem 1 (Matrix-valued Chernoff bound). *Let X_i be $d \times d$ i.i.d. matrix random variables such that* $\mathrm{E}[X_i] = X$, $\|X_i - X\| \leq R$, *and* $\sigma^2 := max\{E((X_i - X)(X_i - X)^*), E((X_i - X)^*(X_i - X))\}$. *Then for all integers ℓ and $t \geq 0$,*

$$\Pr\left(\Big\|\frac{1}{\ell}\sum_{k=1}^{\ell} X_i - X\Big\| \geq t\right) \leq 2de^{-\frac{\ell t^2}{2\sigma^2 + 2Rt/3}}. \tag{2}$$

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Proof of Proposition 1 from Methods. We apply Theorem 1 with $X_i = C^m P_{I_i}$, $X = A$ and $t = \frac{1}{q(n)}$, noting the bounds $R = C^m + 1$, $\sigma^2 \leq (C^m + 1)^2$. Using these in (2) yields 1. with probability at least

$$1 - 2d\exp\left(-\frac{\ell}{4C^{2m}q(n)^2}\right).$$

Choosing m as prescribed and noting that $C^{2m} = n^{O(\frac{\varepsilon_0}{\varepsilon})}$ (where the constant in the exponent may depend on the degree of q) leads to the probability of failure bounded by $\exp(-\ell/(n^{O(\varepsilon_0/\varepsilon)}q(n)^2))$ which for the specified choice of ℓ is exponentially small and in particular can be made less than $n^{-3}/2$ with an appropriate choice of constants.

Property 2 follows from elementary probability: letting Y_j to be a random variable counting the number of times P_j appears in a given term, Y_j has mean $\frac{m}{n}$ and variance bounded by $\frac{m}{n}$ and thus

$$\Pr\left(\left|Y_j - \frac{m}{n}\right| \geq a\sqrt{\frac{m}{n}}\right) \leq e^{-\Omega(a^2)}.$$

For a proper choice of $a = O(\sqrt{\log(n\ell m)})$ the probability is bounded by $\frac{1}{2n^3\ell m}$. By a union bound, the probability of every projection P_j appearing no more than $(a\sqrt{\frac{m}{n}} + \frac{m}{n})$ times in any term of K is bounded below by $1 - \frac{1}{2n^3}$. With the prescribed choices of a and m, $a\sqrt{\frac{m}{n}} + \frac{m}{n}$ is upper bounded by $O((\varepsilon_0/\varepsilon)\log n)$. □

We note that the exponential dependence of the exponent $c(\epsilon)$ on the ground state energy that is stated in our main theorem is due to the dependence of the parameters m, ℓ and κ in Proposition 1 on $1/\varepsilon$. Huang [4] suggested a variant of our algorithm in which our AGSP is replaced by a construction based on ideas of Hastings [1] and Osborne [5], leading to an improved dependence on $1/\varepsilon$ that allows for an algorithm that runs in polynomial time for any value of the ground state energy.

Proof of Lemma 4 from Methods. Aplying Proposition 1 from *Methods* with the polynomial $q(n)/2$, we obtain that for a proper choice of the parameters m and ℓ, and with probability at least $1 - 1/n^3$, the sampling AGSP K will have the desired properties. Note that only P_i acts across the boundary cut $(i, i+1)$ and that we can decompose it as the sum of d^2 terms as $P_i = \sum_{j,k=1}^{d} E_j \otimes F_k$. Since furthermore P_i appears no more than $\kappa \log(n)$ times in each term in K (for some $\kappa = O(\frac{\varepsilon_0}{\varepsilon})$), using the decomposition of P_i within each term P_I of K we can write

$$K = \sum_{j=1}^{d^{2\kappa \log n}\ell} A_j \otimes B_j \tag{3}$$

as the sum of polynomially many terms with A_j acting only to the left of the cut and B_j acting to the right. Define $L := \{A_j\}$ and set $S' := \{A_j|s\rangle : A_j \in L, |s\rangle \in S\}$. Letting $|v\rangle = \sum \lambda_j |a_j\rangle|b_j\rangle$ be a witness for S, we get $|\langle\Gamma|\frac{Av}{\|Av\|}\rangle| \geq 1 - c_\varepsilon/(2q(n))$. Given that $\|K - A\| \leq c_\varepsilon/(2q(n))$, we get $|\langle\Gamma|\frac{Kv}{\|Kv\|}\rangle| \geq 1 - c_\varepsilon/q(n)$, so that $\frac{Kv}{\|Kv\|}$ is a witness for S' achieving the claimed error.

Each step of the construction: generating the randomness needed for K, the computation of the A_j and the construction of S' can be done efficiently since there are only a polynomial number of terms involved, and the matrix product operator representations of the A_j have polynomial size bond dimension and can be efficiently computed. As a result, the set S' has size a fixed polynomial times that of S, and the bond dimension of vectors in S' is also a fixed polynomial times the bond dimension of vectors in S. □

5. Auxiliary lemma

The following two simple lemma are used repeatedly.

Lemma 14. *Suppose a state* $|v\rangle$ *has energy* $\langle v|H|v\rangle \leq \varepsilon_0 + \delta$, *for some* $0 \leq \delta \leq \varepsilon$. *Then* $|\langle v|\Gamma\rangle| \geq 1 - \delta/\varepsilon$.

Proof. Write $|v\rangle = \lambda|\Gamma\rangle + \sqrt{1-\lambda^2}|\Gamma^{\perp}\rangle$ for some unit vector $|\Gamma^{\perp}\rangle$ orthogonal to $|\Gamma\rangle$. $|v\rangle$ has energy

$$\varepsilon_0 + \delta \geq \langle v|H|v\rangle \geq \lambda^2\varepsilon_0 + (1-\lambda^2)\,\varepsilon_1,$$

which gives $\lambda^2 \geq 1 - \delta/\varepsilon$, hence $\lambda \geq 1 - \delta/\varepsilon$. □

Lemma 15. *Let* $0 \leq \delta, \delta' \leq 1$ *and* $|v\rangle$, $|v'\rangle$ *and* $|w\rangle$ *be states such that* $|\langle v|w\rangle| \geq 1 - \delta$ *and* $|\langle v'|w\rangle| \geq 1 - \delta'$. *Then* $|\langle v|v'\rangle| \geq 1 - 2(\delta + \delta')$.

Proof. We have

$$\begin{aligned} |\langle v|v'\rangle| &\geq |\langle v|\Gamma\rangle\langle v'|\Gamma\rangle| - \left((1-|\langle v|\Gamma\rangle|^2)(1-|\langle v'|\Gamma\rangle|^2)\right)^{1/2} \\ &= (1-\delta)(1-\delta') - 2\sqrt{\delta\delta'} \\ &\geq 1 - 2(\delta+\delta'). \end{aligned}$$

□

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