p-ADIC q-EXPANSION PRINCIPLES ON UNITARY SHIMURA VARIETIES

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Abstract. We formulate and prove certain vanishing theorems for p-adic automorphic forms on unitary groups of arbitrary signature. The p-adic q-expansion principle for p-adic modular forms on the Igusa tower says that if the coefficients of (sufficiently many of) the q-expansions of a p-adic modular form f are zero, then f vanishes everywhere on the Igusa tower. There is no p-adic q-expansion principle for unitary groups of arbitrary signature in the literature. By replacing q-expansions with Serre-Tate expansions (expansions in terms of Serre-Tate deformation coordinates) and replacing modular forms with automorphic forms on unitary groups of arbitrary signature, we prove an analogue of the p-adic q-expansion principle. More precisely, we show that if the coefficients of (sufficiently many of) the Serre-Tate expansions of a p-adic automorphic form f on the Igusa tower (over a unitary Shimura variety) are zero, then f vanishes identically on the Igusa tower.

This paper also contains a substantial expository component. In particular, the expository component serves as a complement to Hida's extensive work on p-adic automorphic forms.

Contents

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1. Introduction

The purpose of this paper is twofold. The main purpose is to formulate and prove certain vanishing theorems for p-adic automorphic forms on unitary groups. These vanishing theorems are analogous to the p-adic q-expansion principle for modular forms. We anticipate our vanishing theorems will play a key role in certain constructions of p-adic families of automorphic forms on unitary groups of arbitrary signature, similarly to how p-adic q-expansion principles have played a key role in certain constructions of p-adic families of modular forms.

The second purpose is to provide an expository guide to the theory of p-adic automorphic forms on unitary groups. We aim for our exposition to serve as a complement to H. Hida's extensive work in this area. In particular, we have been careful to provide details that are important in our case but are either not worked out or are not highlighted in the prior work in the area. In addition to providing a basis for our proofs of vanishing theorems, the expository portion of the paper was written with the goal of providing a foundation for further advances in the field.

1.1. **Vanishing theorems.**

1.1.1. *q-expansion principles for modular (and Hilbert modular) forms.* The notion that properties of modular forms are encoded in their q-expansion coefficients is well-established. The algebraic q-expansion principle makes this notion precise for algebraic modular forms. For p-adic modular forms (viewed as sections on the Igusa tower, which lies over the ordinary locus), there is a p-adic q-expansion principle, which states that if all of the coefficients of the q-expansions of a p-adic modular form f are zero, then f vanishes identically on the Igusa tower. (Actually, it is sufficient to check a subset of the q-expansions, namely a q-expansion at one cusp per geometrically irreducible component of the Igusa tower.)

The study of properties of q-expansions has proved indispensable in the development of the theory of p-modular forms. Indeed, the study of p-adic modular forms originated from J.-P. Serre's work, which used q-expansions to *define* p-adic modular forms [Ser73]. Serre's fundamental work in the subject was closely followed by N. Katz's generalization in [Kat73]. While Katz's definition of p-adic modular forms did not use q-expansions, much of his work (on modular forms and, more generally, Hilbert modular forms) nevertheless relies on both an algebraic q-expansion principle ([Kat73, Section 1.6] and [DR80, Theorems (5.4) and (5.5)], which depends crucially on [Rap78] and [Rib75]) and a p-adic q-expansion principle ([Kat78, Theorem (1.9.9)] and [DR80, Theorem (5.13)]). For instance, the q-expansion principles play a key role in Katz's construction of families of p-adic modular forms [Kat77, Kat78]. Similarly, they play a key role in P. Deligne and K. Ribet's construction of a family of p-adic modular forms [DR80]. Finally, we should also mention that the p-adic q-expansion principle is a crucial ingredient in the work of Buzzard and Taylor on the icosahedral Artin conjecture and in generalizations of this type of argument to Hilbert modular surfaces.

1.1.2. *Principles for other groups, including unitary groups.* Given the power of the algebraic and p-adic q-expansion principles for studying p-adic modular forms and given the significance of the related applications (e.g. L-functions and homotopy theory, as discussed in Section 1.2), it is natural to ask whether there are algebraic and p-adic q-expansion principles (or some closely related analogue) for automorphic forms on groups other than GL_2. In other words, if we extend our study of automorphic forms beyond the case of modular forms and Hilbert modular forms, can we expect a q-expansion principle? If not, what (if anything) is an appropriate alternative?

There are q-expansion principles, or partial results in this direction, in a number of cases. For Siegel modular forms, i.e. automorphic forms on symplectic groups, there is an algebraic q-expansion principle in [CF90]. By [Hid04, Corollary 8.17], there is a p-adic q-expansion principle for Siegel modular forms. By [Lan13, Proposition 7.1.2.14], there is an algebraic q-expansion principle for scalar-valued automorphic forms on unitary groups of signature (a, a) for any positive integer a. As mentioned in the last paragraph of [Hid04], there is a p-adic q-expansion principle for automorphic forms on unitary groups of signature (n, n), although Hida states "we leave for the reader a precise formulation of this fact." The proofs of all of these q-expansion principles rely primarily on results on the geometry of the underlying moduli space (i.e. a Shimura variety or Igusa tower).

For automorphic forms on unitary groups of signature (a, b) with $a \neq b$, the underlying geometry of the associated moduli spaces prevents the existence of a q-expansion principle, although there are analogues. In the case of automorphic forms on unitary groups of signature (a, b), when the corresponding Shimura varieties are non-compact, the usual q-expansion is replaced by a Fourier-Jacobi expansion, a generalization of the Fourier expansion, in which the coefficients are themselves functions formed from theta-functions. Nevertheless, there is an algebraic *Fourier-Jacobi Principle* for unitary groups [Lan13, Proposition 7.1.2.14]. (This Fourier-Jacobi Principle gives the algebraic q-expansion principle for unitary groups of signature (a, a).)

While it is natural to ask for a "p-adic Fourier-Jacobi expansion principle" for unitary groups of arbitrary signature, a slightly different - but analogous - principle, a "Serre-Tate expansion principle" follows more easily from the existing literature. The main result of this paper is the formulation and proof of the Serre-Tate Expansion Principle (in Theorem 5.3). Algebraic q-expansions and algebraic Fourier-Jacobi expansions are expansions of a modular (or automorphic) form at a cusp. On the other hand, a Serre-Tate expansion is the expansion of a modular form at a CM point. There is a canonical choice of coordinates for the local ring at the CM point; these are called *Serre-Tate deformation coordinates*. Roughly speaking our main result (stated precisely in Theorem 5.3) says that given suitable conditions on the prime p, if f is an automorphic form on a unitary group and for each irreducible component C of the associated Igusa tower, a Serre-Tate expansion of f at some CM point in C is 0, then f is vanishes identically on the Igusa tower. The proof relies on Hida's description of the geometry of the Igusa tower.

1.2. **Anticipated applications.** As noted above, the use of q-expansion principles in the construction of p-adic families of modular forms is well-established. In [Kat78], N. Katz used the q-expansion principle for Hilbert modular forms to study congruences between values of different Hilbert modular forms, which led to the construction of certain p-adic families of Hilbert modular forms. Similarly, in [Eis13], the second author used the q-expansion principle to construct p-adic families of automorphic forms on unitary groups of signature (a, a). Katz's p-adic families of Hilbert modular forms are the main ingredient in his construction of p-adic L-functions for CM fields [Kat78]. Analogously, the second author constructed the p-adic families of automorphic forms in [Eis13] to complete a step in the construction p-adic L-functions (for unitary groups) proposed in [HLS06].

We plan to use the Serre-Tate Expansion Principle in Theorem 5.3 analogously to how the q-expansion principle is used in contexts in which q-expansions exist. More precisely, in a joint paper in progress, we are using the Serre-Tate expansion principle introduced in this paper to construct p-adic families of automorphic forms on unitary groups of signature (a, b) with $a \neq b$. As explained in [Eis14], the lack of such a principle in the literature was an obstacle faced by the second author in her effort to extend her results on p-adic families of automorphic forms to unitary groups of arbitrary signature. This paper eliminates that obstacle and fills in a hole in the literature. We also plan to use the expository portion of this paper as part of the foundation for our construction of these families.

One advantage of expansions around CM points over q-expansions is that they can be used for compact as well as non-compact Shimura varieties. The Serre-Tate expansion has been used before by Hida (for example, to define his idempotent in [Hid04]) and also appears in work of Brooks (for Shimura curves) and Burungale and Hida (for Hilbert modular varieties) with applications to special values of p-adic L-functions.

In a more speculative direction, we note the potential for applications to homotopy theory. Certain p-adic families of modular forms, studied in terms of their q-expansions, were used to defined an invariant (the *Witten genus*) in homotopy theory [Hop95, Hop02, AHR10]. The Witten genus is a p-adic modular form valued invariant that occurs in the theory of *topological modular forms*. Recently, there have been attempts to construct an analogue of the Witten genus in the theory of *topological automorphic forms*, where there is conjecturally an invariant taking values in the space of p-adic automorphic forms on unitary groups of signature $(1, n)$ [Beh09]. Vanishing theorems analogous to the q-expansion principle will likely play an analogously important role in this context.

1.3. Structure of the paper.

We now provide a brief overview of the paper. Section 2 introduces Shimura varieties for unitary groups and the associated moduli problem. We work with these Shimura varieties throughout most of the paper. Section 3 reviews the theory of classical automorphic forms on unitary groups, from several perspectives. Section 4 introduces Hida's theory of p-adic automorphic forms (i.e. over the ordinary locus). This section includes details about the Igusa tower, as well as the space of p-adic automorphic forms (defined as sections of a particular line bundle over the Igusa tower). Finally, Section 5 covers the main results of this paper, namely the *Serre-Tate expansion principle*, an analogue of the q-expansion principle, for p-adic automorphic forms on unitary groups of arbitrary signature. In addition, this section discusses how Serre-Tate expansions behave with respect to pullbacks. We are using this result in a paper in preparation that constructs families of p-adic automorphic forms on unitary groups of arbitrary signature.

1.4. Notation and conventions.

We now establish some notation and conventions that we will use throughout the paper.

First, we establish some notation for fields. Fix a totally real number field K^+ and an imaginary quadratic extension F of $\mathbb{Q}$. Define K be the composition of K^+ and F. Let c denote complex conjugation on K, i.e. the generator of $\mathrm{Gal}(K/K^+)$. We denote by Σ the set of archimedean places of K^+, and we denote by Σ_K the set of archimedean places of K. We typically use τ to denote an element of Σ or Σ_K. A reflex field will be denoted by E (with subscripts to denote different reflex fields when there is more than one reflex field appearing in the same context). Given a field L, we denote the ring of integers in L by $\mathcal{O}_L$. We write $\mathbb{A}$ to denote the adeles over $\mathbb{Q}$, and we write $\mathbb{A}^\infty$ to denote the adeles away from the archimedean places.

Fix a rational prime p that splits as $p = w \cdot w^c$ in the imaginary quadratic extension $F/\mathbb{Q}$. We make this assumption in order to ensure that our unitary group at p is a product of (restrictions of scalars of) general linear groups. Instead, we could assume that every place of K^+ above p splits in the quadratic extension K/K^+ and choose a CM type for K. In addition, we restrict our attention to the case when the prime p is *unramified* in K. This ensures that the Shimura varieties we consider have smooth integral models over $\mathcal{O}_E \otimes \mathbb{Z}_{(p)}$ (where $\mathcal{O}_E$ is the ring of integers in the reflex field E of these Shimura varieties) when no level structure at p is imposed.

To help the reader keep track of each setting, we mostly adhere to the following conventions for fonts used to denote schemes, integral models, and formal completions throughout the paper (except for the formal scheme $\mathrm{Ig}^{\mathrm{ord}}$ defined in Section 4). Schemes over $\mathbb{Q}$ are in normal font, their integral models are in mathcal font, and their formal completions are in mathfrak font.

2. Unitary Shimura varieties

In this section, we introduce unitary Shimura varieties. In Section 2.1, we introduce PEL data and conventions for unitary groups, with which we work throughout the paper. Section 2.2 introduces the PEL moduli problem, and Section 2.3 specializes to the setting over $\mathbb{C}$.

2.1. **PEL data and unitary groups.** By a *PEL datum*, we mean a tuple $(K, c, L, \langle , \rangle, h)$ consisting of

- the CM field K equipped with the involution c introduced in Section 1.4,
- a $\mathbb{Z}$-lattice L with an action of $\mathcal{O}_K$,
- a non-degenerate Hermitian pairing $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}$ satisfying $\langle k \cdot v_1, v_2 \rangle = \langle v_1, k^c \cdot v_2 \rangle$ for all $v_1, v_2 \in L$ and $k \in \mathcal{O}_K$,
- an $\mathbb{R}$-algebra endomorphism
$$h : \mathbb{C} \to \mathrm{End}_{K \otimes_{\mathbb{Z}} \mathbb{R}}(L \otimes_{\mathbb{Z}} \mathbb{R})$$
such that $(v_1, v_2) \mapsto \langle v_1, h(i) \cdot v_2 \rangle$ is symmetric and positive definite and such that $\langle h(z) v_1, v_2 \rangle = \langle v_1, h(\overline{z}) v_2 \rangle$.

Furthermore, for considering the moduli problem over a p-adic ring and for defining p-adic modular forms, we require:

- $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is self-dual under the alternating Hermitian pairing $\langle \cdot, \cdot \rangle_p$ on $L \otimes_{\mathbb{Z}} \mathbb{Q}_p$.

We define the signature and the reflex field of the PEL datum as in [Lan13, Section 1.2.5]. More precisely, given a PEL datum $(K, c, L, \langle , \rangle, h)$, we define a vector space over K by
$$V := L \otimes_{\mathbb{Z}} \mathbb{R}.$$
Note that $h \times_{\mathbb{R}} \mathbb{C}$ gives rise to a decomposition $V \otimes_{\mathbb{R}} \mathbb{C} = V_1 \oplus V_2$ (where $h(z)$ acts by z on V_1 and by $\bar{z}$ on V_2). We have decompositions $V_1 = \oplus_{\tau \in \Sigma_K} V_{1,\tau}$ and $V_2 = \oplus_{\tau \in \Sigma_K} V_{2,\tau}$. As defined in [Lan13, Definition 1.2.5.2], the *signature* of $(V, \langle , \rangle, h)$ is the tuple of pairs $(a_{+\tau}, a_{-\tau})_{\tau \in \Sigma_K}$ such that $a_{+\tau} = \dim_{\mathbb{C}} V_{1,\tau}$ and $a_{-\tau} = \dim_{\mathbb{C}} V_{2,\tau}$ for all $\tau \in \Sigma_K$. Let
$$n = a_{+\tau} + a_{-\tau}.$$
Note that n is independent of τ and furthermore $a_{\pm\tau} = a_{\mp\tau^c}$. As defined in [Lan13, Definition 1.2.5.4], the *reflex field* of $(V, \langle , \rangle, h)$ is the field of definition of V_1 as an $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{C}$-module. Henceforth, we denote the reflex field by E. To the PEL datum $(K, c, L, \langle , \rangle, h)$, we associate algebraic groups $GU = GU(L, \langle , \rangle)$, $GU_+ = GU_+(L, \langle , \rangle)$, $U = U(L, \langle , \rangle)$, and $SU = SU(L, \langle , \rangle)$ defined over $\mathbb{Z}$, whose R-points (for any $\mathbb{Z}$-algebra R) are given by
$$\begin{aligned}
GU(R) &:= \{(g, \nu) \in \mathrm{End}_{\mathcal{O}_K \otimes_{\mathbb{Z}} R}(L \otimes_{\mathbb{Z}} R) \times R^\times \mid \langle g \cdot v_1, g \cdot v_2 \rangle = \nu \langle v_1, v_2 \rangle\} \\
GU_+(R) &:= \{(g, \nu) \in GU(R) \mid \nu > 0\} \\
U(R) &:= \{g \in \mathrm{End}_{\mathcal{O}_K \otimes_{\mathbb{Z}} R}(L \otimes_{\mathbb{Z}} R) \mid \langle g \cdot v_1, g \cdot v_2 \rangle = \langle v_1, v_2 \rangle\} \\
SU(R) &:= \{g \in U(R) \mid \det g = 1\}.
\end{aligned}$$

Note that ν is called a *similitude factor*. We denote by Levi the Levi subgroup of GU that preserves both L_1 and L_2, where $L = L_1 \oplus L_2$ is the decomposition of the lattice L coming from the decomposition $V = V_1 \oplus V_2$. Since we will only be working with one of the groups GU, GU_+, U, and SU at a time, no confusion should result from the fact that we will also denote the corresponding Levi subgroup of these other groups by Levi.

2.2. **PEL moduli problem.** The goal of this section is to introduce PEL-type unitary Shimura varieties from a moduli-theoretic perspective. We will restrict our attention to cases where these Shimura varieties have no level structure and good reduction at p.

We now define a moduli problem for abelian varieties equipped with extra structures (more precisely, polarizations, endomorphisms and level structure) and which will be representable by unitary Shimura varieties that have integral models. For each open compact subgroup $\mathcal{U} \subset GU(\mathbb{A}^\infty)$, consider the moduli problem

$$(S, s) \mapsto \{(A, i, \lambda, \alpha)\}$$

which assigns to every connected, locally noetherian scheme S over E together with a geometric point s of S the set of tuples (A, i, λ, α), where

- A is an abelian variety over S of dimension $g := [K : \mathbb{Q}] \cdot n$,
- $i : K \hookrightarrow (\mathrm{End}(A)) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an embedding of $\mathbb{Q}$-algebras
- $\lambda : A \to A^\vee$ (where $A^\vee$ denotes the dual abelian variety) is a polarization satisfying $\lambda \circ i(k^*) = i(k)^\vee \circ \lambda$ for all $k \in K$,
- α is a $\pi_1(S, s)$-invariant $\mathcal{U}$-orbit of $K \otimes_{\mathbb{Q}} \mathbb{A}^\infty$-equivariant isomorphisms

$$L \otimes_{\mathbb{Z}} \mathbb{A}^\infty \xrightarrow{\sim} V_f A,$$

  which takes the Hermitian pairing $\langle \cdot, \cdot \rangle$ on L to an $(\mathbb{A}^\infty)^\times$-multiple of the λ-Weil pairing on the rational Tate module $V_f A$.

Note that Lie A is a locally free $\mathcal{O}_S$-module of rank g and has an induced action of K via i. The tuple (A, i, λ, α) must satisfy the following *determinant condition*:

$$\det_E(k|V_1) = \det_{\mathcal{O}_S}(k|\text{Lie } A)$$

for all $k \in K$. Two tuples (A, i, λ, α) and $(A', i', \lambda', \alpha')$ are equivalent if there exists an isogeny $A \to A'$ taking i to i', λ to a rational multiple of λ' and α to α'. We note that the definition is independent of the choice of geometric point s of S. We can extend the definition to non-connected schemes by choosing a geometric point for each connected component.

If the compact open subgroup $\mathcal{U}$ is neat (in particular, if it is sufficiently small) as defined in [Lan13, Definition 1.4.1.8], then this moduli problem is representable by a smooth, quasi-projective scheme $M_{\mathcal{U}}/E$. It is an exercise left to the reader to identify for every embedding $E \hookrightarrow \mathbb{C}$ the complex points of $M_{\mathcal{U}} \times_E \mathbb{C}$ with the points of the locally symmetric complex variety corresponding to (GU, h). We merely sketch it in this paper; for more details, see [Kot92].

From now on, assume that $\mathcal{U} = \mathcal{U}^p \mathcal{U}_p$ is neat and that $\mathcal{U}_p \subset GU(\mathbb{Q}_p)$ is hyperspecial. We can construct an integral model of $M_{\mathcal{U}}$ by considering an integral version of the above moduli problem. To a pair (S, s), where S is now a scheme over $\mathcal{O}_E$, we assign the set of tuples $(A, i, \lambda, \alpha^p)$, where

- A is an abelian variety over S of dimension g,
- $i : \mathcal{O}_K \hookrightarrow (\mathrm{End}(A)) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is an embedding of $\mathbb{Z}_{(p)}$-algebras
- $\lambda : A \to A^\vee$ is a prime-to-p polarization satisfying $\lambda \circ i(k^*) = i(k)^\vee \circ \lambda$ for all $k \in \mathcal{O}_K$,
- α^p is a $\pi_1(S, s)$-invariant $\mathcal{U}^p$-orbit of $K \otimes_{\mathbb{Q}} \mathbb{A}_f^p$-equivariant isomorphisms

$$L \otimes_{\mathbb{Z}} \mathbb{A}_f^p \xrightarrow{\sim} V_f^p A,$$

which takes the Hermitian pairing $\langle\cdot,\cdot\rangle$ on L to an $(\mathbb{A}_f^p)^\times$-multiple of the λ-Weil pairing on V_f^pA (the Tate module away from p).

In addition, the tuple (A, i, λ, α) must satisfy the following *determinant condition*:

$$\det_E(k|V_1) = \det_{\mathcal{O}_S}(k|\mathrm{Lie}A)$$

for all $k \in \mathcal{O}_K$. Two tuples (A, i, λ, α) and $(A', i', \lambda', \alpha')$ are equivalent if there exists a prime-to-p isogeny $A \to A'$ taking i to i', λ to a prime-to-p rational multiple of λ' and α to α'.

This moduli problem is representable by a smooth, quasi-projective scheme $\mathcal{M}_{\mathcal{U}}$ over $\mathcal{O}_E \otimes \mathbb{Z}_{(p)}$. We have a canonical identification

$$M_{\mathcal{U}} = \mathcal{M}_{\mathcal{U}} \times_{\mathrm{Spec}\ (\mathcal{O}_E\otimes\mathbb{Z}_{(p)})} \mathrm{Spec}\ E.$$

As the level $\mathcal{U}^p$ varies, the inverse system $\mathcal{M}$ of Shimura varieties $\mathcal{M}_{\mathcal{U}}$ has a natural action of $GU(\mathbb{A}^{\infty,p})$. (More precisely, $g \in GU(\mathbb{A}^{\infty,p})$ acts by precomposing the level structure α with it.) Since our interest is in the p-adic theory, we will suppress the level $\mathcal{U}$ from the notation from now on, though it should be easily deduced from context.

2.3. **Abelian varieties and Shimura varieties over $\mathbb{C}$.** In this section, we specialize to working over $\mathbb{C}$.

2.3.1. *Abelian varieties over* $\mathbb{C}$. Recall that an abelian variety A over $\mathbb{C}$ of dimension g is a smooth projective variety over $\mathbb{C}$ with group structure and whose $\mathbb{C}$-points are of the form V/Λ, where Λ is a $\mathbb{Z}$-lattice in V. Furthermore, because V/Λ comes from a complex abelian variety A, it is polarizable, i.e. there exists a nondegenerate, positive definite Hermitian form

$$\lambda_{\mathbb{C}} : V \times V \to \mathbb{C} \quad \text{s.t. } \lambda_{\mathbb{C}}(\Lambda, \Lambda) \subset \mathbb{Z}.$$

We call each such Hermitian form $\lambda_{\mathbb{C}}$ a *polarization* of V/Λ. It may be better to think of a polarization as an alternating form $\lambda_{\mathbb{R}} : V \times V \to \mathbb{R}$ satisfying $\lambda(iu, iv) = \lambda(u, v)$ for all $u, v \in V$ and

$$\lambda_{\mathbb{C}}(u, v) = \lambda_{\mathbb{R}}(u, iv) + i\lambda_{\mathbb{R}}(u, v).$$

Because any abelian variety over $\mathbb{C}$ admits a polarization, it is enough to characterize a pair $(A, \lambda_{\mathbb{C}})$ by considering the following triple:

(1) Let $\Lambda = H_1(A, \mathbb{Z})$ be the free $\mathbb{Z}$-module of rank $2g$;
(2) Identify $A(\mathbb{C}) \cong \mathrm{Lie}\, A(\mathbb{C})/H_1(A(\mathbb{C}), \mathbb{Z})$ and consider the alternating form on $H_1(A, \mathbb{Z})$ induced by $\lambda_{\mathbb{C}}$ denoted by $\langle\cdot,\cdot\rangle$;
(3) Consider the $\mathbb{R}$-algebra homomorphism $\mathbb{C} \to \mathrm{End}_{\mathbb{R}}(\Lambda \otimes \mathbb{R}) = \mathrm{End}_{\mathbb{R}}(H_1(A(\mathbb{C}), \mathbb{R})) = \mathrm{End}_{\mathbb{R}}(\mathrm{Lie}\, A)$ describing the complex structure on $\mathrm{Lie}\, A$.

2.3.2. *Shimura varieties over* $\mathbb{C}$. For any real embedding $\tau \in \Sigma$, we let $\tilde{\tau}$ denote a choice of an extension to a complex embedding in Σ_K. Let $\mathfrak{h}_\tau$ denote the subset of elements $I \in \mathrm{End}_{K\otimes_{\mathbb{Z}}\mathbb{R}}(V)$ which satisfy

(1) $I^2 = -1$
(2) $I^c = -I$
(3) $(w, v) \mapsto \langle w, Iv\rangle$ is a positive definite symmetric form on V
(4) For every embedding $\tilde{\tau}$, $\dim_{\mathbb{C}}(V \otimes_{K,\tilde{\tau}} \mathbb{C})^{I=i} = a_{+\tau}$.

Letting $I = h(i)$, these properties define the $\mathbb{R}$-algebra homomorphism $h : \mathbb{C} \to \mathrm{End}_{K\otimes_{\mathbb{Z}}\mathbb{R}}(V)$ defining the complex structure on V. Note that any two choices of homomorphisms h, h' such that $h(i), h'(i) \in \mathfrak{h}_\tau$ are conjugate under $SU(\mathbb{R})$, or equivalently under $GU_+(\mathbb{R})$. Fixing

h, let C_τ denote its centralizer in $GU_+(\mathbb{R})$. If $\mathcal{U} \subset GU(\mathbb{A}^\infty)$ is a neat open compact subgroup, then define

$$M_{\mathcal{U},\tau}(\mathbb{C}) = GU(\mathbb{Q})\backslash(GU(\mathbb{A}^\infty)/\mathcal{U} \times GU_+(\mathbb{R})/C_\tau).$$

It is a moduli space for quadruples (A, i, λ, α) where

(1) A is an abelian variety over $\mathbb{C}$;
(2) $i : K \hookrightarrow \mathrm{End}^0(A)$ is a homomorphism such that $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$ for all $b \in K$ as well as the following: Recall that the vector space $V_\mathbb{C}$ can be decomposed as $V_1 \oplus V_2$ where $h(i)$ acts as i on V_1 and as $-i$ on V_2. Furthermore, the imaginary quadratic field F acts on V_i via $\tau : F \hookrightarrow \mathbb{C}$ or its conjugate embedding τ^c. Thus V_i decomposes into V_i^τ and $V_i^{\tau^c}$. Then V_1^τ is the subspace where $h(i)$ acts as i and F acts via τ, and $V_1^{\tau^c}$ is the subspace where $h(i)$ acts as $-i$ and F acts via τ^c. We require $\mathrm{Lie}\, A$ to be isomorphic to V_1 so that $\mathrm{Lie}\, A$ decomposes into $\mathrm{Lie}\, A^+$ and $\mathrm{Lie}\, A^-$ depending on the action of F.
(3) $\lambda : A \to A^\vee$ is a polarization
(4) $\overline{\alpha}$ is a $\mathcal{U}$-orbit of isomorphisms of skew-Hermitian K-vector spaces (in the sense of Kottwitz) $V \otimes_\mathbb{Q} \mathbb{A}^\infty \cong H_1(A, \mathbb{A}^\infty)$, i.e. the Tate module of A. Note that the alternating pairing on the Tate module is the λ-Weil pairing.

Every quadruple (A, i, λ, α) satisfying the above restrictions gives rise to an element $GU(\mathbb{A}^\infty)/\mathcal{U} \times \mathfrak{h}_\tau$ as follows. Note that the isomorphism α allows us to identify V with $H = H_1(A, \mathbb{Q})$ over any finite place of $\mathbb{Q}$. Over $\mathbb{R}$, we have that $H_1(A, \mathbb{Q})_\mathbb{R} \cong \mathrm{Lie}\, A$, and the complex structure on $\mathrm{Lie}\, A$ induces a complex structure on H, thus we can get a decomposition $H_1 \oplus H_2$. By the conditions on i, we have that $H_i \cong V_i$ as $K \otimes_\mathbb{R} \mathbb{C}$-modules, and thus $H_1(A, \mathbb{R}) \cong V_\mathbb{R}$ as well. Since the complex structure on $H_1(A, \mathbb{R})$ is obtained by conjugating h by some rigidification of $H_1(A, \mathbb{R})$ on $GU(\mathbb{R})$, we see that the possible complex structures on $H_1(A, \mathbb{R})$ is exactly the subset $\mathfrak{h}_\tau$.

By the Hasse principle, we can conclude that $H_1(A, \mathbb{Q})$ and V are isomorphic over $\mathbb{Q}$. Choose an isomorphism $\alpha_\mathbb{Q}$ and consider the automorphism $(\alpha_\mathbb{Q} \otimes \mathbb{A}^\infty) \circ \alpha$ of $V \otimes \mathbb{A}^\infty$. This defines an element of $GU(\mathbb{A}^\infty)$, but since α is only well-defined up to its orbit in $\mathcal{U}$, such an isomorphism determines an element of $GU(\mathbb{A}^\infty)/\mathcal{U}$.

Thus, the tuple (A, i, λ, α) along with a choice of isomorphism $\alpha_\mathbb{Q}$ determines an element of $GU(\mathbb{A}^\infty)/\mathcal{U} \times \mathfrak{h}_\tau$. Forgetting the isomorphism $\alpha_\mathbb{Q}$ is equivalent to taking the quotient by the left action of $GU_+(\mathbb{Q})$. Thus, the quadruple (A, i, λ, α) determines a point on $GU_+(\mathbb{Q})\backslash(GU(\mathbb{A}^\infty)/\mathcal{U} \times \mathfrak{h}_\tau = M_{\mathcal{U},\tau}(\mathbb{C})$. Note that the set of complex points of the Shimura variety $M_\mathcal{U}$ is the (disjoint) product over all $\tau \in \Sigma_K$ of $M_{\mathcal{U},\tau}(\mathbb{C})$.

3. Classical automorphic forms

In this section we will first recall the classical definition of automorphic forms on unitary groups over $\mathbb{C}$ following [Shi00], and then describe equivalent viewpoints that let us generalize to work over other base rings than $\mathbb{C}$.

3.1. Classical definition of complex automorphic forms on unitary groups.

Consider the domain $\mathcal{H}$ for $GU_+(\mathbb{R})$:

$$\mathcal{H} = \prod_{\tau\in\Sigma} \mathcal{H}_{a_{+\tau}\times a_{-\tau}} \text{ with } \mathcal{H}_{a_{+\tau}\times a_{-\tau}} = \{z \in \mathrm{Mat}_{a_{-\tau}\times a_{+\tau}}(\mathbb{C}) \,|\, 1 - {}^t z^c z \text{ is positive definite}\}.$$

If we write $g = \begin{pmatrix} a_{g,\tau} & b_{g,\tau} \\ c_{g,\tau} & d_{g,\tau} \end{pmatrix}_{\tau\in\Sigma} \in GU_+(\mathbb{R})$, where $a_{g,\tau} \in \mathrm{GL}_{a_{+\tau}}(\mathbb{R})$ and $d_{g,\tau} \in \mathrm{GL}_{a_{-\tau}}(\mathbb{R})$, then the action of g on $\mathcal{H}$ is given by

$$gz = ((a_{g,\tau} z_\tau + b_{g,\tau})(c_{g,\tau} z_\tau + d_{g,\tau})^{-1})_{\tau\in\Sigma} \text{ for } z_\tau \in \mathcal{H}_{a_{+\tau}\times a_{-\tau}}.$$

Because $GU_{+,\tau}(\mathbb{R})$ acts transitively on $\mathcal{H}_{a_{+\tau}\times a_{-\tau}}$ and there is a point with stabilizer isomorphic to C_τ, we can identify $\mathcal{H}$ with $\prod_{\tau\in\Sigma}\mathfrak{h}_\tau$ (see [Shi00][12.1]).

In order to define the desired transformation properties that automorphic forms should satisfy, we need to introduce a few more definitions. Using the above notation, for $g\in GU_+(\mathbb{R})$ and $z=(z_\tau)_{\tau\in\Sigma}\in\mathcal{H}$ the *factors of automorphy* for each $\{\tilde{\tau},\tilde{\tau}^c\}\subset\Sigma_K$ above $\tau\in\Sigma$ are defined by

$$\mu_{\tilde{\tau}}(g,z) := c_{g,\tau}z_\tau + d_{g,\tau} \text{ and } \mu_{\tilde{\tau}^c}(g,z) := \overline{b}_{g,\tau}{}^t z_\tau + \overline{a}_{g,\tau},$$

and the *scalar factors of automorphy* are

$$j_\tau(g,z) := \det(\mu_\tau(g,z)) \text{ for } \tau\in\Sigma_K.$$

Remark 1. By [Shi78, Equation (1.19)], for all $\tau\in\Sigma$ and $g\in GU_+(K_\tau^+)$,

$$\det(\mu_{\tilde{\tau}^c}(g,z)) = \det(g)^{-1}\,\nu(g)^{a_\tau}\det(\mu_{\tilde{\tau}}(g,z)).$$

Define

$$M_g(z) := (\mu_{\tilde{\tau}^c}(g,z),\mu_{\tilde{\tau}}(g,z))_{\tau\in\Sigma} \quad \in \prod_{\tau\in\Sigma}\mathrm{Mat}_{a_{+\tilde{\tau}}\times a_{+\tilde{\tau}}}\times\mathrm{Mat}_{a_{-\tilde{\tau}}\times a_{-\tilde{\tau}}}$$

If $\rho:\mathrm{Levi}(\mathbb{C})\left(\cong\prod_{\tau\in\Sigma_K}\mathrm{GL}_{a_{+\tau}}(\mathbb{C})\times\mathrm{GL}_{a_{-\tau}}(\mathbb{C})\right)\to\mathrm{GL}(X)$ is a rational representation into a finite-dimensional complex vector space X, $f:\mathcal{H}\to X$ a map and $g\in GU_+(\mathbb{R})$, then denote by $f\|_\rho g:\mathcal{H}\to X$ and $f|_\rho g:\mathcal{H}\to X$ the maps given by

$$(f\|_\rho g)(z) := \rho(M_g(z))^{-1}f(gz)$$
$$f|_\rho g := f\|_\rho\left(\nu(g)^{-1/2}g\right).$$

for all $z\in\mathcal{H}$. Note that for all $g\in U(\mathbb{C})$,

$$f|_\rho g = f\|_\rho g.$$

Associated to an $\mathcal{O}_F$-lattice L_F with an action of $GU_+(\mathbb{Q})$ and an integral $\mathcal{O}_F$-ideal $\mathfrak{c}$, we define the subgroup

$$\Gamma(L_F,\mathfrak{c}) := \{g\in GU_+(\mathbb{Q})\,|\,{}^tL_F g = {}^tL_F \text{ and } {}^tL_F(1-g)\subset\mathfrak{c}\,{}^tL_F\}.$$

Then a *congruence subgroup* Γ of $GU_+(\mathbb{Q})$ is a subgroup of $GU_+(\mathbb{Q})$ that contains $\Gamma(L_F,\mathfrak{c})$ as subgroup of finite index for some choice of $(L_F,\mathfrak{c})$ as above.

Definition 1. Let Γ be a congruence subgroup of $GU_+(\mathbb{Q})$, X a finite-dimensional complex vector space and $\rho:\mathrm{Levi}(\mathbb{C})\to\mathrm{GL}(X)$ a rational representation. A function $f:\mathcal{H}\to X$ is called a *(holomorphic) automorphic form* of weight ρ with respect to Γ if it satisfies the following properties

(1) f is holomorphic,
(2) $f\|_\rho\gamma = f$ for every γ in Γ,
(3) if (a_+,a_-)=(1,1), then f is holomorphic at every cusp.

We will call a function $f:\mathcal{H}\to X$ that satisfies property (2), but not necessarily (1) and (3) an *automorphic function*.

Remark 2. Note that if $(a_+,a_-)\neq(1,1)$, Koecher's principle, see [Lan14, Thm. 2.5] for a very general version, implies that an automorphic form is automatically holomorphic at the boundary.

Remark 3. Sometimes, in the definition of an automorphic form, the second condition of Definition 1 is replaced by $f|_\rho\gamma = f$ for every γ in Γ. The condition that arises from geometry, though, is $f||_\rho\gamma = f$. Since our main results and proofs are geometric (and since we want this definition of automorphic forms to agree with the geometric definitions we give later), we require $f||_\rho\gamma = f$ instead of $f|_\rho\gamma = f$ in this paper.

3.1.1. *Weights of an automorphic form.* The irreducible algebraic representations of Levi $\cong \prod_{\tau\in\Sigma_K} \mathrm{GL}_{a_{+\tau}} \times \mathrm{GL}_{a_{-\tau}}$ over $\mathbb{C}$ are in one to one correspondence with dominant weights of a maximal torus of T (over $\mathbb{C}$). More precisely, let T be the product of the diagonal tori $T_{a_{+\tau}} \times T_{a_{-\tau}}$ for $\tau \in \Sigma_K$. For $1 \leq i \leq a_{+\tau} + a_{-\tau}$, let ε_i^τ in $X(T) := \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{G}_m)$ be the character defined by

$$T(\mathbb{C}) = \prod_{\tau\in\Sigma_K} T_{a_{+\tau}}(\mathbb{C}) \times T_{a_{-\tau}}(\mathbb{C}) \ni \mathrm{diag}(x_1^\tau, \cdots, x_{a_{+\tau}+a_{-\tau}}^\tau)_{\tau\in\Sigma_K} \mapsto x_i^\tau \in \mathbb{G}_m(\mathbb{C}).$$

These characters form a basis of $X(T)$, and we choose $\Delta = \{\alpha_i^\tau := \varepsilon_i^\tau - \varepsilon_{i+1}^\tau\}_{\tau\in\Sigma_K, 1\leq i<a_{+\tau}+a_{-\tau}}$ as a basis for the root system of Levi. Then the *dominant weights* of T with respect to Δ are $X(T)_+ = \{\kappa \in X(T) \mid \langle\kappa, \check{\alpha}\rangle \geq 0 \, \forall \alpha \in \Delta\}$, and they can be identified as follows:

$$X(T)_+ \cong \{(n_1^\tau, \cdots, n_{a_{+\tau}+a_{-\tau}}^\tau)_{\tau\in\Sigma_K} \in \prod_{\tau\in\Sigma_K} \mathbb{Z}^{a_{+\tau}+a_{-\tau}} : n_i^\tau \leq n_{i+1}^\tau \forall i\}.$$

For such a dominant weight κ, let $\rho_\kappa : \mathrm{Levi} \to M_\rho$ denote the algebraic representation of highest weight κ. We call f an automorphic form of weight κ, where $\kappa \in X(T)_+$, if f is an automorphic form of weight ρ_κ.

3.1.2. *Unitary domains and moduli problems.* We will now reformulate Definition 1 in order to generalize it in Section 3.3. We quickly recall the correspondence between quadruples described in Section 2.2 and points of $\mathcal{H}$.

Fix a PEL datum $(K, c, L, \langle,\rangle, h)$ and a neat open compact subgroup $\mathcal{U} \subset GU(\mathbb{A}^\infty)$; let $M_{\mathcal{U}}(\mathbb{C})$ denote the complex Shimura variety of level $\mathcal{U}$ associated to the PEL datum. Let $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ and let $\Gamma_{\mathcal{U}} = \mathcal{U} \cap GU(\mathbb{Q})$. We now describe the identification between elements $z \in \Gamma\backslash\mathcal{H}$ and abelian varieties $\underline{A} \in M_{\mathcal{U}}(\mathbb{C})$ given by [Shi00]. Shimura defines for $z \in \mathcal{H}$ an $\mathbb{R}$-linear isomorphism $p_z : V \to \mathbb{C}^g$ which induces a Riemann form on $\mathbb{C}^g$.

$$E_z(p_z(x), p_z(y)) = \langle x, y\rangle \quad x, y \in V.$$

This implies that $A_z = \mathbb{C}^g/p_z(L)$ is an abelian variety together with a polarization λ_z corresponding to E_z. For $k \in K$, define $i_z(k)$ to be the element of $\mathrm{End}_{\mathbb{Q}}(A_z)$ induced by the action of $h(k)$ and let α_z denote the $\mathcal{U}$-orbit of isomorphisms $V \otimes \mathbb{A}^\infty \cong H_1(A_z, \mathbb{A}^\infty)$ induced by p_z. Altogether, p_z gives rise to the Riemann form E_z, the following commutative diagram:

(1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & V & \longrightarrow & V/L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}^g & \longrightarrow & A & \longrightarrow & 0, \end{array}$$

and $\underline{A}_z := (A_z, i_z, \lambda_z, \alpha_z)$. Shimura [Shi00, Theorem 4.8] (together with [Mil04, Proposition 4.1]) further proves that

Theorem 3.1. *For each $z \in \mathcal{H}$, $\underline{A}_z \in M_{\mathcal{U}}(\mathbb{C})$ for some neat open $\mathcal{U}$. Conversely, if $\underline{A} \in M_{\mathcal{U}}(\mathbb{C})$ for some $\mathcal{U}$, then there is a $z \in \mathcal{H}$ such that $\underline{A} \cong \underline{A}_z$. Furthermore $\underline{A}_z$ and $\underline{A}_w$ for $z, w \in \mathcal{H}$ are isomorphic if and only if $w = \gamma z$ for some $\gamma \in \mathcal{U} \cap GU_+(\mathbb{Q})$.*

3.1.3. *Second definition of classical automorphic functions.* For $\underline{A} \in M_{\mathcal{U}}(\mathbb{C})$, let $\underline{\Omega} = H^1(A,\mathbb{Z})\otimes \mathbb{C}$ which has a decomposition into $\underline{\Omega}^{\pm}$ depending on the (induced) action of $h(i)$ on $\underline{\Omega}$. Set

$$\mathcal{E}^{\pm}_{\underline{A}_\tau} = \mathrm{Isom}_{\mathbb{C}}(\mathbb{C}^{a_{\pm\tau}}, \underline{\Omega}^{\pm}_{\tau}) \quad \text{and} \quad \mathcal{E}_{\underline{A}} = \bigoplus_{\tau\in\Sigma_K} (\mathcal{E}^{+}_{\underline{A}_\tau} \oplus \mathcal{E}^{-}_{\underline{A}_\tau}).$$

The group $\mathrm{GL}_{a_{\pm\tau}}(\mathbb{C})$ acts on $\mathcal{E}^{\pm}_{\underline{A}_\tau}$ by

$$(g.f)(x) = f({}^t g x) \text{ for } g \in \mathrm{GL}_{a_{\pm\tau}}(\mathbb{C}), f \in \mathcal{E}^{\pm}_{\underline{A}_\tau}, x \in \mathbb{C}^{a_{\pm\tau}},$$

which induces a (diagonal) action of $\mathrm{Levi}(\mathbb{C})$ on $\mathcal{E}_{\underline{A}}$.

For $z \in \mathcal{H}$, the map p_z induces a choice of basis on $H_1(A_z,\mathbb{Z})$ via the identification with $p_z(L)$, which by duality induces a choice of basis $\mathcal{B}_z$ of $\underline{\Omega}$. This is equivalent to giving an element l_z of $\mathcal{E}_{\underline{A}_z}$.

Lemma 3.2. *Let $\rho : \mathrm{Levi}(\mathbb{C}) \to \mathrm{GL}(X)$ be a rational representation. Then there exists a one-to-one correspondence between automorphic functions of weight ρ with respect to $\Gamma_{\mathcal{U}}$ and the set of functions F from pairs $(\underline{A}, l)$, where $\underline{A} \in M_{\mathcal{U}}(\mathbb{C})$ and $l \in \mathcal{E}_{\underline{A}}$, to X satisfying*

$$F(\underline{A}, gl) = \rho({}^t g)^{-1} F(\underline{A}, l) \text{ for all } g \in \mathrm{Levi}(\mathbb{C}). \tag{2}$$

The bijection is given by sending a function F to the automorphic function $f_F : z \mapsto F(\underline{A}_z, l_z)$.

Proof. (sketch) We will first show that the map $F \mapsto (f_F : z \mapsto F(\underline{A}_z, l_z))$ is well defined and afterwards construct an inverse for it. To do the former, let $z \in \mathcal{H}$, $\gamma \in \Gamma_{\mathcal{U}}$. We need to show that $F(\underline{A}_z, l_z) = \rho(M_\gamma(z))^{-1} F(\underline{A}_{\gamma z}, l_{\gamma z})$. It follows from [Shi00, page 27] that ${}^t M_\gamma(z)$ maps $p_{\gamma z}(L)$ to $p_z(L)$ and defines an isomorphism between $\underline{A}_{\gamma z}$ and $\underline{A}_z$. Note that under this isomorphism, $l_{\gamma z}$ of $\underline{\Omega}$ gets mapped to ${}^t M_\gamma(z)^{-1} l_z$. Hence using property (2), we obtain

$$F(\underline{A}_{\gamma z}, l_{\gamma z}) = F(\underline{A}_z, {}^t M_\gamma(z)^{-1} l_z) = \rho(M_\gamma(z)) F(\underline{A}_z, l_z)$$

as desired.

We define the inverse of $F \mapsto f_F$ as follows: Let f be an automorphic function of weight ρ with respect to $\Gamma_{\mathcal{U}}$, and $(\underline{A}, l)$ where $\underline{A} \in M_{\mathcal{U}}(\mathbb{C})$ and $l \in \mathcal{E}_{\underline{A}}$. Then, by Theorem 3.1, there exists $z \in \mathcal{H}$ such that $\underline{A}_z$ is isomorphic to $\underline{A}$, and there exists a unique $g \in \mathrm{Levi}(\mathbb{C})$ such that $l = g l_z$. We define $F_f(\underline{A}, l) = \rho({}^t g)^{-1} f(z)$. By the transformation property of automorphic function we obtain analogously to above that $f \mapsto F_f$ is well-defined and F_f satisfies (2). Moreover, $f \mapsto F_f$ is obviously an inverse of $F \mapsto f_F$. □

Automorphic functions of weight ρ with respect to congruence subgroups Γ that strictly contains $\Gamma_{\mathcal{U}}$ can be characterized in the same style as follows:

Lemma 3.3. *Let $\rho : \mathrm{Levi}(\mathbb{C}) \to \mathrm{GL}(X)$ be a rational representation, a Shimura variety $M_{\mathcal{U}}(\mathbb{C})$ where $\mathcal{U}$ is a neat open compact subgroup, and Γ a congruence subgroup of $GU_+(\mathbb{Q})$ containing $\Gamma_{\mathcal{U}} = \mathcal{U} \cap GU_+(\mathbb{Q})$. Then there exists a one to one correspondence between automorphic functions of weight ρ with respect to Γ and the set of functions F from pairs $(\underline{A}, l)$, where $\underline{A} = (A, i, \lambda, \alpha) \in M_{\mathcal{U}}(\mathbb{C})$ and $l \in \mathcal{E}_{\underline{A}}$, to X satisfying*

$$F(\underline{A}, gl) = \rho({}^t g)^{-1} F(\underline{A}, l) \text{ for all } g \in \mathrm{Levi}(\mathbb{C}),$$

and such that for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$, we have

$$F(\underline{A}_z, l_z) = \rho(M_\gamma(z))^{-1} F(\underline{A}_{\gamma z}, l_{\gamma z}). \tag{3}$$

It suffices to check condition (3) for a set of representatives of Γ/Γ'.

This lemma follows easily from Lemma 3.2 and Definition 1.

3.2. **Algebraic definition of classical automorphic functions.** Finally, we would like to view automorphic forms as functions on abelian varieties on $M_{\mathcal{U}}(\mathbb{C})$. For this, we define the *contracted product* $\mathcal{E}_{\underline{A},\rho} = \mathcal{E}_{\underline{A}} \times^{\rho({}^t\cdot)^{-1}} X$ of $\mathcal{E}_{\underline{A}}$ and X to be the product $\mathcal{E}_{\underline{A}} \times X$ modulo the equivalence relation given by $(l,v) \sim \left(gl, \rho({}^tg)^{-1}v\right)$ for $g \in \mathrm{Levi}(\mathbb{C})$. Note that for $g \in GU_+(\mathbb{Q})$, the identification $H_1(A_z,\mathbb{Z}) \otimes \mathbb{C} = \prod_{\tau\in\Sigma_K} \mathbb{C}^n = H_1(A_{gz},\mathbb{Z}) \otimes \mathbb{C}$ induces an identification $\iota_g : \mathcal{E}_{\underline{A}_z} \to \mathcal{E}_{\underline{A}_{gz}}$, and we can define the isomorphism $i_g : \mathcal{E}_{\underline{A}_z,\rho} \to \mathcal{E}_{\underline{A}_{gz},\rho}$ by $(l,v) \to (\iota_g(l), \rho(M_g(z))v)$. Note that i_g is the identity for $g \in \Gamma_{\mathcal{U}}$. It is an easy exercise to see that Lemma 3.3 can be reformulated as follows.

Lemma 3.4. *Let $\rho : \mathrm{Levi}(\mathbb{C}) \to \mathrm{GL}(X)$ be a rational representation, $M_{\mathcal{U}}(\mathbb{C})$ and Γ a congruence subgroup of $GU_+(\mathbb{Q})$ containing $\Gamma_{\mathcal{U}}$. Then there exists a one-to-one correspondence between automorphic functions of weight ρ and level Γ and the set of functions $\widetilde{F}$ from $\underline{A} \in M_{\mathcal{U}}(\mathbb{C})$ to $\mathcal{E}_{\underline{A},\rho}$ satisfying*

$$i_\gamma\left(\widetilde{F}(\underline{A}_z)\right) = \widetilde{F}(\underline{A}_{\gamma z}) \text{ for all } z \in \mathcal{H}, \gamma \in \Gamma. \tag{4}$$

Remark 4. Note that by Theorem 3.1 giving a function $\widetilde{F}$ from $M_{\mathcal{U}}(\mathbb{C})$ to $\mathcal{E}_{\underline{A},\rho}$ satisfying (4) is the same as giving a global section of the vector bundle $\mathcal{E}_{\underline{A}_z,\rho}$ over $M_{\mathcal{U}}(\mathbb{C})$.

3.3. **(Classical) algebraic automorphic forms.** In Section 3.1, we considered automorphic forms over $\mathbb{C}$. Building on the discussion from Section 3.1, we now consider automorphic forms over other base rings. The approach in this section is similar to the approach in [Eis12, Section 2.5] and [Kat78, Section 1.2]. Note that [Eis12] only considers the case in which the signature at each $\tau \in \Sigma_K$ is $(a_{+\tau}, a_{-\tau})_{\tau\in\Sigma_K}$ with $a_{+\tau} = a_{-\tau}$, but the definitions from [Eis12, Section 2.5] carry over to the general case with only trivial modifications.

For any neat open compact subgroup $\mathcal{U}$, consider the integral model $\mathcal{M}_{\mathcal{U}}/\mathcal{O}_E$ introduced in Section 2.3.2. For any scheme S over $\mathrm{Spec}(\mathcal{O}_E)$, we put

$$\mathcal{M}_{\mathcal{U},S} := \mathcal{M}_{\mathcal{U}} \times_{\mathcal{O}_E} S.$$

When $S = \mathrm{Spec}(T)$ for a ring T, we will often write $\mathcal{M}_T$ instead of $\mathcal{M}_{\mathrm{Spec}(T)}$.

For any S-point $\underline{A} = (A, i, \lambda, \alpha^p)$ of $\mathcal{M}_{\mathcal{U}}$, let $\underline{\Omega}_{A/S}$ denote the locally free $\mathcal{O}_S$-module defined as the pullback of the relative differentials. We have a decomposition $\underline{\Omega}_{A/S} = \oplus_{\tau\in\Sigma_K}(\underline{\Omega}^+_{A/S,\tau} \oplus \underline{\Omega}^-_{A/S,\tau})$ where $\underline{\Omega}^\pm_{A/S,\tau}$ is rank $a_{+\tau}$ and $a_{-\tau}$, respectively. Note that in matrix form, $\mathrm{Levi} = \prod_{\tau\in\Sigma_K} \mathrm{GL}_{a_{+\tau}} \times \mathrm{GL}_{a_{-\tau}}$. Let $\mathrm{Levi}^\pm = \prod_{\tau\in\Sigma_K} \mathrm{GL}_{a_{\pm,\tau}}$

$$\mathcal{E}^\pm_{\underline{A}/S} := \bigoplus_{\tau\in\Sigma_K} \mathrm{Isom}_{\mathcal{O}_S}\left(\mathcal{O}_S^{a_{\pm\tau}}, \underline{\Omega}^\pm_{A/S,\tau}\right) \qquad \text{and} \qquad \mathcal{E}_{\underline{A}/S} := \mathcal{E}^+_{\underline{A}/S} \oplus \mathcal{E}^-_{\underline{A}/S},$$

respectively. Let R be a $\mathcal{O}_E$-algebra, and consider an algebraic representation ρ of L (over R) into a finite locally free R-module M_ρ. For any such ρ, define a sheaf $\mathcal{E}_{\underline{A}/S,\rho}$

$$\begin{aligned}\mathcal{E}_{(\underline{A}/S,\rho)} &= \mathcal{E}_{\underline{A}/S} \times^\rho M \\ &:= \left(\mathcal{E}_{\underline{A}/S} \times M\right) / (\ell, m) \sim \left(g\ell, \rho({}^tg^{-1})m\right).\end{aligned}$$

(Define the contracted produce locally and then glue.)

Definition 2 (First Equivalent Definition of Algebraic Automorphic Forms)**.** An automorphic form of weight ρ level $\mathcal{U}$ defined over R is a function f

$$(\underline{A}, \ell) \mapsto f(\underline{A}, \ell) \in \mathcal{M}_{\mathcal{U}}(R')$$

defined for all R-algebras R', R'-points $\underline{A} \in M_{\mathcal{U}}$, and global sections $\ell \in \mathcal{E}_{\underline{A}/R'}$, such that all of the following hold:

(1) $f(\underline{A}, \ell)$ depends only on the R'-isomorphism class of $(\underline{A}, \ell)$;

(2) $f(\underline{A}, \alpha\ell) = \rho\left(\left({}^t\alpha\right)^{-1}\right) f(\underline{A}, \ell)$ for all $\alpha \in \mathrm{Levi}(R')$ and all $\ell \in \mathcal{E}_{\underline{A}/R'}(R')$;

(3) The formation of $f(\underline{A}, \ell)$ commutes with extension of scalars $R_1 \to R_2$ for any R-algebras R_1 and R_2. More precisely, if $R_1 \to R_2$ is a ring homomorphism of R-algebras, then

$$f(\underline{A} \times_{R_1} R_2, \ell \otimes_{R_1} R_2) = f(\underline{A}, \ell) \otimes_{R_1} 1_{R_2} \in \mathcal{M}_{\mathcal{U}}(R_2).$$

Below in Definition 3, we give a definition of automorphic forms that is equivalent to Definition 2.

Definition 3 (Second Equivalent Definition of Algebraic Automorphic Forms)**.** An automorphic form of weight ρ defined over an $\mathcal{O}_E$-algebra R is a function $\tilde{f}$

$$\underline{A} \mapsto \tilde{f}(\underline{A}) \in \mathcal{E}_{\underline{A}/R}(R')$$

defined for all R-algebras R' and $\underline{A} \in \mathcal{M}_{\mathcal{U},R'}$ such that all of the following hold:

(1) $\tilde{f}(\underline{A})$ depends only on the R'-isomorphism class of $\underline{A}$;

(2) The formation of $\tilde{f}(\underline{A})$ commutes with extension of scalars $R_1 \to R_2$ for any R-algebras R_1 and R_2. More precisely, if $R_1 \to R_2$ is a ring homomorphism of R-algebras, then

$$\tilde{f}(\underline{A} \times_{R_1} R_2) = \tilde{f}(\underline{A}) \otimes_{R_1} 1_{R_2}.$$

Remark 5. The equivalence between Definition 2 and Definition 3 is given by

$$\tilde{f}(\underline{A}) = (\ell, f(\underline{A}, \ell)).$$

for all abelian varieties $\underline{A}/R$ (corresponding to $R \to M$) and global sections $\ell \in \mathcal{E}_{\underline{A}/R}$.

Let $\mathcal{A} = (A, i, \lambda, \alpha^p)^{\mathrm{univ}}$ denote the universal abelian variety over $\mathcal{M}_{\mathcal{U},S}$ equipped with $e : \mathcal{A} \to \mathcal{M}_{\mathcal{U},S}$. Denote the pullback via the identity section of the sheaf of differentials $\underline{\Omega}_{\mathcal{A}/\mathcal{M}}$ and define $\mathcal{E}_{\mathcal{U}} = \mathrm{Isom}_{\mathcal{O}_{\mathcal{M}_{\mathcal{U}}}}(\mathcal{O}_{\mathcal{M}_{\mathcal{U}}}, \underline{\Omega}_{\mathcal{A}/\mathcal{M}})$. For any algebraic representation ρ of Levi over a locally free finite R-module M_ρ, define $\mathcal{E}_{\mathcal{U},\rho} := \mathcal{E} \times^\rho M_\rho$.

Definition 4 (Third Equivalent Definition of Algebraic Automorphic Forms)**.** An automorphic form of weight ρ defined over R is a global section of the sheaf $\mathcal{E}_{\mathcal{U},\rho}$ on $\mathcal{M}_{\mathcal{U},R}$.

Remark 6. When we are working with a representation ρ which are uniquely determined by its highest weight κ, we shall sometimes write $\mathcal{E}_{\mathcal{U},\kappa}$ in place of $\mathcal{E}_{\mathcal{U},\rho}$.

3.3.1. *Comparison with classical definition of complex automorphic forms.* Having defined algebraic automorphic forms over general base rings, we will show that in the special case of the base ring being $\mathbb{C}$ the definition coincides with the classical definition of complex automorphic forms given in Section 3.1.

For an integer N, we define $\mathcal{U}_N$ to be a compact open subgroup of $GU(\mathbb{A}^\infty)$ such that

$$GU_+(\mathbb{Q}) \cap \mathcal{U}_N = \Gamma(N) := \{(g, \nu) \in GU_+(\mathbb{Q}) : g \equiv 1 \bmod N\}.$$

Proposition 3.5. *Let N be a large enough integer so that $\mathcal{U}_N$ is neat, and let ρ be an algebraic representation of the Levi* Levi *over $\mathbb{C}$. Then there is bijection between the (algebraic) automorphic forms of weight ρ defined in Definition 4 as global sections of $\mathcal{E}_\rho$ on $M_{\mathcal{U}_N}(\mathbb{C})$ and the (holomorphic) automorphic forms of weight ρ with respect to $\Gamma(N)$ as defined in Definition 1 in Section 3.1.*

Proof. (sketch) By Theorem 3.1, $M_{\mathcal{U}_N}(\mathbb{C}) \simeq \Gamma\backslash\mathcal{H}$ as complex manifolds. By GAGA and Lemma 3.4 together with Remark 4, we conclude that the global sections of $\mathcal{E}_\rho = \mathcal{E}_{\mathcal{U},\rho}$ on $M_{\mathcal{U}_N}(\mathbb{C})$ are in one-to-one correspondence with holomorphic automorphic forms of weight ρ with respect to $\Gamma(N)$. □

Corollary 3.6. *Let ρ be an algebraic representation of the Levi* Levi *over $\mathbb{C}$. Then there is bijection between the (algebraic) automorphic forms of weight ρ defined in Definition 4 as global section of $\mathcal{E}_\rho$ on $M_{\mathbb{C}}$ and the (holomorphic) automorphic forms of weight ρ as defined in Definition 1 in Section 3.1 (of arbitrary level).*

This corollary follows from Proposition 3.5 by observing that every congruence subgroup Γ' contains a congruence subgroup $\Gamma(N)$ for some large enough integer N, and every compact open subgroup $\mathcal{U}$ of $GU(\mathbb{A}^\infty)$ contains some compact open subgroup $\mathcal{U}_N$.

4. p-adic theory

Section 4.1 introduces the Igusa tower, a tower of finite étale Galois coverings of the ordinary locus of $\mathcal{M}$. Section 4.2 introduces p-adic automorphic forms, which arise as sections of a line bundle over the Igusa tower.

4.1. The Igusa tower over the ordinary locus.

In this section, we will construct a tower of finite étale Galois coverings of the ordinary locus of the integral model $\mathcal{M}$. In order to guarantee that the ordinary locus over the special fiber of $\mathcal{M}$ is nonempty, we make the following assumption: the prime p splits completely in the reflex field E. In this case, the ordinary locus is open and dense in the special fiber of $\mathcal{M}$. Choose a place P of E above p and let E_P be the corresponding completion of E, with ring of integers $\mathcal{O}_{E_P}$ and residue field k. By abuse of notation, we will still denote the base change of $\mathcal{M}$ to $\mathcal{O}_{E_P}$ by $\mathcal{M}$. Let S be a scheme of characteristic p.

Definition 4.1.1. *An abelian variety A/S of dimension g is ordinary if for all geometric points s of S, the fiber $A[p](s)$ has p^g elements.*

For every abelian variety A/S, the Hasse invariant $\mathrm{Ha}(A/S)$ is a section of $\omega_{A/S}^{\otimes(p-1)}$, where $\omega_{A/S}$ is the top exterior power of the pushforward to S of the sheaf of invariant differentials on A. It is easy to show that an abelian variety A is ordinary if and only if $\mathrm{Ha}(A/S)$ is invertible. (We now sketch an argument for this, as in [Sch13, Lemma III.2.5]: the Hasse invariant, which corresponds to pullback along the Verschiebung isogeny, is invertible if and only if Verschiebung is an isomorphism on tangent spaces, which happens if and only if Verschiebung is finite etale. A degree computation shows that this is equivalent to the condition that A be ordinary.) We define the ordinary locus

$$\overline{\mathcal{M}}^{\mathrm{ord}} \subset \overline{\mathcal{M}} := \mathcal{M} \times_{\mathcal{O}_{E_P}} k$$

to be the complement of the zero set of the Hasse invariant. Since p splits completely in E, the nonemptiness of $\overline{\mathcal{M}}^{\mathrm{ord}}$ follows from [Wed99]. We also define the ordinary locus $\mathcal{M}^{\mathrm{ord}}$ over $\mathcal{O}_{E_P}$ to be the complement of the zero set of a lift of the Hasse invariant. In addition, we define $\mathcal{S}^{\mathrm{ord}}$ over the ring of Witt vectors $\mathbb{W}$ to be an irreducible component of the complement of the zero set of a lift of the Hasse invariant, i.e. $\mathcal{S}^{\mathrm{ord}}$ is one connected component of $\mathcal{M}^{\mathrm{ord}} \times_{\mathcal{O}_{E_P}} \mathbb{W}$. In other words, $\mathcal{S}^{\mathrm{ord}}$ is the ordinary locus of one connected component $\mathcal{S}$ of $\mathcal{M}$.

Remark 7. We note that we could define the ordinary locus in an alternate way, using the stratification of $\overline{\mathcal{M}}$ in terms of the isogeny class of the p-divisible group $\mathcal{G} := \mathcal{A}_{\overline{\mathcal{M}}}[p^\infty]$, where $\mathcal{A}_{\overline{\mathcal{M}}}$ denotes the universal abelian variety over $\overline{\mathcal{M}}$. The isogeny class of the p-divisible group $\mathcal{G}$ (equipped with all its extra structures) defines a stratification of $\overline{\mathcal{M}}$ with locally closed strata and the ordinary locus, corresponding to the constant isogeny class $(\mu_{p^\infty})^g \times (\mathbb{Q}_p/\mathbb{Z}_p)^g$, is the unique open stratum.

Let $\mathcal{A} := \mathcal{A}_{\mathcal{M}^{\text{ord}}}$ be the universal ordinary abelian variety over $\mathcal{M}^{\text{ord}}$. Pick an $\mathcal{O}_{E_p}$-valued point x of $\mathcal{M}^{\text{ord}}$, with closed geometric point $\bar{x}$, an $\overline{\mathbb{F}}_p$-valued point of $\mathcal{M}^{\text{ord}}$. We can identify L_p (defined as in Section 2.1) with the p-adic Tate module of the p-divisible group $\mathcal{G}_x$. Choose such an identification $L_p \simeq T_p\mathcal{A}_x[p^\infty]$, compatible with the $\mathcal{O}_K$-action and with the Hermitian pairings on both factors. The kernel of the reduction map

$$T_p\mathcal{A}_x[p^\infty] \to T_p\mathcal{A}_{\bar{x}}[p^\infty]^{\acute{e}t}$$

determines an $\mathcal{O}_K$-submodule $\mathcal{L} \subset L_p$ as the kernel. Recall that L_p is self-dual under the Hermitian pairing $\langle\cdot,\cdot\rangle$. Define $\mathcal{L}^\vee := L_p/\mathcal{L}$.

The choice of $\bar{x}$ determines a connected component of $\mathcal{M}^{\text{ord}}$, namely the connected component containing $\bar{x}$. The choice of $\bar{x}$ also determines the lattice $\mathcal{L}^\vee$. The different connected components can be labeled by different lattices $\mathcal{L}^\vee$ (corresponding to different choices of $\bar{x}$).

Remark 8. By considering the primes in K^+ above p individually, we can write down an explicit formula for $\mathcal{L}^\vee$ (denoted $\mathcal{L}$ in [Hid11]), using the fact that each such prime splits from K^+ to K. The exact formula for $\mathcal{L}$ as an $\mathcal{O}_K$-module will depend on the set of signatures of the unitary similitude group $GU(\mathbb{R})$. More precisely, for each embedding $\tau : K \hookrightarrow \mathbb{C}$, let $(a_{+\tau}, a_{-\tau})$ be the signature of GU at the infinite place τ. Choose an isomorphism $\iota_p : \mathbb{C} \tilde{\to} \bar{\mathbb{Q}}_p$. By composing with ι_p, each τ determines a place of K above p. Let $p = \prod_{i=1}^r \mathfrak{p}_i$ be the decomposition of p into prime ideals of K^+. Each $\mathfrak{p}_i$ splits in K as $\mathfrak{p}_i = \mathfrak{P}_i\mathfrak{P}_i^c$, where $\mathfrak{P}_i$ lies above the prime w of F. The i-term of L_p (obtained from the decomposition $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p = \oplus_{i=1}^r(\mathcal{O}_{K_{\mathfrak{P}_i}} \oplus \mathcal{O}_{K_{\mathfrak{P}_i^c}}))$ can be identified with

$$\mathcal{O}^n_{K_{\mathfrak{P}_i}} \oplus \mathcal{O}^n_{K_{\mathfrak{P}_i^c}}.$$

Then the determinant condition implies that

$$\mathcal{L} \simeq \oplus_{i=1}^r(\mathcal{O}^{a_{+\tau_i}}_{K_{\mathfrak{P}_i}} \oplus \mathcal{O}^{a_{-\tau_i}}_{K_{\mathfrak{P}_i^c}}),$$

where τ_i is a place inducing $\mathfrak{P}_i$. Note that there is a natural decomposition $\mathcal{L} = \mathcal{L}^+ \oplus \mathcal{L}^-$, coming from the splitting $p = w \cdot w^c$. Also note that $\text{Levi}(\mathbb{Z}_p) \cong \prod_{i=1}^r GL_{a_{+\tau_i}}(\mathcal{O}_{K_{\mathfrak{P}_i}}) \times GL_{a_{-\tau_i}}(\mathcal{O}_{K_{\mathfrak{P}_i^c}})$.

Now, we introduce the *Igusa tower* over the component $\mathcal{S}^{\text{ord}}$. For each $n \in \mathbb{Z}_{\geq 1}$, consider the functor

$$\text{Ig}_n^{\text{ord}} : \{\text{Schemes}/\mathcal{S}^{\text{ord}}\} \to \{\text{Sets}\}$$

that takes an $\mathcal{M}^{\text{ord}}$-scheme S to the set of $\mathcal{O}_K$-linear closed immersions

$$\mathcal{L} \otimes_{\mathbb{Z}} \mu_{p^n} \hookrightarrow \mathcal{A}_S[p^n],$$

where $\mathcal{A}_S := \mathcal{A}_{\mathcal{S}^{\text{ord}}} \times_{\mathcal{S}^{\text{ord}}} S$. This functor is representable by an $\mathcal{S}^{\text{ord}}$-scheme, which we also denote Ig_n^{ord}. For each $m \in \mathbb{Z}_{\geq 1}$, we define $\mathbb{W}_m := \mathbb{W}/p^m\mathbb{W}$ and

$$\mathcal{S}_m^{\text{ord}} := \mathcal{S}^{\text{ord}} \times_{\mathbb{W}} \mathbb{W}_m.$$

We also consider the functor

$$\text{Ig}_{n,m}^{\text{ord}} : \left\{\text{Schemes}/\mathcal{S}_m^{\text{ord}}\right\} \to \{\text{Sets}\}$$

that takes an $\mathcal{S}_m^{\text{ord}}$-scheme S to the set of $\mathcal{O}_K$-linear closed immersions

$$\mathcal{L} \otimes_{\mathbb{Z}} \mu_{p^n} \hookrightarrow \mathcal{A}_S[p^n],$$

where $\mathcal{A}_S := \mathcal{A}_{\mathcal{S}^{\mathrm{ord}}} \times_{\mathcal{S}^{\mathrm{ord}}} S$. We define

$$\mathrm{Ig}_n^{\mathrm{ord}} := \varprojlim_m \mathrm{Ig}_{n,m}^{\mathrm{ord}}$$
$$\mathrm{Ig}^{\mathrm{ord}} := \varprojlim_n \mathrm{Ig}_n^{\mathrm{ord}},$$

with the maps given by forgetful functors. The tower $\mathrm{Ig}^{\mathrm{ord}}$ finite étale Galois coverings is called the *Igusa tower*. Note that for each $n \geq 1$, $\mathrm{Ig}_{n,m}^{\mathrm{ord}}$ is a finite étale covering of $\mathcal{S}_m^{\mathrm{ord}}$ whose Galois group is the group of $\mathcal{O}_K$-linear automorphisms of $\mathcal{L}/p^n\mathcal{L}$. Also, note that $\mathrm{Ig}^{\mathrm{ord}}$ is a formal scheme that is an étale covering of $\mathcal{S}_\infty^{\mathrm{ord}} := \varprojlim_m \mathcal{S}_m^{\mathrm{ord}}$.

We now give a different way of thinking about $\mathrm{Ig}^{\mathrm{ord}}$. Let $\mathfrak{M}^{\mathrm{ord}}$ be the formal completion of $\mathcal{S}^{\mathrm{ord}}$ along its special fiber. This is a formal scheme over $\mathbb{W}$. For p-divisible groups over $\mathfrak{M}^{\mathrm{ord}}$ there is a connected-étale exact sequence, so it makes sense to define the connected part $\mathcal{A}_{\mathfrak{M}^{\mathrm{ord}}}[p^\infty]^\circ$ of $\mathcal{A}_{\mathfrak{M}^{\mathrm{ord}}}[p^\infty]$. Then the formal completion $\mathrm{Ig}_n^{\mathrm{ord}}$ can be identified with the formal scheme $\mathrm{Isom}_{\mathfrak{M}^{\mathrm{ord}}}(\mathcal{L} \otimes_{\mathbb{Z}} \mu_{p^n}, \mathcal{A}_{\mathfrak{M}^{\mathrm{ord}}}[p^n]^\circ)$. Using the duality induced by the Hermitian pairing on L_p and by λ on $\mathcal{A}_{\mathfrak{M}^{\mathrm{ord}}}[p^\infty]$ (and noting that duality interchanges the connected and étale parts), we can further identify $\mathrm{Ig}_n^{\mathrm{ord}}$ with the formal scheme $\mathrm{Isom}_{\mathfrak{M}^{\mathrm{ord}}}\left(\mathcal{L}^\vee/p^n\mathcal{L}^\vee, \mathcal{A}_{\mathfrak{M}^{\mathrm{ord}}}[p^n]^{\acute{e}t}\right)$. This is finite étale over $\mathfrak{M}^{\mathrm{ord}}$.

4.1.2. *Irreducibility.* In this section, we show that the Igusa tower $\mathrm{Ig}^{\mathrm{ord}}$ is not irreducible, but we also sketch how one can pass to a partial SU-tower that is irreducible.

Notice that $\mathrm{Ig}_n^{\mathrm{ord}}$ can be identified with

$$\mathrm{Isom}_{\mathcal{S}^{\mathrm{ord}}}\left(\mathcal{L}^\vee/p^n\mathcal{L}^\vee, \mathcal{A}_{\mathfrak{M}^{\mathrm{ord}}}[p^n]^{\acute{e}t}\right). \tag{5}$$

Consider the compatible sequence of determinant morphisms of sheaves $\det : \mathrm{Ig}_n^{\mathrm{ord}} \to (\mathcal{O}_K/p^n\mathcal{O}_K)^\times$ over $\mathfrak{M}^{\mathrm{ord}}$. Let

$$v \in \varprojlim_n \mathrm{Isom}_{\mathcal{S}^{\mathrm{ord}}}\left(\mathcal{L}^\vee/p^n\mathcal{L}^\vee, \mathcal{A}_{\mathfrak{M}^{\mathrm{ord}}}[p^n]^{\acute{e}t}\right)$$

be a generator. Let $\mathrm{Ig}_n^{\mathrm{ord},SU}$ be the inverse image of $v \pmod{p^n}$ under det.

Theorem 4.1. *(Hida)* $\mathrm{Ig}_n^{\mathrm{ord},SU}$ *is a geometrically irreducible component of* $\mathrm{Ig}^{\mathrm{ord}}$.

Proof. (Sketch) We have an étale cover $\mathrm{Ig}_n^{\mathrm{ord},SU}$ of the smooth irreducible variety $\mathcal{S}^{\mathrm{ord}}$ over $\mathbb{W}$. One of Hida's strategies for proving the irreducibility of the étale cover in this situation is considering a compatible group action of a product $\mathcal{G}_1 \times \mathcal{G}_2$ on $\mathrm{Ig}_n^{\mathrm{ord},SU}$ and $\mathcal{S}^{\mathrm{ord}}$, in such a way that $\mathcal{G}_1 \subset \mathrm{Aut}(\mathcal{S}^{\mathrm{ord}})$ fixes and $\mathcal{G}_2 \subset \mathrm{Aut}(\mathrm{Ig}_n^{\mathrm{ord},SU}/\mathcal{S}^{\mathrm{ord}})$ acts transitively on the connected components of $\mathrm{Ig}_n^{\mathrm{ord},SU}$. The group $\mathcal{G}_1$ can be identified with the finite adelic points $SU(\mathbb{A}^\Sigma)$ away from certain bad places Σ containing p. This group will not have any finite quotient and therefore will preserve the connected components of the Igusa tower. The group $\mathcal{G}_2$ can be identified with the Levi subgroup $\mathrm{Levi}_1(\mathbb{Z}_p) := \mathrm{Levi}(\mathbb{Z}_p) \cap SU(\mathbb{Z}_p)$, where $\mathrm{Levi}(\mathbb{Z}_p)$ is as in Remark 8. The action of $\mathrm{Levi}_1(\mathbb{Z}_p)$ on the connected components is transitive, since $\mathrm{Ig}_n^{\mathrm{ord},SU}/\mathcal{S}^{\mathrm{ord}}$ is an $\mathrm{Levi}_1(\mathbb{Z}_p)$-torsor via the action on the Igusa level structure.

Then, choosing x to be a point on $\mathrm{Ig}_n^{\mathrm{ord},SU}$ (which amounts to the same thing as choosing a CM point $\tilde{x}$ on the ordinary locus $\mathcal{S}^{\mathrm{ord}}$), Hida considers the group $\mathcal{T}_x$ generated by $\mathcal{G}_1$ and the stabilizer of x in $\mathcal{G}_1 \times \mathcal{G}_2$. He shows one can choose the point x such that $\mathcal{T}_x$ is dense in $\mathcal{G}_1 \times \mathcal{G}_2$, which amounts to choosing a point whose stabilizer has p-adically dense image in $\mathrm{Levi}_1(\mathbb{Z}_p)$. This density is obtained as a by-product of the fact that $\tilde{x}$ has many extra endomorphisms over $\mathbb{Q}$ and the fact that $\mathrm{Levi}_1(\mathbb{Z}_p) \cap SU(\mathbb{Q})$ is dense in $\mathrm{Levi}_1(\mathbb{Z}_p)$. On one hand, $\mathcal{T}_x$ is dense in a group acting transitively on the connected components of $\mathrm{Ig}_n^{\mathrm{ord},SU}$; on the other hand, $\mathcal{T}_x$ fixes the connected components by definition. This shows that $\mathrm{Ig}_n^{\mathrm{ord},SU}$ has only one connected component to start with and, therefore, that it is irreducible. □

4.2. p-adic automorphic forms.

4.2.1. *Definition.* Assume the tame level group $\mathcal{U}$ is neat. Recall that $\mathrm{Ig}_n^{\mathrm{ord}}$ parametrizes quintuples $(A,\lambda,i,\alpha,j)/R$ where $(A,\lambda,i,\alpha)\in\mathcal{M}_{\mathcal{U},R}$ together with a

$$j:\mu_{p^n}\otimes_{\mathbb{Z}}\mathcal{L}\hookrightarrow A[p^n].$$

Note that the existence of j implies that if p is nilpotent in the base ring R then A must be ordinary. Additionally, $\mathrm{Ig}_{n,\infty}^{\mathrm{ord}}$ is a Galois cover of the ordinary locus of the Shimura variety with Galois group equal to Levi.

Recall that on $\mathcal{M}_{\mathcal{U},\mathbb{W}}$, there is an automorphic vector bundle $\mathcal{E}_{\mathcal{U},\rho}$ associated to an algebraic representation ρ of Levi over a locally free finite module M_ρ over the $\mathbb{W}$. Recall that a section f of $\mathcal{E}_{\mathcal{U},\rho}$ can be thought of as a morphism $f:\mathcal{E}_{\mathcal{U}}\to M_\rho$ such that

$$f(x,hl)=\rho(h)f(x,l)\quad h\in\text{Levi}.$$

Equivalently, if N denotes the unipotent radical of the Borel of Levi and $\pi:\mathcal{E}_{\mathcal{U}}\to\mathcal{M}_{\mathcal{U}}$, then

$$\mathcal{E}_{\mathcal{U},\rho}=\mathcal{E}_{\mathcal{U},\kappa}=(\pi_*\mathcal{E}_{\mathcal{U}})^N[\kappa],$$

where κ denotes the highest weight of ρ. It is a locally free coherent $\mathcal{O}_{\mathcal{M}}$-module.

Recall that the space of modular forms over $\mathbb{W}$ of weight ρ_κ and some neat level $\mathcal{U}$ can be thought of as

$$H^0(\mathcal{M}_{\mathcal{U}/\mathbb{W}},\mathcal{E}_\kappa).$$

Let $\mathrm{Ha}\in H^0(\mathcal{M}_{\mathcal{U}/\overline{\mathbb{F}}_p},\det(\mathcal{E}_{\mathrm{Std}})^{p-1})$ be the Hasse invariant. Since $\det(\mathcal{E}_{\mathrm{Std}})^{p-1}$ is ample, we can lift a power of the Hasse invariant Ha_{p-1}^t for some sufficiently large positive integer t to obtain a global section $\widetilde{\mathrm{Ha}}_{p-1}^t\in H^0(\mathcal{M},\det(\mathcal{E}_{\mathrm{Std}})^{t(p-1)})$. Note that $\det(\mathcal{E}_{\mathrm{Std}})^{t(p-1)}=\mathcal{E}_{\underline{t(p-1)}}$ where $\underline{t(p-1)}=(t(p-1)\cdots t(p-1))$. Recall that $\mathcal{M}_{/\mathbb{W}_m}^{\mathrm{ord}}=\mathcal{M}[1/\widetilde{\mathrm{Ha}}_{p-1}^t]_{/\mathbb{W}_m}$ be the ordinary locus of $\mathcal{M}_{/\mathbb{W}_m}$. Let $V_\kappa^{\mathrm{ord}}(\mathbb{W}_m):=H^0(\mathcal{M}_m^{\mathrm{ord}},\mathcal{E}_\kappa)$. By definition, we have

$$V_\kappa^{\mathrm{ord}}(\mathbb{W}_m)=\lim_n\frac{H^0(\mathcal{M}_{/\mathbb{W}_m},\mathcal{E}_{\kappa+n\underline{t(p-1)}})}{\widetilde{\mathrm{Ha}}_{p-1}^{nt}}.$$

Recall that the Igusa tower $(\mathrm{Ig}_{n,m}^{\mathrm{ord}})_{n,m\geq0}$ consists of finite covers of $\mathrm{Ig}_{0,m}^{\mathrm{ord}}$ with finite etale maps $\pi_{n',n}:\mathrm{Ig}_{n',m}^{\mathrm{ord}}\to\mathrm{Ig}_{n,m}^{\mathrm{ord}}$ for $n'\geq n\geq0$.

Define

$$V_{n,m}=H^0(\mathrm{Ig}_{n,m}^{\mathrm{ord}},\mathcal{O}_{\mathrm{Ig}_{n,m}^{\mathrm{ord}}}),$$

Let $V_{\infty,m}:=\varinjlim_n V_{m,n}$ and $V:=V_{\infty,\infty}:=\varprojlim_m V_{\infty,m}$.

Definition 4.3. We call $V^N=V_{\infty,\infty}^N$ the *space of p-adic modular forms.*

Remark 9. In Definition 4.3, we have defined p-adic modular forms over the non-compactified Shimura variety. Typically, p-adic automorphic forms are defined over the compactified Shimura variety. Any toroidal compactification would suffice, because Lan proved that the sheaves on any toroidal compactification descend to the minimal compactification. We plan to address problems concerning compactifications of the Igusa tower in future work. For the present paper, though, we are interested in local properties of automorphic forms, namely their local behavior at CM points. Thus, compactifications have no bearing on the main results of this paper.

Let $\mathbb{T} = T(\mathbb{Z}_p)$ and let $\Lambda_T := \mathbb{Z}_p[[\mathbb{T}]]$. The Galois action of $\mathbb{T}$ on $V^N_{\infty,m}$ makes the space of p-adic modular forms a discrete Λ_T-module.

Let $n \geq m$. Recall that each element $f \in V^N_{m,n}$ can be viewed as a function

$$(\underline{A}, j) \mapsto f(\underline{A}, j) \in \mathbb{W}_m, \tag{6}$$

where $\underline{A}$ consists of an abelian variety $A/\mathbb{W}_m$ together with a polarization, an endomorphism, and a level structure, and $j : A[p^n]^{\acute{e}t} \xrightarrow{\sim} (\mathbb{Z}/p^n\mathbb{Z}) \otimes \mathcal{L}$ is an isomorphism such that $j(A[w^n]^{\acute{e}t}) = (\mathbb{Z}/p^n\mathbb{Z}) \otimes \mathcal{L}^+$ and $j(A[(w^c)^n]^{\acute{e}t}) = (\mathbb{Z}/p^n\mathbb{Z}) \otimes \mathcal{L}^-$.

Recall that each element g in $\mathrm{Levi}(\mathcal{L}/p^m\mathcal{L}) = \mathrm{GL}(\mathbb{Z}/p^m\mathbb{Z} \otimes \mathcal{L})$ acts on elements f in $V_{m,n}$ via

$$(g \cdot f)(\underline{A}, j) := f(\underline{A}, gj).$$

We may view each element $f \in H^0\left(\mathcal{S}^{\mathrm{ord}}_m, \mathcal{E}_\kappa\right)$ as a function

$$(\underline{A}, \ell) \mapsto f(\underline{A}, \ell) \in \mathrm{Ind}_B^{\mathrm{Levi}}(\kappa)_{\mathbb{W}} = \{f : \mathrm{Levi}/N \to \mathbb{A}^1_{\mathbb{W}} : f(ht) = \kappa(t) f(h), t \in T\},$$

where $\underline{A}$ is as in Equation (6) and $\ell \in \mathrm{Isom}(\mathbb{Z}/p^n\mathbb{Z} \otimes \mathcal{L}^\vee, \underline{\Omega}_{\mathcal{A}/\mathcal{S}^{\mathrm{ord}}_m})$. Additionally, note that there is an action of Levi on $\mathrm{Ind}_B^{\mathrm{Levi}}(\kappa)$ which defines the representation ρ_κ by

$$\mathrm{Levi} \ni h : f(x) \mapsto \rho_\kappa(h) f(x) = f(h^{-1}x).$$

Since $n \geq m$, for A ordinary over $\mathbb{W}_m$, we have

$$\mathrm{Lie}\, A = \mathrm{Lie}\, A[p^n]^\circ.$$

Thus, for the universal abelian variety $\mathcal{A}$, we have

$$\underline{\Omega}_{\mathcal{A}/\mathcal{S}^{\mathrm{ord}}_m} = (\mathrm{Lie}\,\mathcal{A})^\vee = (\mathrm{Lie}\,\mathcal{A}[p^n]^\circ)^\vee \cong (\mathrm{Lie}(\mathcal{A}[p^n]^{\acute{e}t})^\vee)^\vee \cong \mathcal{A}[p^n]^{\acute{e}t} \otimes \mathcal{O}_{\mathcal{S}^{\mathrm{ord}}_m}. \tag{7}$$

By Equation (7), there is an isomorphism

$$\underline{\Omega}_{\mathcal{A}/\mathcal{S}^{\mathrm{ord}}_m} \xrightarrow{\sim} \mathcal{A}[p^n]^{\acute{e}t} \otimes \mathcal{O}_{\mathcal{S}^{\mathrm{ord}}_m}.$$

Thus, we may view each element $f \in H^0\left(\mathcal{S}^{\mathrm{ord}}_m, \mathcal{E}_\kappa\right)$ as a function

$$(\underline{A}, j) \mapsto f(\underline{A}, j) \in \mathrm{Ind}_B^{\mathrm{Levi}}(\kappa)/\mathbb{W}_m,$$

where $\underline{A}$ and j are as in Equation (6).

As explained in [Hid04, Section 8.1.2], there is a unique (up to $\mathbb{W}$-unit multiple) N-invariant element $\ell_{\mathrm{can}} \in \left(\mathrm{Ind}_B^{\mathrm{Levi}}(\kappa)_{\mathbb{W}}\right)^\vee = \mathrm{Hom}_{\mathbb{W}}(\mathrm{Ind}_B^{\mathrm{Levi}}(\kappa), \mathbb{W})$. The element ℓ_{can} generates $\left(\left(\mathrm{Ind}_B^{\mathrm{Levi}}(\kappa)_{\mathbb{W}}\right)^\vee\right)^N$. We may normalize ℓ_{can} so that it is evaluation at the identity in GL_n.

Now, we define a map

$$\begin{aligned}\Psi_{m,n} : H^0\left(\mathcal{S}^{\mathrm{ord}}_m, \mathcal{E}_\kappa\right) &\to V^N_{m,n}[\kappa]\\ f &\mapsto \tilde{f}(\underline{A}, j) := \ell_{\mathrm{can}}(f(\underline{A}, j)).\end{aligned}$$

Note that for each $b \in B$,

$$\begin{aligned}(b\tilde{f})(\underline{A}, j) &= \tilde{f}(\underline{A}, bj)\\ &= \ell_{\mathrm{can}}(f(\underline{A}, bj))\\ &= f(\underline{A}, bj)(1)\\ &= f(\underline{A}, j)(b)\\ &= \kappa(b) f(\underline{A}, j)(1) = \kappa(b)\ell_{\mathrm{can}}(f(\underline{A}, j)).\end{aligned}$$

for all $\underline{A}$ and j as in Equation (6). So $\tilde{f}$ is indeed in $V^N_{m,n}[\kappa]$.

Remark 10. Note that the map Ψ is injective. (Indeed, if $\Psi(f) = 0$, then for all $\underline{A}$ and j, $f(\underline{A}, j)(1) = 0$, so $f(\underline{A}, gj)(1) = 0$ for all $g \in G$, so $f(\underline{A}, j)(g) = 0$ for all $g \in G$, and hence, $f(\underline{A}, j) = 0$.)

We therefore have a map

$$\Psi : H^0\left(\mathcal{S}^{\mathrm{ord}}, \mathcal{E}_\kappa\right) \to V^N_{\infty,\infty}[\kappa].$$

In the sequel, we label $\Psi = \Psi_\kappa$ to reflect the weight.

Remark 11. While it is not the subject of this paper, it is natural to ask about density results concerning the space of p-adic automorphic forms (and spaces of classical automorphic forms within the space of p-adic automorphic forms). For this, we refer the reader to [Hid04, Chapter 8] and [HLTT13, Proposition 6.2]. Also, when thinking of p-adic automorphic forms as classes in the completed cohomology of Shimura varieties, Theorem IV.3.1 of [Sch13] shows that every class can be p-adically interpolated from classical automorphic forms. This statement is stronger than all previous results, since it also applies to torsion classes which contribute to completed cohomology.

5. Serre-Tate expansions

The goal of this section is to establish a p-adic analogue of the q-expansion principle for automorphic forms as a consequence of Hida's irreducibility result for the Igusa tower.

Classically, q-expansions arise by localization at a cusp, i.e., the q-expansion of a scalar-valued form f is the image of f in the complete local ring at the cusp, regarded as a power series in q, for q a canonical choice of the local parameter at the cusp. In these terms, the q-expansion principle states that localization is injective, and it is an immediate consequence of the fact that the space is connected. Alternatively, when working over the whole Shimura variety, it becomes necessary to choose a cusp on each connected component, and consider all localizations at once.

In this paper, we work over a connected component $\mathcal{S}$ of $\mathcal{M}$, but we replace cusps with integral ordinary CM points (i.e. points of $\mathcal{S}^{\mathrm{ord}}$ defined over $\mathbb{W}$ corresponding to abelian varieties with complex multiplication). The crucial observation is that given an integral ordinary CM point x_0, the choice of a lift x of x_0 to the Igusa tower uniquely determines a choice of Serre-Tate local parameters at x_0, i.e. x defines an isomorphism of the p-adic completion of the complete local ring at x_0 with a power series ring over $\mathbb{W}$. We call the power series corresponding to the localization at x of an automorphic form its t-expansion, for t denoting the Serre-Tate local parameters.

5.1. Localization.

Let $\bar{x}_0 \in \mathcal{S}^{\mathrm{ord}}(k)$, and $x_0 \in \mathcal{S}^{\mathrm{ord}}(\mathbb{W})$ any integral lift of $\bar{x}_0$. (Without loss of generality, we may chose x_0 to be a CM point, or even the canonical CM lift of $\bar{x}_0$; see Remark 16.) We write $\mathcal{O}^\wedge_{\mathcal{S}^{\mathrm{ord}},x_0}$ for the complete local ring of $\mathcal{S}^{\mathrm{ord}}$ at x_0, and define $\mathcal{R}_{\mathcal{S}^{\mathrm{ord}},x_0}$ to be the p-adic completion of $\mathcal{O}^\wedge_{\mathcal{S}^{\mathrm{ord}},x_0}$.

More explicitly, $\mathcal{R}_{\mathcal{S}^{\mathrm{ord}},x_0}$ can be constructed as follows. For each $m \geq 1$, we write $x_{0,m}$ for the reduction of x_0 modulo p^m, regarded as a point of $\mathcal{S}^{\mathrm{ord}}_m := \mathcal{S}^{\mathrm{ord}} \times_{\mathbb{W}} \mathbb{W}/p^m\mathbb{W}$ (in particular, $\bar{x}_0 = x_{0,1}$). Let $\mathcal{O}^\wedge_{\mathcal{S}^{\mathrm{ord}}_m,x_0}$ denote for the complete local ring of $\mathcal{S}^{\mathrm{ord}}_m$ at $x_{0,m}$. The p-adic completion $\mathcal{R}_{\mathcal{S}^{\mathrm{ord}},x_0}$ of $\mathcal{O}^\wedge_{\mathcal{S}^{\mathrm{ord}},x_0}$ can be identified with $\varprojlim_m \mathcal{O}^\wedge_{\mathcal{S}^{\mathrm{ord}}_m,x_0}$. Alternatively, $\mathcal{R}_{\mathcal{S}^{\mathrm{ord}},x_0}$ can also be identified with $\mathcal{O}^\wedge_{\mathcal{S}^{\mathrm{ord}},\bar{x}_0}$.

Let $x \in \mathrm{Ig}^{\mathrm{ord}}$ denote a compatible system of integral points $x = (x_n)_{n\geq 0}$ on the Igusa tower above x_0; i.e. for each $n \geq 0$, x_n is an integral point in $\mathrm{Ig}^{\mathrm{ord}}_n$, $\mathrm{Ig}^{\mathrm{ord}}_0 = \mathcal{S}^{\mathrm{ord}}$, with x_n mapping to x_{n-1} under the natural projections. Given the point $x_0 \in \mathcal{S}^{\mathrm{ord}}$, a choice of $x \in \mathrm{Ig}^{\mathrm{ord}}$ lying above x_0 is equivalent to a choice of an Igusa structure of infinite level (i.e. a choice of compatible isomorphisms as in Equation (5)) on the corresponding ordinary abelian variety. For all m, as n varies, the natural projections $j = j_{m,n} : \mathrm{Ig}^{\mathrm{ord}}_{m,n} \to \mathcal{S}^{\mathrm{ord}}_m$ induce a compatible system of isomorphisms

$$j_x : \mathcal{O}^{\wedge}_{\mathcal{S}^{\mathrm{ord}}_m, x_0} \xrightarrow{\sim} \mathcal{O}^{\wedge}_{\mathrm{Ig}^{\mathrm{ord}}_{m,n}, x_n}$$

which allow us to canonically identify $\mathcal{O}^{\wedge}_{\mathrm{Ig}^{\mathrm{ord}},x} := \varprojlim_m \varinjlim_n \mathcal{O}^{\wedge}_{\mathrm{Ig}^{\mathrm{ord}}_{m,n}, x_{m,n}}$ with $\mathcal{R}_{\mathcal{S}^{\mathrm{ord}},x_0}$.

We define $\mathrm{loc}_x : V^N \to \mathcal{R}_{\mathcal{S}^{\mathrm{ord}},x_0}$ as the localization at $x \in \mathrm{Ig}^{\mathrm{ord}}$ composed with j_x^{-1}. With abuse of language, we still refer to $\mathrm{loc}_x(f) \in \mathcal{R}_{\mathcal{S}^{\mathrm{ord}},x_0}$ as the localization of f at x, for all $f \in V^N$.

Remark 12. We compare localizations at different points $x \in \mathrm{Ig}^{\mathrm{ord}}$ above x_0. For any $g \in \mathrm{Levi}(\mathbb{Z}_p)$, we have $j \circ g = j$. Thus, for all $f \in V^N$ we have

$$\mathrm{loc}_{gx}(f) = \mathrm{loc}_x(g \cdot f),$$

for any $g \in \mathrm{Levi}(\mathbb{Z}_p)$ and $x \in \mathrm{Ig}^{\mathrm{ord}}$.

Proposition 5.1.1. Let $x \in \mathrm{Ig}^{\mathrm{ord}}_\infty$. The map $\mathrm{loc}_x : V^N \to \mathcal{R}_{\mathcal{S}^{\mathrm{ord}},x_0}$ is injective.

Proof. Were the Igusa covers $\mathrm{Ig}^{\mathrm{ord}}_n$ irreducible over $\mathcal{S}^{\mathrm{ord}}$, the statement would immediately follow. As it happens $\mathrm{Ig}^{\mathrm{ord}}_n$ is not irreducible, thus a priori the vanishing under the localization map loc_x only implies the vanishing on the connected component containing x. As the torus $T(\mathbb{Z}_p) \subset \mathrm{Levi}(\mathbb{Z}_p)$ acts transitively on the connected components of $\mathrm{Ig}^{\mathrm{ord}}$ over $\mathcal{S}^{\mathrm{ord}}$, it suffices to prove that for all $f \in V^N$, the identity $\mathrm{loc}_x(f) = 0$ implies $\mathrm{loc}_{gx}(f) = 0$ for all $g \in T(\mathbb{Z}_p)$. This is obviously true for $f \in V^N[\kappa]$, for any weight κ. Since the subspaces $V^N[\kappa]$ span a dense subspace of V^N (as κ varies), the statement holds for all V^N. □

5.2. **Serre-Tate coordinates.** It follows from the smoothness of $\mathcal{S}^{\mathrm{ord}}$ that for any point x_0 the ring $\mathcal{R}_{\mathcal{S}^{\mathrm{ord}},x_0}$ is (non-canonically) isomorphic a power series ring over $\mathbb{W}$. The goal of this section is to explain how Serre-Tate theory implies that for x a lift of x_0 to $\mathrm{Ig}^{\mathrm{ord}}$, the ring $\mathcal{O}^{\wedge}_{\mathrm{Ig}^{\mathrm{ord}},x}$ is canonically isomorphic to a ring of power series over $\mathbb{W}$.

We recall Serre-Tate theory following [Kat81]. The first theorem describes the deformation space of an ordinary abelian variety, and the second explains how to address the lifting of additional structures (such as a polarization and extra endormorphisms). As an application we deduce a description of the ring $\mathcal{R}_{\mathcal{S}^{\mathrm{ord}},x_0}$, for any $x_0 \in \mathcal{S}^{\mathrm{ord}}(\mathbb{W})$.

We introduce some notation. Let A be an ordinary abelian variety over k, of dimension g. The *physical Tate module* of A, $T_pA(k)$, is the Tate module of the maximal étale quotient of $A[p^\infty]$, i.e

$$T_pA(k) = \varprojlim_n A[p^n](k) = \varprojlim_n A[p^n]^{ét}(k).$$

As A is ordinary, $T_pA(k)$ is a free $\mathbb{Z}_p$-module of rank g. Like above, we denote by $A^\vee$ the dual abelian variety, and we denote by $T_pA^\vee(k)$ the physical Tate module of $A^\vee$.

Let R be an Artinian local ring, with residue field k. A *lifting* (or *deformation*) of A over R is a pair $(\mathcal{A}/R, j)$, consisting of an abelian scheme $\mathcal{A}$ over R, together with an isomorphism $j : \mathcal{A} \otimes_R k \to A$. By abuse of notation we sometime simply write $\mathcal{A}/R$ for the pair $(\mathcal{A}/R, j)$. To each lifting $(\mathcal{A}/R, j)$, as explained in [Kat81, Section 2.0, p. 148], Serre and Tate associated a $\mathbb{Z}_p$-bilinear form

$$q_{\mathcal{A}/R} : T_pA(k) \times T_pA^\vee(k) \to \hat{\mathbb{G}}_m(R).$$

Theorem 5.1 ([Kat81] Theorem 2.1, p. 148)**.** *Let the notation be as above.*

(1) *The map* $(\mathcal{A}/R, j) \mapsto q_{\mathcal{A}/R}$ *is a bijection from the set of isomorphism classes of liftings of* A *over* R *to the group* $\mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k) \otimes T_pA^\vee(k), \hat{\mathbb{G}}_m(R))$.
(2) *The above construction defines an isomorphism of functors between the deformation space* $\mathcal{M}_{A/k}$ *and* $\mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k) \otimes T_pA^\vee(k), \hat{\mathbb{G}}_m)$.

In the following we refer to the above isomorphism as the *Serre-Tate isomorphism.*

Let A and B be ordinary abelian varieties over k, and let $f : A \to B$ be a k-isogeny. We write $f^\vee : B^\vee \to A^\vee$ for the dual isogeny. A theorem of Drinfield ([Kat81, Lemma 1.1.3, p.141]) proves that for any Artinian local ring R, and pair of liftings $(\mathcal{A}/R, j_A)$,$(\mathcal{B}/R, j_B)$ of A, B respectively, if there exists a isogeny $\phi : \mathcal{A} \to \mathcal{B}$ lifting f (i.e. satisfying $f = j_B \circ (\phi \otimes 1_k) j_A^{-1}$) then ϕ is unique. Yet, in general such a lifting of f will not exists. Theorem 5.2 gives a necessary and sufficient condition for the existence of ϕ in terms of the Serre-Tate isomorphism.

Theorem 5.2 ([Kat81] Theorem 2.1, Part 4, p.149)**.** *Let the notation be as above. Given* $\mathcal{A}$ *and* $\mathcal{B}$ *lifting* A *and* B*, respectively, over an Artinian local ring* R*. A morphism* $f : A \to B$ *lifts to a morphism* $\mathcal{A} \to \mathcal{B}$ *if and only if* $q_{\mathcal{A}/R} \circ (1 \times f^\vee) = q_{\mathcal{B}/R} \circ (f \times 1)$.

We apply the above results in our setting. Let $\bar{x}_0 \in \mathcal{S}^{\mathrm{ord}}(k)$ be an ordinary point in positive characteristic, and $\underline{A} \coloneqq \underline{\mathcal{A}}_{\bar{x}_0}$ be the associated abelian variety over k, together with its additional structures. Then, the physical Tate module $T_pA(k)$ of A is a free $\mathcal{O}_{K,p}$-module, and the polarization λ of A induces a conjugate-linear isomorphism $T_p(A)(k) \tilde{\to} T_pA^\vee(k)$. Furthermore, the formal completion $\mathcal{S}^{\mathrm{ord}}{}^\wedge_{\bar{x}_0}$ of $\mathcal{S}^{\mathrm{ord}}$ at $\bar{x}_0$ is canonically identified with the closed subspace of the deformation space $\mathcal{M}_{A/k}$ consisting of all deformations of A which are (can be) endowed with additional structures lifting those of A.

Proposition 5.2.1. The restriction of the Serre-Tate isomorphism induces an isomorphism between the formal neighborhood $\mathcal{S}^{\mathrm{ord}}{}^\wedge_{\bar{x}_0}$ and $\oplus_{i=1}^r \mathrm{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{P}_i}A(k) \otimes T_{\mathfrak{P}_i^c}A(k), \hat{\mathbb{G}}_m)$.

Proof. For any local Artinian ring R and $x \in \mathcal{S}^{\mathrm{ord}}(R)$, the associated abelian scheme $\underline{\mathcal{A}}_x$ lifts A together with its additional structure to R. We deduce that the corresponding bilinear form $q_{\mathcal{A}_x}$ is symmetric (i.e. satisfies $q_{\mathcal{A}_x}(1 \times \lambda^\vee) = q_{\mathcal{A}_x^\vee}(\lambda \times 1)$) and c-hermitian (i.e. $q_{\mathcal{A}_x}(1 \times b^c) = q_{\mathcal{A}_x}(b \times 1)$ for all $b \in \mathcal{O}_K$, where c denotes complex conjugation). In particular, $q_{\mathcal{A}_x}$ is uniquely determined by its restrictions $q_{x,i} : T_{\mathfrak{P}_i}A(k) \times T_{\mathfrak{P}_i^c}A(k) \to \hat{\mathbb{G}}_m(R)$, for $i = 1, \dots r$. To conclude we observe that any such collection of morphisms $(q_i)_{i=1,\cdots,r}$ extends to a unique symmetric c-hermitian bilinear form on $T_pA(k)$. □

We finally describe how the choice of an Igusa level structure (of infinite level) on $\underline{A} = \underline{\mathcal{A}}_{x_0}$ determines a choice of local parameters.

Let $\mathcal{O}_{K,p} \coloneqq \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We consider the $\mathcal{O}_{K,p}$-modules introduced in Section 4.1:

$$\mathcal{L} = \mathcal{L}^+ \oplus \mathcal{L}^- \simeq \oplus_{i=1}^r (\mathcal{O}_{K_{\mathfrak{P}_i}}^{a_{v_i}+} \oplus \mathcal{O}_{K_{\mathfrak{P}_i^c}}^{a_{v_i}-}),$$

where the decomposition comes from the splitting $p = w \cdot w^c$, and for each $i = 1, \dots, r$, v_i is a place inducing $\mathfrak{p}_i$. For each i, we write $\mathcal{L}_i^+ \subset \mathcal{L}^+$ for the submodule corresponding to the place v_i, $\mathcal{L}_i^+ \simeq \mathcal{O}_{K_{\mathfrak{P}_i}}^{a_{v_i}+}$. Similarly, we define $\mathcal{L}_i^- \subset \mathcal{L}^-$, $\mathcal{L}_i^- \simeq \mathcal{O}_{K_{\mathfrak{P}_i}}^{a_{v_i}-}$. Then, $\mathcal{L}^+ = \oplus_{i=1}^r \mathcal{L}_i^+$ and $\mathcal{L}_i^- = \oplus_{i=1}^r \mathcal{L}_i^-$.

We define the $\mathcal{O}_{K,p}$-module

$$\mathcal{L}^2 \simeq \oplus_{i=1}^r \mathcal{L}_i^+ \otimes \mathcal{L}_i^-$$

naturally regarded as a submodule of $\mathcal{L} \otimes \mathcal{L}$.

Proposition 5.2.2. A choice of an Igusa structure x of infinite level on $\underline{A}$ determines a unique isomorphism between ${\mathcal{S}^{\mathrm{ord}}}^{\wedge}_{\bar{x}_0}$ and $\hat{\mathbb{G}}_m \otimes \mathcal{L}^2$. In particular, for each x we have a canonical isomorphism

$$\beta_x : W[[t]] \otimes (\mathcal{L}^2)^\vee \tilde{\to} \mathcal{R}_{\mathcal{S}^{\mathrm{ord}}, x_0}.$$

Proof. The choice of an Igusa structure of infinite level $\iota : \mu_{p^\infty} \otimes \mathcal{L} \hookrightarrow A[p^\infty]$ is equivalent to the choice of an $\mathcal{O}_{K,p}$-linear isomorphism $T_p(\iota^\vee) : T_p A(k) \tilde{\to} \mathcal{L}^\vee$, satisfying

$$T_p(\iota^\vee)(T_w A(k)) = (\mathcal{L}^+)^\vee \text{ and } T_p(\iota^\vee)(T_{w^c} A(k)) = (\mathcal{L}^-)^\vee.$$

More precisely, $T_p(\iota^\vee)$ induces isomorphisms $T_{\mathfrak{P}_i} A(k) \simeq (\mathcal{L}_i^+)^\vee$ and $T_{\mathfrak{P}_i^c} A(k) \simeq (\mathcal{L}_i^-)^\vee$, for all $i = 1, \ldots, r$. The first isomorphism in the statement is obtained from the Serre-Tate isomorphism by composition with the trivializations induced by $T_p(\iota^\vee)$. The second one follows from the first for t the canonical parameter on $\hat{\mathbb{G}}_m$. □

We denote by $\mathbb{I}$ the identity on $W[[t]]$.

Remark 13. For all $g \in \mathrm{Levi}(\mathbb{Z}_p)$, we have $\beta_{gx} = \beta_x \circ (\mathbb{I} \times g)$.

Finally, we can state an appropriate analogue of the q-expansion principle for p-adic automorphic forms.

Theorem 5.3 ("t-expansion principle" or "Serre-Tate expansion principle")**.** *Let x_0 be a point of $\mathcal{S}^{\mathrm{ord}}(\mathbb{W})$, x be a point on the infinite Igusa cover lying above x_0, and f be a p-adic automorphic form of any weight. We define the t-expansion of f at x as*

$$f_x(t) := \beta_x^{-1}(\mathrm{loc}_x(f)) \in W[[t]] \otimes (\mathcal{L}^2)^\vee.$$

Then, the t-expansion $f_x(t)$ vanishes if and only if f vanishes.

Remark 14. For any point x on the Igusa tower, the map

$$V \to W[[t]] \otimes (\mathcal{L}^2)^\vee \text{ defined by } f \mapsto f_x(t)$$

is $\mathrm{Levi}(\mathbb{Z}_p)$-equivariant, i.e. for any $g \in \mathrm{Levi}(\mathbb{Z}_p)$

$$(g \cdot f)_x(t) = (\mathbb{I} \otimes g)(f_x(t)),$$

for all global functions f on the infinite Igusa tower.

Remark 15. For all $g \in M(\mathbb{Z}_p)$: $f_{gx}(t) = (g \cdot f)_x(t)$.

Remark 16. As a consequence of the Serre-Tate Theorem (i.e. Theorem 5.1), any point $\bar{x}_0 \in \mathcal{S}^{\mathrm{ord}}(k)$ always lifts to a point $x_0 \in \mathcal{S}^{\mathrm{ord}}(\mathbb{W})$. In fact, it even admits a canonical lift y_0, namely the one corresponding to the form $q = 0$. The abelian variety $\mathcal{A}_{y_0}$ is the unique deformation of $\mathcal{A}_{\bar{x}_0}$ to which all endomorphisms lift. Thus, in particular, $\mathcal{A}_{y_0}$ is a CM abelian variety with ordinary reduction. The point y_0 is called the canonical CM lift of $\bar{x}_0$.

Remark 17. Given $x_0 \in \mathcal{S}^{\mathrm{ord}}(\mathbb{W})$, let $\bar{x}_0 \in \mathcal{S}^{\mathrm{ord}}(k)$ denote its reduction modulo p. Then, reduction modulo p gives a canonical bijection between the points on the Igusa tower over x_0 and those above $\bar{x}_0$.

Remark 18. For f a vector-valued automorphic form of weight κ, one may also consider the localization of f at a point of $x_0 \in \mathcal{S}^{\mathrm{ord}}$. Such a localization is an element in a free module M_κ over $\mathcal{R}_{\mathcal{S}^{\mathrm{ord}}, x_0}$, with rank depending on the weight κ. Even with canonical choices of trivializations of the modules M_κ's, this approach only allows one to establish congruences when the two ranks agree, e.g. when the difference among the weights is parallel. A crucial advantage of Hida's realization of (vector-valued) automorphic forms as function on the infinite Igusa tower is that one can define congruences for any weight. The t-expansion principle as stated above implies that those congruence relations can be detected by the coefficients of associated power series, as in the classical setting; i.e. given two p-adic automorphic forms f, f' of any weight $f \equiv f'(\bmod p^r)$ if and only if $f_x(t) \equiv f'_x(t)(\bmod p^r)$, for each $r \in \mathbb{N}$.

5.3. **Restriction of t-expansions.** As an example of how Serre-Tate coordinates may be used to understand certain operators on p-adic automorphic forms, we consider the case of the restriction map from our unitary Shimura variety to a lower dimensional unitary Shimura subvariety, i.e. the pullback map on global sections of the automorphic sheaves.

We start by setting some notation. Let $L = \oplus_{i=1}^s W_i$ be a self-dual decomposition of the lattice L, and denote by $\langle , \rangle_i$ the pairing on W_i induced by $\langle , \rangle$ on L. For each $i = 1, \dots, s$, let $GU_i = GU(W_i, \langle , \rangle_i)$, a unitary group of signature (a_{+i}, a_{-i}). Then, the signatures $(a_{+i}, a_{-i})_{i=1,\dots,s}$ form a partition of the signature (a_+, a_-).

We define $G' \subset \prod_i GU_i$ to be the subgroup of elements with the same similitude factor; i.e. if $\nu_i : GU_i \to \mathbb{G}_m$ denote the similitude factors, then $G' = \nu^{-1}(\mathbb{G}_m)$, for $\nu = \prod_i \nu_i$ and $\mathbb{G}_m \subset \mathbb{G}_m^s$ embedded diagonally. Then, there is a natural inclusion $G' \to GU$ of algebraic groups compatible with the partition of the signature.

Like above, we continue to denote by Levi the Levi subgroup of $Res_{K/\mathbb{Q}}(GL_g)$ associated with the signature (a_+, a_-) of GU. Then, the partition $(a_{+i}, a_{-i})_{i=1,\dots,s}$ of the signature (a_+, a_-) defines a unique Levi subgroup Levi′ of Levi. We identify the maximal torus T of Levi with the maximal torus T' in Levi′. Note that under such identification the order on $X^*(T)$ does not agree with that on $X^*(T')$. In particular, for any character κ of $T = T'$, if κ is dominant in $X^*(T)$ then it is also dominant in $X^*(T')$, but the converse is false. For κ a dominant weight of T (and thus also of T'), let ρ_κ (resp. ρ'_κ) be the unique irreducible representation of Levi (resp. Levi′) with highest weight κ. Then ρ'_κ is an irreducible constituent of the restriction of ρ_κ from Levi to Levi′ (although in general other irreducible representations ρ' of Levi′ also appear). We fix the projection $\rho_\kappa \to \rho'_\kappa$.

Finally, we choose a level away from p, and consider the corresponding closed embedding

$$\theta : \mathcal{M}' \hookrightarrow \mathcal{M}$$

of the Shimura variety $\mathcal{M}'$ associated with G' into the Shimura variety $\mathcal{M}$ associated with GU. A point $x \in \mathcal{M}$ is in the image of θ if and only if the corresponding abelian variety $\underline{\mathcal{A}}_x$ decomposes as a cartesian product of abelian varieties, of dimensions prescribed by the above partitions (and compatibly with the additional structures).

Remark 19. We describe an example. Let V_n be an n-dimensional $\mathcal{O}_F$-lattice equipped with a Hermitian pairing $\langle , \rangle$, and assume that associated group $GU_n = GU(V_n, \langle , \rangle)$ has real signature $(1, n)$. We fix the partition $\{(1, n-1), (0, 1)\}$ of the signature $(1, n)$, and we realize V_{n-1} as a direct summand of V_n. We choose a neat level Γ hyperspecial at p and denote by Sh_n the simple Shimura variety of level Γ associated with GU_n. Sh_n is a classifying space for polarized abelian varieties of dimension n, equipped with a compatible action of $\mathcal{O}_F$. We write $\mathcal{A}_n$ for the universal abelian scheme on Sh_n. Then, for each elliptic curve $E_0/\mathbb{Z}_p$ with complex multiplication by $\mathcal{O}_K$ (corresponding to the choice of a connected component of the 0-dimensional Shimura variety associated with $GU(0,1)$), the morphism θ is defined by $\theta^*\mathcal{A}_n = \mathcal{A}_{n-1} \times E_0$, and its image $\theta(\mathrm{Sh}_{n-1})$ is a divisor in Sh_n.

Let $\mathcal{E}_\kappa/\mathcal{M}$ (resp. $\mathcal{E}'_\kappa/\mathcal{M}'$) be the automorphic sheaf of weight κ. Then, on $\mathcal{M}'$ we have a canonical morphism of sheaves $\theta^*\mathcal{E}_\kappa \to \mathcal{E}'_\kappa$. We call the induced map on global sections pre-composed with pullback under θ

$$\mathrm{res}_\kappa : H^0(\mathcal{M}, \mathcal{E}_\kappa) \to H^0(\mathcal{M}', \theta^*\mathcal{E}_\kappa) \to H^0(\mathcal{M}', \mathcal{E}'_\kappa)$$

the *weight κ restriction.*

We now assume that p splits completely in all the reflex fields E_i (and thus also in E), where E_i is the reflex field for the integral model $\mathcal{M}_i$ associated with the group GU_i. Then the ordinary loci $\mathcal{M}_i^{\mathrm{ord}}$, $\mathcal{M}'^{\mathrm{ord}}$ and $\mathcal{M}^{\mathrm{ord}}$ are non-empty. In particular, a split abelian variety $A = \prod_i A_i$ is ordinary if and only if each of its constituents A_i are ordinary.

Each connected component $\mathcal{S}'$ of $\mathcal{M}'$ can be identified with a product of connected component $\mathcal{S}_i$ of $\mathcal{M}_i$. We choose connected components $\mathcal{S}'$ of $\mathcal{M}'$, and identify $\mathcal{S}' = \prod_{i=1}^s \mathcal{S}_i$. Then, there is a unique connected component $\mathcal{S}$ of $\mathcal{M}$ such that $\theta(\mathcal{S}') \subset \mathcal{S}$. We write $\mathcal{S}_i^{\mathrm{ord}}$ (resp. $\mathcal{S}'^{\mathrm{ord}}$ and $\mathcal{S}^{\mathrm{ord}}$) for the ordinary locus of $\mathcal{S}$ (resp. $\mathcal{S}'$ and $\mathcal{S}$). Thus, $\theta(\mathcal{S}'^{\mathrm{ord}}) \subset \mathcal{S}^{\mathrm{ord}}$, and we may identify $\mathcal{S}'^{\mathrm{ord}} = \prod_i \mathcal{S}_i^{\mathrm{ord}}$. By abuse of notation we still denote by res_κ the restriction of such map to global sections over the ordinary loci; i.e.

$$\mathrm{res}_\kappa : H^0(\mathcal{S}, \mathcal{E}_\kappa) \to H^0(\mathcal{S}', \theta^* \mathcal{E}_\kappa) \to H^0(\mathcal{M}', \mathcal{S}'_\kappa).$$

Corresponding to our choice of connected components, there are two $\mathcal{O}_{F,p}$-linear decomposition $\mathcal{L}^+ = \oplus_{i=1}^s \mathfrak{L}_i^+$ and $\mathcal{L}^- = \oplus_{i=1}^s \mathfrak{L}_i^-$, the ranks of the summands determined by the partition $(a_{+i}, a_{-i})_{i=1,\dots,s}$ of the signature (a_+, a_-). For each $i = 1, \dots, s$, we write $\mathfrak{L}_i = \mathfrak{L}_i^+ \oplus \mathfrak{L}_i^-$. Thus, $\mathcal{L} = \oplus_{i=1}^s \mathfrak{L}_i$.

For every level $n \geq 1$, the homomorphism $\theta : \mathcal{S}'^{\mathrm{ord}} \to \mathcal{S}^{\mathrm{ord}}$ lifts canonically to a compatible system of homomorphisms $\Theta = (\Theta_n)_n$ among the Igusa towers,

$$\Theta_n : \prod_i \mathrm{Ig}_{n,i}^{\mathrm{ord}} \to \mathrm{Ig}_n^{\mathrm{ord}},$$

where $\mathrm{Ig}_{n,i}^{\mathrm{ord}}$ denotes the n-th level of the Igusa tower over $\mathcal{S}'^{\mathrm{ord}}$. Given a split abelian variety $A = \prod_i A_i$ corresponding to a point x in the image of $\mathcal{S}'^{\mathrm{ord}}$, an Igusa structure $\iota : \mu_{p^n} \otimes \mathcal{L} \hookrightarrow A[p^n]$ of level n on A is in the image of Θ_n if and only if it satisfies the conditions $\iota(\mu_{p^n} \otimes \mathfrak{L}_i) \subset A_i[p^n]$ for all i. In particular, for each n, the morphism Θ_n defines a closed embedding of the Igusa cover $\mathrm{Ig}_n'^{\mathrm{ord}} = \prod_i \mathrm{Ig}_{n,i}^{\mathrm{ord}}$ over $\mathcal{S}'^{\mathrm{ord}}$ in $\mathcal{S}'^{\mathrm{ord}} \times_{\mathcal{S}^{\mathrm{ord}}} \mathrm{Ig}_n^{\mathrm{ord}}$. We note that for non-trivial partitions of the signature (a_+, a_-), this is not an isomorphism.

Let Θ^* denote the pullback on global functions of the Igusa tower, $\Theta^* : V^N \to V'^{N'}$. (Here, N and N' denote the $\mathbb{Z}_p$-points of the unipotent radical subgroups of Levi and Levi$'$ respectively. So $N' = N \cap \mathrm{Levi}'(\mathbb{Z}_p)$.)

Remark 20. For any weight κ, let $\Psi_\kappa : H^0(\mathcal{S}, \mathcal{E}_\kappa) \to V^N$ and $\Psi'_\kappa : H^0(\mathcal{S}', \mathcal{E}'_{\kappa'}) \to V'^{N'}$ be the inclusions defined in section 4.2.1. For any $f \in H^0(\mathcal{S}, \mathcal{E}_\kappa)$:

$$\Theta^*(\Psi_\kappa(f)) = \Psi'_\kappa(\mathrm{res}_\kappa(f)).$$

In the following, we refer to the pullback of Θ^* as *restriction.*

Our goal is to give a simple description of Θ^* in Serre-Tate coordinates, and deduce an explicit criterium for the vanishing of the restriction of a p-adic automorphic form in terms of vanishing of some of the coefficients in its t-expansion.

We choose an integral ordinary point x_0 in the image of θ, i.e. $\theta((x_0^i)_i) = x_0$, and write $\bar{x}_0$ for its reduction modulo p. We also choose a lift x of x_0 to the infinite Igusa tower in the image of Θ, i.e. $\Theta((x^i)_i) = x$.

Proposition 5.3.1. Under the Serre-Tate isomorphism, the map $\theta_{\bar{x}_0} : \mathcal{S}'^{\mathrm{ord}\wedge}_{\bar{x}_0} \to \mathcal{S}^{\mathrm{ord}\wedge}_{\bar{x}_0}$ corresponds to the collection, for $j = 1, \dots, r$, of the natural inclusions

$$\oplus_{i=1}^s \mathrm{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{P}_j} A^i(k) \otimes T_{\mathfrak{P}_j^c} A^i(k), \hat{\mathbb{G}}_m) \subset \mathrm{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{P}_j} A(k) \otimes T_{\mathfrak{P}_j^c} A(k), \hat{\mathbb{G}}_m).$$

Finally, for each $i = 1, \dots, s$, we define $\mathcal{O}_{K,p}$-modules $\mathfrak{L}_i^2 \subset \mathfrak{L}_i^{\otimes 2}$ similarly to $\mathcal{L}^2 \subset \mathcal{L}^{\otimes 2}$ in section 5.2. We consider the $\mathcal{O}_{F,p}$-module $\mathfrak{L}^2 = \oplus_{i=1}^s \mathfrak{L}_i^2$. By the definition $\mathfrak{L}^2$ is a direct summed of $\mathcal{L}^2$, we write $\epsilon : \mathfrak{L}^2 \to \mathcal{L}^2$ for the natural inclusion.

Then, associated with the point x on the Igusa cover of infinite level, we have isomorphisms

$$\beta^*_{x^i} : (\mathcal{S}^{\mathrm{ord}}_i)^\wedge_{\bar{x}^i_0} \xrightarrow{\sim} \hat{\mathbb{G}}_m \otimes \mathfrak{L}^2_i,\ i = 1, \cdots, s, \text{ and } \beta^*_x : \mathcal{S}^{\mathrm{ord}\wedge}_{\bar{x}} \xrightarrow{\sim} \hat{\mathbb{G}}_m \otimes \mathcal{L}^2.$$

We write ${\beta'}^*_x : \mathcal{S}'^{\mathrm{ord}\wedge}_{\bar{x}} \xrightarrow{\sim} \hat{\mathbb{G}}_m \otimes \mathfrak{L}^2$ for the isomorphism induced by collection $(\beta^*_{x^i})_{i=1,\ldots,s}$.

Proposition 5.3.2. The notation is the same as above. The map $\beta^*_x \circ \theta_{x_0} \circ ({\beta'}^*_x)^{-1}$ agrees with the inclusion $\mathbb{I} \otimes \epsilon : \hat{\mathbb{G}}_m \otimes \mathfrak{L}^2 \to \hat{\mathbb{G}}_m \otimes \mathcal{L}^2$.

Equivalently, in terms of the local Serre-Tate coordinates, the morphism Θ^*_x agrees with the map $\mathbb{I} \otimes \epsilon^\vee : \mathbb{W}[[t]] \otimes (\mathcal{L}^2)^\vee \twoheadrightarrow \mathbb{W}[[t]] \otimes (\mathfrak{L}^2)^\vee$, i.e. $(\Theta^* f)_x(t) = (\mathbb{I} \otimes \epsilon^\vee)(f_x(t))$,

$$\begin{array}{ccccc}
\mathcal{R}^\wedge_{\mathcal{S}^{\mathrm{ord}}, x_0} & \xrightarrow{\Theta^*_x} & \mathcal{R}^\wedge_{\mathcal{S}'^{\mathrm{ord}}, x_0} & \xrightarrow{\sim} & \otimes_i \mathcal{R}^\wedge_{\mathcal{S}^{ord}_i, x^i_0} \\
\downarrow{\scriptstyle \beta_x} & & \downarrow{\scriptstyle \beta'_x} & & \downarrow{\scriptstyle \otimes \beta^i_x} \\
\mathbb{W}[[t]] \otimes (\mathcal{L}^2)^\vee & \xrightarrow{\mathbb{I} \otimes \epsilon^\vee} & \mathbb{W}[[t]] \otimes (\mathfrak{L}^2)^\vee & \xrightarrow{\sim} & \otimes_i (\mathbb{W}[[t]] \otimes (\mathfrak{L}^2_i)^\vee)
\end{array}$$

Corollary 5.4. *The restriction $\Theta^* f$ of a p-adic automorphic form f on GU to the subgroup G' vanishes if and only if $(\mathbb{I} \otimes \epsilon^\vee)(f_x(t)) = 0$.*

Remark 21. For any $g \in \mathrm{Levi}(\mathbb{Z}_p)$, if x is a point of the infinite Igusa tower in the image of Θ then gx is in the image of Θ if and only if $g \in \mathrm{Levi}'(\mathbb{Z}_p)$. If $g \in \mathrm{Levi}'(\mathbb{Z}_p)$ then

$$\Theta^*(g \cdot f) = g \cdot \Theta^* f \text{ and } (\mathbb{I} \otimes \epsilon^\vee \circ g)(f_x(t)) = (\mathbb{I} \otimes \epsilon^\vee)(f_{gx}(t)).$$

Remark 22. We briefly consider what happens when we choose to replace the projection $\theta^* \mathcal{E}_\kappa \to \mathcal{E}'_\kappa$, with a projection onto a different automorphic sheaf $\mathcal{E}_{\kappa'}$, for κ' the highest weight of an irreducible sub-representation of the restriction of ρ_κ from Levi to Levi$'$, other than κ. Note that κ' is not a dominant weight for T. We write

$$\mathrm{res}_{\kappa,\kappa'} : H^0(\mathcal{S}, \mathcal{E}_\kappa) \to H^0(\mathcal{S}', \theta^* \mathcal{E}_\kappa) \to H^0(\mathcal{S}', \mathcal{E}'_{\kappa'})$$

for the associated map on the space of p-adic automorphic forms. For $\kappa' \neq \kappa$, the maps $\Theta^* \circ \Psi_\kappa$ and $\Psi'_{\kappa'} \circ \mathrm{res}_{\kappa,\kappa'}$ do not agree.

We assume κ' is conjugate to κ under the action of the Weil group $W_{\mathrm{Levi}}(T)$, i.e. $\kappa' = \kappa^\sigma$ for some $\sigma \in W_{\mathrm{Levi}}(T)$. We choose $g = g_\sigma \in N_{\mathrm{Levi}}(T)(\mathbb{Z}_p)$ lifting σ (note that $g \notin \mathrm{Levi}'(\mathbb{Z}_p)$). Then, for all $f \in H^0(\mathcal{S}, \mathcal{E}_\kappa)$ we have

$$\Theta^*(g \cdot \Psi_\kappa(f)) = \Psi'_{\kappa'}(\mathrm{res}_{\kappa,\kappa'}(f)).$$

Thus, for any point x on the infinite Igusa cover in the image of Θ, we deduce

$$\Psi'_{\kappa'}(\mathrm{res}_{\kappa,\kappa'}(f))_x(t) = \Theta^*(g \cdot \Psi_\kappa(f))_x(t) = (\mathbb{I} \times \epsilon^\vee)(g \cdot \Psi_\kappa(f))_x(t) = (\mathbb{I} \times \epsilon^\vee \circ g)(\Psi_\kappa(f)_x(t)).$$

Note that we also have

$$\Psi'_{\kappa'}(\mathrm{res}_{\kappa,\kappa'}(f))_x(t) = (\mathbb{I} \times \epsilon^\vee)(g \cdot \Psi_\kappa(f))_x(t) = (\mathbb{I} \times \epsilon^\vee)(\Psi_\kappa(f)_{gx}(t)),$$

where the point gx is no longer in the image of Θ.

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