ON THE COHOMOLOGY OF CERTAIN PEL TYPE SHIMURA VARIETIES

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Abstract. In this paper, we study the local geometry at a prime p of PEL type Shimura varieties for which there is a hyperspecial level subgroup. We consider the Newton polygon stratification of the special fiber at p of Shimura varieties, and show that each Newton polygon stratum can be described in terms of the products of the reduced fibers of the corresponding PEL type Rapoport-Zink spaces with certain smooth varieties (which we call Igusa varieties), and of the action on them of a p-adic group which depends on the stratum. We then extend our construction to characteristic zero and, in the case of bad reduction at p, use it to compare the vanishing cycles sheaves of the Shimura varieties to the ones of the Rapoport-Zink spaces. As a result of this analysis, in the case of proper Shimura varieties, we obtain a description of the l-adic cohomology of the Shimura varieties, in terms of the l-adic cohomology with compact supports of the Igusa varieties and of the Rapoport-Zink spaces, for any prime $l \neq p$.

Contents

1. Introduction

PEL type Shimura varieties arise as moduli spaces of polarized abelian varieties endowed with the action of an algebra over $\mathbb{Q}$. These varieties play an important role in the Langlands' program as in some case the global Langlands correspondences are expected to be realized inside their l-adic cohomology.

In [26] Rapoport and Zink introduce some local analogues of PEL type Shimura varieties. These are rigid analytic spaces which arise as moduli spaces of Barsotti-Tate groups with additional structures and which provide a p-adic uniformization of the corresponding Shimura varieties. The l-adic cohomology of the Rapoport-Zink spaces is described by a conjecture of Kottwitz, which is "heuristically compatible" with the corresponding conjecture of Langlands and predicts that in some cases the l-adic cohomology of the Rapoport-Zink spaces realizes the local Langlands correspondences.

In [10], Harris and Taylor prove the local Langlands correspondence for GL_n by studying a certain class of PEL type Shimura varieties. In their work, they study the geometry of the reduction modulo p of the Shimura varieties by introducing the analogue in this context of Igusa curves, which they call Igusa varieties. These are smooth moduli spaces of abelian varieties in characteristic p which arise as finite étale covers of the fibers mod p of Shimura varieties with good reduction at p, and which are isomorphic (up to inseparable morphisms) to the smooth components of the fibers mod p of the Shimura varieties with bad reductions at p. This analysis relies on the fact that the deformation theory of the abelian varieties classified by the Shimura varieties they consider is controlled by Barsotti-Tate groups which are one dimensional.

For general PEL type Shimura varieties, the above assumption on the dimension of the pertinent Barsotti-Tate groups does not hold. It follows that, in the general context, while it is still possible to define the Igusa moduli problems the resulting varieties cover only certain subvarieties inside the reduction modulo p of a Shimura variety, not the entire space. On the other hand, these subvarieties are is in some sense "orthogonal" to the ones uniformized by the Rapoport-Zink spaces. More precisely, the Igusa varieties describe the loci where the isomorphism classes of the p-divisible parts of the abelian varieties are constant, while the Rapoport-Zink spaces uniformize the loci corresponding to p-prime isogeny classes of abelian varieties. Together, they cover the loci where the p-divisible parts of the abelian varieties have prescribed isogeny class, i.e. the Newton polygon strata of the reduction of the Shimura varieties.

It is our idea to study the local geometry at p of PEL type Shimura varieties via the Newton polygon stratification of their fibers mod p. By combining together the approach of Harris and Taylor and the one of Rapoport and Zink, we obtain a description of the geometry and cohomology of each stratum in terms of the geometry and cohomology of the products of the Igusa varieties with the reduced fibers of Rapoport-Zink spaces. Moreover, in the cases of bad reduction at p, our construction enable us to compare the vanishing cycles sheaves on the Shimura varieties to the ones on the Rapoport-Zink spaces.

In the case of proper Shimura varieties, we can apply the theory of vanishing cycles to translate these results into a formula describing the l-adic cohomology of the Shimura varieties, regarded as a virtual representation of $G(\mathbb{A}^\infty)$, the adelic points of the associated algebraic group $G/\mathbb{Q}$, and of W_{E_v}, the Weil group of the localization of the reflex field E at a prime v above p, in terms of the l-adic cohomology of the corresponding Rapoport-Zink spaces and Igusa varieties.

In [20], we carried out this plan for a special class of of PEL type Shimura varieties. In this paper, we extend those results to all PEL type Shimura varieties for which there exists a hyperspecial level subgroup (i.e. a level subgroup which is guaranteed to give good reduction) at a chosen prime p. In particular, we focus on the solution of the technical problems which arise when extending the class of Shimura varieties considered. We remark that all the main results of [20] can be recovered as special cases of the results in this paper, although with different (but equivalent) formulations. (In [20], the simplifying assumptions effected definitions and construction in a way that shadowed some of the results.)

Let us outline in detail the structure of this paper.

In section 2 we introduce the class of PEL type Shimura varieties we study. In section 3, we recall the definitions of the Newton polygon stratification of the reduction of the Shimura varieties modulo p and of Oort's foliation of the Newton polygon strata. Both constructions are borrowed from the general theory of moduli spaces of abelian varieties in positive characteristic and are originally due respectively to Grothendieck and Katz in [15], and Oort in [22]. In the context of Shimura varieties, their definitions require some adjustments (generalizations of the Newton polygon stratification are due to Kottwitz in [19], and Rapoport and Richartz in [25]). In section 4 we introduce the Igusa varieties as finite étale covers of the leaves of Oort's foliation. They arise as moduli spaces of abelian varieties in positive characteristic and their notion is originally due to Harris and Taylor in [10], who adapted Igusa's moduli problems for elliptic curves to abelian varieties. In our context, the generalization of the construction of Harris and Taylor relies on the notion of slope filtration for a Barsotti-Tate group (see [15]) and in particular on a result of Zink in [27]. The Igusa varieties are naturally equipped with two group actions, one which reflects the group action away from p on the Shimura varieties and the other which reflects changes of level in the Igusa structure. In section 5 we introduce the moduli spaces of Barsotti-Tate groups defined by Rapoport and Zink as the local analogues of PEL type Shimura varieties. We show how the geometry and cohomology of the Newton polygon strata of the Shimura varieties can be understood in terms of the geometry and cohomology of the reduced fibers of the Rapoport-Zink spaces and of the Igusa varieties. In particular, for each Newton polygon stratum, we construct and study a system of finite surjective morphisms from the product of the Igusa varieties with truncations of the reduced fiber of the corresponding Rapoport-Zink space onto the stratum. This concludes our analysis of the fibers in positive characteristic of Shimura varieties with good reduction. In sections 6 and 7, we focus our attention to the cases of bad reduction. Using Katz's and Mazur's notion of level structure, we construct some integral models in characteristic zero and investigate the possibility of extending the previous maps in positive characteristic to morphisms between formal schemes in character zero. This problem is strictly related to comparing the vanishing cycles of the Shimura varieties to the vanishing cycles of the corresponding Rapoport-Zink spaces (in the sense of Berkovich's [2] and [3]). In the case of proper Shimura varieties, our results on the reduction modulo p and on the vanishing cycles can be piece together into understanding the l-adic cohomology of the Shimura varieties, for any prime $l \neq p$ (see section 8).

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2. PEL type Shimura varieties

Let p be a prime number. The focus of our study is a certain class of PEL type Shimura varieties (introduced by Kottwitz in [18]) which have good reduction at all places of the reflex field lying above the prime p, in the cases when no level structure at p is considered.

Following [18] (Sec. 5, pp. 389–392), we consider the PEL moduli problems associated to the PEL data $(B, *, V, \langle\, , \rangle)$, where:

- B is a finite dimensional simple algebra over $\mathbb{Q}$;

- $*$ is a positive involution on B over $\mathbb{Q}$;
- V is a nonzero finitely generated left B-module;
- $\langle\,,\rangle$ is a non degenerate $\mathbb{Q}$-valued $*$-hermitian alternating pairing on V;

satisfying the following three conditions:

(1) there exists a $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B$ in B whose p-adic completion is a maximal order in $B_{\mathbb{Q}_p}$ and which is preserved by $*$;
(2) there exists a lattice Λ in $V_{\mathbb{Q}_p}$ which is self-dual for $\langle\,,\rangle$ and is preserved by $\mathcal{O}_B$;
(3) *(unramified hypothesis)* $B_{\mathbb{Q}_p}$ is a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$.

We remark that if F is the center of B, then condition (3) particular that the number field F is unramified at p.

To the above data we associate the algebraic group $G/\mathbb{Q}$ of the B-linear automorphisms of V which preserve the pairing $\langle\,,\rangle$ up to scalar multiple, and its subgroup $G_1/\mathbb{Q}$ of the automorphisms of V which preserve $\langle\,,\rangle$. More precisely, let C denotes the $\mathbb{Q}$-algebra $\mathrm{End}_B(V)$; it is a simple algebra over F endowed with an involution $\#$ coming from the pairing $\langle\,,\rangle$ on V. Then, for any $\mathbb{Q}$-algebra R

$$G(R) = \{x \in (C \otimes_{\mathbb{Q}} R)^{\times} \,|\, xx^{\#} \in R^{\times}\}$$

and

$$G_1(R) = \{x \in (C \otimes_{\mathbb{Q}} R)^{\times} \,|\, xx^{\#} = 1\}.$$

Finally, we also fix the datum of a morphism $h : \mathbb{C} \to C_{\mathbb{R}}$ such that $h(\bar{z}) = h(z)^{\#}$, for all $z \in \mathbb{C}$ and such that the symmetric $\mathbb{R}$-valued form $\langle \cdot, h(i)\cdot\rangle$ on $V_{\mathbb{R}}$ is positive definite.

The choice of such morphism h determines a decomposition of the $B_{\mathbb{C}}$-module $V_{\mathbb{C}}$ as $V_{\mathbb{C}} = V_1 \oplus V_2$, where V_1 (resp. V_2) is the subspace of $V_{\mathbb{C}}$ on which $h(z)$ acts as z (resp. $\bar{z}$). It follows from the definition that V_1, V_2 are $B_{\mathbb{C}}$-submodule of $V_{\mathbb{C}}$.

We denote by $E \subset \mathbb{C}$ the field of definition of the isomorphism class of the complex representation V_1 of B; E is called the reflex field.

Before introducing the PEL moduli problems associated to the above data, let us recall that they fall in three families (cases A, C and D) which are distinguished as follows. We consider the restriction of the involution $*$ of B to its center F, and denote by F_0 its fixed field, which is a totally real field since $*$ is positive on F. Then the group $G_1/\mathbb{Q}$ is obtained from an algebraic group G_0/F_0 by restriction of scalars from F_0 to $\mathbb{Q}$. Let n be the positive integer defined by $[F : F_0](\dim_F C)^{1/2}/2$.

If $F = F_0$, in which case we call the involution $*$ of the first kind, the associated algebraic group G_0 is of type A_{n-1}, and we refer to this as case A.

Otherwise, F is a totally complex quadratic extension of F_0, in which case $*$ is said of the second kind, and the group G_0 is either an orthogonal group in $2n$ variables (i.e. of type D_n) or a symplectic group in $2n$ variables (i.e. of type C_n); we refer to these as cases D and C, respectively.

In the case D, we assume $p \neq 2$. This assumption assures the smoothness of the p-adic integral models we define below ([18], Sec. 5, p. 391).

Finally, let us point out that calling the moduli spaces we define below Shimura varieties is a loose use of the terminolgy. In fact, in some cases (i.e. when the Hasse principle for $H^1(\mathbb{Q}, G)$ fails), these moduli spaces give Shimura varieties only after we pass to a connected component (see [18], Sec. 8, pp. 398–400).

Let $\mathbb{A}^\infty$ denote the ring of the finite adeles of $\mathbb{Q}$. To any open compact subgroup $U \subset G(\mathbb{A}^\infty)$, we associate a contravariant set-valued functor F_U on the category of locally Noetherian schemes S over E. We remark that it is enough to define $F_U(S)$ in the case when the scheme S is connected. Further more, we shall first define $F_U(S) = F_U(S,s)$ for a choice of a geometric point $s \in S$ and then observe that such set is independent on the choice of $s \in S$ (see [18], Sec. 5, p. 391).

To any connected locally Noetherian scheme S over E and a geometric point $s \in S$, the functor F_U associates the set of isomorphism classes of quadruples $(A, \lambda, i, \bar{\mu})$ where:

- A is an abelian scheme over S;
- $\lambda : A \to A^\vee$ is a polarization;
- $i : B \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ a morphism of $\mathbb{Q}$-algebras such that $\lambda \circ i(b^*) = i(b)^\vee \circ \lambda$ and $\det(b, \mathrm{Lie}(A)) = \det(b, V_1)$, for all $b \in B$;
- $\bar{\mu}$ is a $\pi_1(S,s)$-invariant U-orbit of isomorphisms of $B \otimes_{\mathbb{Q}} \mathbb{A}^\infty$-modules $\mu : V \otimes_{\mathbb{Q}} \mathbb{A}^\infty \to VA_s$ which takes the pairing $\langle\, ,\rangle$ on $V \otimes_{\mathbb{Q}} \mathbb{A}^\infty$ to a $(\mathbb{A}^\infty)^\times$-scalar multiple of the λ-Weil pairing.

Two quadruples $(A, \lambda, i, \bar{\mu})$ and $(A', \lambda', i', \bar{\mu}')$ are equivalent if there exists an isogeny $\beta : A \to A'$ which takes λ to a $\mathbb{Q}^\times$-multiple of λ', i to i' and $\bar{\mu}$ to $\bar{\mu}'$ (see [18], Sec. 5, p. 390).

We say that an open compact subgroup U of $G(\mathbb{A}^\infty)$ is sufficiently small if there exists a prime x in $\mathbb{Q}$ such that the projection of U in $G(\mathbb{Q}_x)$ contains no elements of finite order other than 1. If U is sufficiently small then the functor F_U is represented by a smooth quasi-projective scheme X_U defined over E (see [18], Sec. 5, p. 391).

As the level U varies, the Shimura varieties X_U form an inverse system, naturally endowed with an action of $G(\mathbb{A}^\infty)$. For all integers $i \geq 0$ and any prime l, we write

$$H^i(X, \mathbb{Q}_l) = \varinjlim_U H^i(X_U \times_E \bar{E}, \mathbb{Q}_l)$$

for the i-th cohomology group of the Shimura varieties, regarded as an l-adic representation of $G(\mathbb{A}^\infty) \times Gal(\bar{E}/E)$.

We now restrict our attention to levels U of the following form. Let K_0 be the maximal open compact subgroup of $G(\mathbb{Q}_p)$ defined by

$$K_0 = Stab_{G(\mathbb{Q}_p)}(\Lambda) = \{x \in C_{\mathbb{Q}_p} |\, x(\Lambda) \subset \Lambda,\, xx^\# \in \mathbb{Z}_p^\times\}.$$

For any open compact subgroup $U^p \subset G(\mathbb{A}^{\infty,p})$, we define

$$U^p(0) = U^p \times K_0 \subset G(\mathbb{A}^\infty).$$

It is an open compact subgroup of $G(\mathbb{A}^\infty)$. Moreover, if U^p is sufficiently small (i.e. there exists a prime $x \neq p$ in $\mathbb{Q}$ such that the projection of U^p in $G(\mathbb{Q}_x)$ contains no elements of finite order other than 1), then $U^p(0)$ is also sufficiently small.

For any level U^p away from p, we call the associated Shimura variety $X_{U^p(0)}$ a Shimura variety with no level structure at p. In [18], Kottwitz proves that these varieties admit smooth quasi-projective models over $\mathcal{O}_{E,(p)} = \mathcal{O}_E \otimes \mathbb{Z}_{(p)}$, which arise as follows.

We define a set-valued functor $\mathcal{F}_{U^p(0)}$ on the category of pairs (S,s), where S is a connected locally Noetherian $\mathcal{O}_{E,(p)}$-schemes and s is a geometric point on S. We set $\mathcal{F}_{U^p(0)}(S,s)$ to be the set of equivalence classes of quadruples $(A, \lambda, i, \bar{\mu}^p)$ where:

- A is an abelian scheme over S;

- $\lambda : A \to A^\vee$ is a prime-to-p polarization;
- $i : \mathcal{O}_B \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ a morphism of $\mathbb{Z}_{(p)}$-algebras such that $\lambda \circ i(b^*) = i(b)^\vee \circ \lambda$ and $\det(b, \mathrm{Lie}(A)) = \det(b, V_1)$ for all $b \in \mathcal{O}_B$;
- $\bar{\mu}^p$ is a $\pi_1(S, s)$-invariant U^p-orbit of isomorphisms of $B \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules $\mu^p : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \to V^p A_s$ which takes the pairing $<,>$ on $V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ to a $(\mathbb{A}^{\infty,p})^\times$-scalar multiple of the λ-Weil pairing (we denote by $V^p A_s$ the Tate space of A_s away from p).

Two quadruples $(A, \lambda, i, \bar{\mu}^p)$ and $(A', \lambda', i', (\bar{\mu}^p)')$ are equivalent if there exists a prime-to-p isogeny $\beta : A \to A'$ which takes λ to a $\mathbb{Z}_{(p)}^\times$-multiple of λ', i to i' and $\bar{\mu}$ to $(\bar{\mu}^p)'$.

The set $\mathcal{F}_{U^p(0)}(S, s)$ is canonically independent on s, and thus it gives rise to a set-valued functor on the category of locally Noetherian $\mathcal{O}_{E,(p)}$-schemes.

For a sufficiently small U^p, the functor $\mathcal{F}_{U^p(0)}$ on the category of locally Noetherian $\mathcal{O}_{E,(p)}$-schemes is represented by a smooth quasi-projective scheme $\mathcal{X}_{U^p(0)}$ over $\mathcal{O}_{E,(p)}$ (see [18], Sec. 5, p. 391). Further more, there is a canonical isomorphism

$$X_{U^p(0)} = \mathcal{X}_{U^p(0)} \times_{\mathrm{Spec}\, \mathcal{O}_{E,(p)}} \mathrm{Spec}\, E.$$

As the level U^p away from p varies, the varieties $\mathcal{X}_{U^p(0)}$ form an inverse system naturally endowed with an action of $G(\mathbb{A}^{\infty,p})$, which is compatible under the above identification with the action of $G(\mathbb{A}^\infty)$ on the Shimura varieties X_U.

We now fix a prime v of E above p, and denote by E_v the completion of E at v. Because of the unramified hypothesis, E_v is an unramified finite extension of $\mathbb{Q}_p$. For any level U^p (resp. U), we then regard $\mathcal{X}_{U^p(0)}$ (resp. X_U) as schemes over the ring of integers $\mathcal{O}_{E_v}$ of E_v (resp. over E_v).

Let $k/\mathbb{F}_p$ be the residue field of $\mathcal{O}_{E_v}$, $\#k = q = p^f$. We choose an algebraic closure $\bar{\mathbb{F}}_p$ of k and write $\sigma : \bar{\mathbb{F}}_p \to \bar{\mathbb{F}}_p$ for the Frobenius map $x \mapsto x^p$.

For any level $U^p \subset G(\mathbb{A}^{\infty,p})$, we denote the reduction of $\mathcal{X}_{U^p(0)}$ in characteristic p as $\bar{X}_{U^p(0)} = \mathcal{X}_{U^p(0)} \times_{\mathcal{O}_{E_v}} k$, and write $Fr = Fr_{\bar{X}_{U^p}}$ for the absolute Frobenius on $\bar{X}_{U^p(0)}$. Then, the f-th power of Fr on $\bar{X}_{U^p(0)}$ is the morphism $\bar{X}_{U^p(0)} \to \bar{X}_{U^p(0)}$ corresponding to the quadruple $(\mathcal{A}^{(q)}, \lambda^{(q)}, \iota, \bar{\mu}^p)$, for $(\mathcal{A}, \lambda, \iota, \bar{\mu}^p)$ the universal family over $\bar{X}_{U^p(0)}$.

3. The Newton polygon stratification and Oort's foliation

Let U^p be a sufficiently small open compact subgroup of $G(\mathbb{A}^{\infty,p})$. We write $\mathcal{X} = \mathcal{X}_{U^p(0)}$ and $\bar{X} = \bar{X}_{U^p(0)}/k$ for its reduction in characteristic p. Let $\mathcal{A}$ be the universal abelian variety over $\bar{X}$ and write $\mathcal{G} = \mathcal{A}[p^\infty]$. Then $\mathcal{G}$ is a Barsotti-Tate group endowed with a quasi-polarization and a compatible action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$, the p-adic completion of $\mathcal{O}_B$ in $B_{\mathbb{Q}_p}$. (We say that an action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ of a polarized Barsotti-Tate group is *compatible* if it satisfies the determinant condition as in [18].) In the following, we call a Barsotti-Tate group endowed with a quasi-polarization and of a compatible action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ a Barsotti-Tate group with *additional structures.*

It follows from Serre-Tate theorem that the Barsotti-Tate group $\mathcal{G}$ controls the geometry of the Shimura variety in characteristic p. Thus, we analyze the variety $\bar{X}$ by studying the behavior of $\mathcal{G}/\bar{X}$.

Following [19] and [25], we denote by $B(G)$ the set of σ-conjugacy classes in $G(L)$, where $L = \mathbb{Q}_p^{nr} \supset E_v$ is the complete maximal unramified extension of $\mathbb{Q}_p$. (We recall that any two elements $b_1, b_2 \in G(L)$ are said to be σ-conjugate if there exists

$g \in G(L)$ such that $b_1 = gb_2g^{-\sigma}$.) Equivalently, $B(G)$ is the set of isogenies classes over $\bar{\mathbb{F}}_p \supset k$ of polarized Barsotti-Tate groups, endowed with a compatible action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$. (We call an isogeny between two Barsotti-Tate groups with additional structures any isogeny between the underlying Barsotti-Tate groups which preserves the additional structures, i.e. which preserves the quasi-polarization up to a $\mathbb{Z}_p^\times$-multiple and which commutes with the $\mathcal{O}_{B_{\mathbb{Q}_p}}$-modules structures.) For any Barsotti-Tate group G with additional structures, we denote by $b(G)$ the corresponding isogeny class, viewed as an element in $B(G)$.

In [25], Rapoport and Richartz construct a partial ordering on the set $B(G)$, which we denote by $\geq$, and prove that for any element $b \in B(G)$, the set

$$\bar{X}^{[b]} = \{x \in \bar{X} \,|\, b(\mathcal{G}_{\bar{x}}) \geq b\} \subset \bar{X}$$

is a closed subset of $\bar{X}$.

Let us denote also by $\bar{X}^{[b]}$ the corresponding reduced closed subschemes of $\bar{X}$, for all $b \in B(G)$. They form a stratification by closed subschemes of $\bar{X}$, indexed by the elements in $B(G)$. We denote by $\bar{X}^{(b)} = \bar{X} - \cup_{b'>b}\bar{X}^{[b']}$ the corresponding open strata. They are the reduced subschemes overlying the loci where the Barsotti-Tate group $\mathcal{G}$ has constant isogeny class,

$$\bar{X}^{(b)} = \{x \in \bar{X} \,|\, b(\mathcal{G}_{\bar{x}}) = b\} \subset \bar{X}.$$

Let us remark that, as the level U^p varies, the above stratification is preserved by the natural projection between Shimura varieties $\bar{X}_{U^p(0)} \to \bar{X}_{V^p(0)}$ (for $U_p \subset V_p$) and also by the action of $G(\mathbb{A}^{\infty,p})$, i.e. by the morphisms $g : \bar{X}_{U^p(0)} \to \bar{X}_{gU^p(0)g^{-1}}$ for all $g \in G(\mathbb{A}^{\infty,p})$.

Finally, it is obvious by the definitions that the Newton polygon stratification of $\bar{X}$ (i.e. the stratification defined by considering the isogeny class of the underlying Barsotti-Tate groups) is always coarser than the above stratification. While in general the stratification introduced by Rapoport and Richartz is finer, in the case of our interest, the two stratification are actually the same (see [25], Thm. 3.8, p. 173).

We now proceed by studying the open strata $\bar{X}^{(b)}$, for $b \in B(G)$. In [22], Oort introduces a foliation of the Newton polygon strata of moduli spaces of abelian varieties in positive characteristic p. He defines it by considering the loci where the isomorphism class of the Barsotti-Tate group of the abelian varieties is constant. These are closed subspaces of the Newton polygon strata which give rise to smooth schemes, when endowed with the induced reduced structure.

As we already remarked, in the context of the reduction of PEL type Shimura varieties, the Barsotti-Tate groups of the abelian varieties classified by the moduli problem are naturally endowed with additional structures inherited by the ones on the abelian varieties. In view of this, we refine the definition of Oort's foliation in this context as follows.

Proposition 1. *Let $\Sigma/\bar{\mathbb{F}}_p \supset k$ be a polarized Barsotti-Tate group endowed with a compatible action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$, and denote by $b \in B(G)$ its isogeny class. We define*

$$C_\Sigma = \{x \in \bar{X} | \mathcal{G}_{\bar{x}} \simeq \Sigma_{\overline{k(x)}}\} \subset \bar{X}^{(b)} \times_k \bar{\mathbb{F}}_p.$$

This is closed subset of the stratum $\bar{X}^{(b)}$ and as a subscheme of $\bar{X}^{(b)} \times_k \bar{\mathbb{F}}_p$ endowed with the induced reduced structure is smooth.

Proof. Let us assume for the moment that the leaves are closed subsets of the corresponding Newton polygon stratum, and endow them with the induced reduced structure. Then it is a direct consequence of the definition that they are smooth. In fact, it follows from the Serre-Tate theorem that the complete local rings at the geometric closed points $x \in C_\Sigma$ are the same for all x (as they depend only on the isomorphisms class of $\Sigma/\bar{\mathbb{F}}_p$). In particular, all points $x \in C_\Sigma$ are smooth.

We now show that the leaves are closed inside the corresponding Newton polygon stratum. In [22] (Thm. 2.2, p. 273 and Thm. 3.3, p. 275) Oort considers both the cases of Barsotti-Tate groups and polarized Barsotti-Tate groups. (The case of non polarized Barsotti-Tate groups was also discussed in [20], Prop. 2.7, p. 219 and Prop. 4.7, p. 265.) In order to prove that these generalized leaves are closed inside the Newton polygon stratum, one may adapt the arguments given in those cases to the general case, by simply restricting the consideration to morphisms between Barsotti-Tate groups which commutes with the additional structures. (It suffices to remark that the condition that a morphisms of Barsotti-Tate groups commutes with the additional structures is a closed condition on the base.)

Otherwise, one may deduce the general statement from the old ones by observing that the each leaf C_Σ is the union of the connected components of the corresponding old leaves, meaning the ones obtained by fixing the isomorphism class of the underlying Barsotti-Tate groups, whose generic point η satisfies the condition that there exists an isomorphism of Barsotti-Tate groups with additional structures $\mathcal{G}_\eta \times_{K(\eta)} \bar{K} \simeq \Sigma_{\bar{K}}$, for some algebraically closed field $\bar{K}$ containing $K(\eta)$, the field of definition of the generic point η (cfr. proof of Thm 3.3 in [22]).

Thus, in particular, also the property of smoothness can be established as a consequence of the analogous property of the old leaves. □

We remark that the refined foliations of the Newton polygon strata we have introduced are finer than the regular foliations defined in [22], i.e. the foliations obtained by fixing the isomorphism class of the underlying Barsotti-Tate groups. In particular, as remarked in the above proof, the refined leaves are union of connected components of the corresponding regular one.

It is an easy consequence of the definition, that, as the level U^p varies, the leaves of Oort's foliation are preserved by both the natural projection between Shimura varieties $\bar{X}_{U^p(0)} \to \bar{X}_{V^p(0)}$ (for $U_p \subset V_p$) and the action of $G(\mathbb{A}^{\infty,p})$.

The leaves of Oort's foliations play an important role in our study. A key fact to our application is the existence of the slope filtration for the universal Barsotti-Tate group $\mathcal{G}$ when restricted to the leaves. The notion of slope filtration for Barsotti-Tate groups is originally due to Grothendieck who proved its existence for any Barsotti-Tate groups over a field in [9]. This result was later extended by Katz ([15]) to Barsotti-Tate groups over a smooth curve, and more recently by Zink ([27]) over any regular scheme and by Oort and Zink ([23]) over a normal base. It is an easy application of Zink's work to this context that the universal Barsotti-Tate group $\mathcal{G}$ admits a slope filtration when restricted to the leaves (see [20], section 3.2.3, p. 239). Here below we focus our attention on the interaction between slope filtration and additional structures.

Let $\Sigma/\bar{\mathbb{F}}_p$ be a completely slope divisible Barsotti-Tate group with additional structures. We write $b \in B(G)$ for its isogeny class and $1 \geq \lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 0$ for the slopes of its Newton polygon (which depends only on b). We recall that a

Barsotti-Tate group over a algebraically closed field is completely slope divisible if and only if is isomorphic to the direct sum of isoclinic slope divisible Barsotti-Tate groups defined over finite fields (see [23], Cor. 1.5, p. 187). Thus Σ admits a slope decomposition: $\Sigma = \oplus_i \Sigma^i$, where for each i, $1 \leq i \leq k$, Σ^i is a slope divisible isoclinic Barsotti-Tate group of slope λ_i. We fix an integer $B \geq 1$ such that, for all $i = 1, \dots, k$, $F^B p^{-\lambda_i B} : \Sigma^i \to \Sigma^{i(p^B)}$ are well defined isogenies. Then, they are indeed isomorphisms. We write $\nu = \oplus_i F^B p^{-\lambda_i B} : \Sigma \simeq \Sigma^{(p^B)}$ for the corresponding isomorphism on Σ.

We are interested in understanding how the additional structures of Σ reflect on its isoclinic components. From the equality $Isog(\Sigma) = \prod_i Isog(\Sigma^i)$, we observe that the datum of an action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ on Σ is equivalent to the data of an action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ on each isoclinic piece Σ^i, for all i. Further more, let us consider the datum of a polarization $\ell : \Sigma \to \Sigma^\vee$ on Σ. The decomposition $\Sigma = \oplus_i \Sigma^i$ give rise to a decomposition of the dual Barsotti-Tate group $\Sigma^\vee = \oplus_i (\Sigma^i)^\vee$, where each $(\Sigma^i)^\vee$ is a slope divisible isoclinic Barsotti-Tate group, of slope $1 - \lambda_i$. Thus, the datum of a quasi-polarization ℓ on Σ is equivalent to the data of some $\mathcal{O}_{B_{\mathbb{Q}_p}}$-equivariant isomorphisms $\ell^i : \Sigma^i \to (\Sigma^j)^\vee$, for all i, j such that $\lambda_i + \lambda_j = 1$, with the property that $(\ell^i)^\vee = c(\ell)\ell^j$, for some constant $c(\ell) \in \mathbb{Z}_{(p)}^\times$ independent of i. (In particular, the existence of a polarization on Σ implies that its Newton polygon is symmetric, i.e. if λ is a slope of $N(\Sigma)$ with multiplicity r_i, then $1-\lambda$ is also a slope of $N(\Sigma)$ with the same multiplicity.) Finally, we remark that, for any quasi-self-isogeny $\rho = \oplus_i \rho^i$ of Σ, ρ preserves the additional structures on Σ if and only if the quasi-self-isogenies ρ^i are $\mathcal{O}_{B_{\mathbb{Q}_p}}$-equivariant and $(\rho^j)^\vee \ell^i \rho^i = c(\rho) p^{2\deg(\rho)} \ell^i$, for some $c(\rho) \in \mathbb{Z}_{(p)}^\times$ and all i, j such that $\lambda_i + \lambda_j = 1$ (or equivalently such that $i + j = k + 1$).

Let $C = C_\Sigma$ be the leaf inside the stratum $\bar{X}^{(b)}$ associated to a completely slope divisible Barsotti-Tate group Σ with additional structures, and consider the universal Barsotti-Tate group $\mathcal{G}$ over C. Then $\mathcal{G}$ is a completely slope divisible Barsotti-Tate group with slope filtration

$$0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_k = \mathcal{G}$$

where the sub-quotients $\mathcal{G}^i = \mathcal{G}_i / \mathcal{G}_{i-1}$ are isoclinic slope divisible Barsotti-Tate groups of slope λ_i (see [20], section 3.2.3, p. 239). Since any morphism between completely slope divisible Barsotti-Tate group preserves the slope filtration, we deduce that both the filtrating Barsotti-Tate groups $\mathcal{G}_i$ and the sub-quotients $\mathcal{G}^i$ inherit an action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$. Further more, let $\mathcal{G}^\vee / C$ be the Barsotti-Tate group dual to $\mathcal{G}$ and denote by $\ell : \mathcal{G} \to \mathcal{G}^\vee$ the quasi-polarization on $\mathcal{G}$. Then, we can endow $\mathcal{G}^\vee$ with an action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$, defined by the isogenies $(a^*)^\vee$, for all $a \in \mathcal{O}_{B_{\mathbb{Q}_p}}$, and regard ℓ as a $\mathcal{O}_{B_{\mathbb{Q}_p}}$-equivariant isomorphism. Furthermore, the dual Barsotti-Tate group $\mathcal{G}^\vee$ is also completely slope divisible and for each i, $1 \leq i \leq k$, we can identify its i-th isoclinic sub-quotient

$$(\mathcal{G}^\vee)^i = (\mathcal{G}^j)^\vee,$$

for $j \in \{1, \dots, k\}$ such that $\lambda_i + \lambda_j = 1$. Thus, the datum of an quasi-polarization $\ell : \mathcal{G} \to \mathcal{G}^\vee$ gives rise to some $\mathcal{O}_{B_{\mathbb{Q}_p}}$-equivariant isomorphisms

$$\ell^i : \mathcal{G}^i \to (\mathcal{G}^\vee)^i = (\mathcal{G}^j)^\vee,$$

for all i, j such that $\lambda_i + \lambda_j = 1$, with the property that $(\ell^i)^\vee = c(\ell)\ell^j$, for some constant $c(\ell) \in \mathbb{Z}_{(p)}^\times$.

We remark that the two data are not equivalent, i.e. it is not true in general that any such collection of isomorphisms glue together to a quasi-polarization on $\mathcal{G}$ (nor that an action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ on the isoclinic subquotients piece together to an action on $\mathcal{G}$). Nevertheless, it is true that any quasi-isogeny between two completely slope divisible Barsotti-Tate groups with slope filtrations commutes with the additional structures if and only if the induced quasi-isogenies among the corresponding slope divisible isoclinic sub-quotients commute with the inherited additional structures.

On the other hand, the following also holds.

Proposition 2. *Let $x \in \bar{X}^{(b)}$, $\Sigma = \oplus_i \Sigma^i$ a complete slope divisible Barsotti-Tate group with additional structures, defined over $\bar{\mathbb{F}}_p$ and $C = C_\Sigma \subset \bar{X}^{(b)}$. Then $x \in C$ if and only if for all $i = 1, \dots, k$ there exist $\mathcal{O}_{B_{\mathbb{Q}_p}}$-equivariant isomorphisms*

$$\mathcal{G}^i{}_{\bar{x}} \simeq \Sigma^i{}_{\overline{k(x)}}$$

which commute with the isomorphisms ℓ^i induced by the quasi-polarizations on $\mathcal{G}_{\bar{x}}$ and Σ, up to $\mathbb{Z}_{(p)}^\times$-multiple.

Proof. One of the two implications is obvious, and the converse follows from the fact that over any perfect field the slope filtration of a Barsotti-Tate group canonically splits (i.e. $\mathcal{G}_{\bar{x}} = \oplus_i \mathcal{G}^i{}_{\bar{x}}$). □

4. Igusa varieties

As in the previous section, we fix a level U^p away from p and consider the reduction mod p, $\bar{X} = \bar{X}_{U^p(0)}$, of the corresponding Shimura variety with no level structure at p. We also fix a complete slope divisible Barsotti-Tate group Σ as above, and denote by $C_b = C_\Sigma \subset \bar{X}^{(b)}$ the corresponding leaf of Oort's foliation ($b = b(\Sigma) \in B(G)$).

For any $m \geq 1$, we write $J_{b,m}$ for the Igusa variety of level m over C_b (sometimes, we also write $J_{b,0} = C_b$). We define the Igusa varieties as the universal spaces for the existence of trivializations of the p^m-torsion of the isoclinic sub-quotients of $\mathcal{G}/C_b$, and we will show that they are finite étale Galois covers of the leaf C_b. (The property of being étale is a direct consequence of the fact that we are trivializing each isoclinic sub-quotient separately.) We recall that the notion of Igusa variety is originally due to Harris and Taylor in [10] (inspired by the work of Igusa, [11]). We adapt their definition to our context.

Definition 3. *For any integer $m \geq 1$, the Igusa variety of level m $J_{b,m} \to C_b$ is the universal space for the existence of isomorphisms*

$$j_{m,i}^{\text{univ}} : \Sigma^i[p^m] \to \mathcal{G}^i[p^m]$$

of finite flat group schemes over C_b such that

1. *they extend étale locally to any level $m' \geq m$;*
2. *they are $\mathcal{O}_{B_{\mathbb{Q}_p}}$-equivariant;*
3. *they commute up to $(\mathbb{Z}/p^m)^\times$-multiple with the isomorphisms induced by the quasi-polarizations on $\mathcal{G}$ and Σ.*

The existence of a universal space for the existence of isomorphisms $\Sigma^i[p^m] \simeq \mathcal{G}^i[p^m]$ satisfying the first condition is proved in [20] (Prop 3.3, p. 240). Thus, in order to conclude the existence of the Igusa varieties, it is enough to observe that the locus where conditions (2) and (3) are satisfied is a closed subset of such space.

It is clear from the definition that the Igusa variety $J_{b,m}$ is equipped with an action of the group of automorphism of Σ, $\Gamma_b = \mathrm{Aut}\,(\Sigma) = \prod_i \mathrm{Aut}\,(\Sigma^i)$.

For any $\gamma = \oplus_i \gamma^i \in \Gamma_b$, we define its action on the Igusa varieties $J_{b,m}$ as

$$(A, \lambda, i, \overline{\mu^p}; j_{m,1}, \dots jm, k) \mapsto (A, \lambda, i, \overline{\mu^p}; j_{m,1} \circ \gamma^1_{|[p^m]}, \dots j_{m,k} \circ \gamma^k_{|[p^m]}),$$

where we denote a point in $J_{b,m}$ by the data of the corresponding abelian variety endowed with additional structures $(A, \lambda, i, \overline{\mu^p})$ and isomorphisms $(j_{m,1}, \dots j_{m,k})$ as in the definition of the Igusa varieties. (It is obvious that if the collection of isomorphisms $\{j_{m,i}\}_i$ satisfies the required properties, so does the collection $\{j_{m,i} \circ \gamma^i_{[p^m]}\}_i$, for any $\gamma \in \Gamma_b$.) It follows from the definition that this action naturally factors through the quotient $\Gamma_{b,m}$ of Γ_b by the subgroup of automorphisms of Σ which restrict to the identity on $\Sigma[p^m]$.

Proposition 4. *For any $m \geq 1$, the Igusa variety $J_{b,m} \to C_b$ is finite étale and Galois, with Galois group $\Gamma_{b,m}$. In particular, the Igusa varieties are smooth.*

Proof. Following [10] (Prop. II.1.7, p.69), it suffices to prove that for any geometric closed point $x \in C_b$ there exist some isomorphisms

$$\mathcal{G}^i \times \mathrm{Spec}\, \mathcal{O}^\wedge_{C_b,x} \simeq \Sigma^i \times \mathrm{Spec}\, \mathcal{O}^\wedge_{C_b,x} \qquad \forall i = 1, \dots, k,$$

which commute with the additional structures.

Let x be a closed geometric point of C_b. It follows from the definition of the leaves and the rigidity of isoclinic Barsotti-Tate groups that the Barsotti-Tate groups underlying $\mathcal{G}^i$ and Σ^i are isomorphic over $\mathrm{Spec}\, \mathcal{O}^\wedge_{C_b,x}$, for all i (see [20], Lemma 3.4, p. 240). Moreover, since $x \in C_b$, we can always choose some isomorphisms

$$\phi^i : \mathcal{G}^i \times \mathcal{O}^\wedge_{C_b,x} \simeq \Sigma^i \times \mathcal{O}^\wedge_{C_b,x} \qquad \forall i = 1, \dots, k,$$

such that their fibers at the point x commute with the additional structures.

Finally, we observe that for any given system of isomorphisms ϕ^i of the underlying Barsotti-Tate groups, the commutativity properties can be expressed as equalities of certain corresponding self-isogenies of the $\Sigma^i \times \mathcal{O}^\wedge_{C_b,x}$. Since all the self-isogenies of the Σ^i over $\mathcal{O}^\wedge_{C_b,x}$ are already defined over $\bar{\mathbb{F}}_p$, any equality among them holds over $\mathcal{O}^\wedge_{C_b,x}$ if and only if it holds at the point x. □

Let us now consider Igusa varieties of different levels. It is easy to see that there are natural projections

$$q_b = q_{b,m',m} : J_{b,m'} \to J_{b,m} \quad \forall m' \geq m,$$

which correspond to restricting the isomorphisms $j_{m',i}$ to the p^m-torsion subgroups, for all i. Moreover, it follows from the definition that the projections q_b are invariant under the action of Γ_b on the Igusa varieties.

Thus, as the level m varies, the Igusa varieties $J_{b,m}$ form a projective system under the morphisms q_b, which is endowed with an action of Γ_b. We show that this action naturally extends to a sub-monoid $S_b \supset \Gamma_b$ of the group T_b of the quasi-self-isogenies of Σ preserving its additional structures.

We define S_b as follows. For any $\rho \in T_b$, we write ρ^i for the quasi-self-isogeny of Σ^i induced by the restriction of ρ. Thus $\rho = \oplus_i \rho^i$. Let us suppose that ρ^{-1} is an isogeny. Then, for each $i = 1, \dots, k$, we define two integers $e_i = e_i(\rho) \geq f_i = f_i(\rho) \geq 0$ to be respectively the minimal and maximal integers such that

$$\ker[p^{f_i}] \subset \ker[{\rho^i}^{-1}] \subset \ker[p^{e_i}].$$

Finally we define

$$S_b = \{\rho \in T_b |\, \rho^{-1} \text{ is an isogeny, } f_{i-1}(\rho) \geq e_i(\rho)\, \forall i \geq 2\}.$$

Let us remark that the inequalities defining S_b are not all independent. More precisely, for any $\rho \in T_b$ and indexes i, j such that $i + j = k + 1$, the corresponding inequalities here above are equivalent (this follows from the property $\rho^\vee \ell \rho = c(\rho) p^{2\deg(\rho)} \ell$). It is easy to see that S_b is a sub-monoid of T_b and that $p^{-1}, fr^{-B} = \oplus_i p^{-\lambda_i B} \in S_b$. Further more, we have that $T_b = \langle S_b, p, fr^B \rangle$ (see [20], Lemma 2.11, p. 222).

The action of Γ_b on the system of Igusa varieties extend to $S_b \subset T_b$.

Lemma 5. *Let $(\mathcal{A}, \lambda, i, \overline{\mu^p}; j_{m,1}, \dots jm, k)$ denote the universal object over the Igusa variety $J_{b,m}$, $\rho = \oplus_i \rho_i \in S_b$ and assume $m \geq e = e_1(\rho)$.*

There exists a unique finite flat subgroup $\mathcal{K}_\rho \subset \mathcal{G}[p^e]$ such that the corresponding subgroups inside the isoclinic sub-quotients $\mathcal{G}^i$ of $\mathcal{G}$ are $j_{m,i}(\ker(\rho_i^{-1}))$, for all i.

Moreover, the quotient abelian variety $\mathcal{A}/\mathcal{K}_\rho$ inherits the additional structures of $\mathcal{A}/J_{b,m}$ and thus gives rise to a morphism between the Igusa varieties

$$\rho : J_{b,m} \to J_{b,m-e}.$$

Proof. The existence of a finite flat subgroups $\mathcal{K}_\rho$ piecing together the subgroups $\mathcal{K}_\rho^i = j_{m,i}(\ker(\rho_i^{-1}))$ can be proved by induction on the number of slopes k. We refer to [20] (Lemma 3.6, p. 243) for a detailed proof. Here we simply recall the construction of $\mathcal{K}_\rho$ for $k = 2$ (the essential idea of the proof being well represented by this case). Let $k = 2$, we write $\iota : \mathcal{G}^1 \to \mathcal{G}$ and $pr : \mathcal{G} \to \mathcal{G}^2$ for the natural inclusion and projection respectively. Then

$$\mathcal{K}_\rho = pr^{-1}_{|\mathcal{G}[p^{e_2}]}(\mathcal{K}_\rho^2) + \iota(\mathcal{K}_\rho^1) \subset \mathcal{G}[p^e].$$

Let us now consider the abelian variety $\mathcal{A}/\mathcal{K}_\rho$. In order to conclude we need to show that it inherits the additional structures of $\mathcal{A}$, via the isogeny $\hat{\rho} : \mathcal{A} \to \mathcal{A}/\mathcal{K}_\rho$.

We observe that it is always possible to define such structures, namely a compatible action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ and a polarization, on the quotient via quasi-isogenies (i.e. to define an action of an element $b \in \mathcal{O}_{B_{\mathbb{Q}_p}}$ on $\mathcal{A}/\mathcal{K}_\rho$ as $\hat{\rho} \circ i(b) \circ \hat{\rho}^{-1}$, where $i(b)$ denotes the action of b on $\mathcal{A}$, and similarly a polarization). Therefore, it suffices to show that these quasi-isogenies are well defined isogenies, and this can be checked by looking at their restrictions to the p^∞-torsion subgroup $\mathcal{G}/\mathcal{K}_\rho \subset \mathcal{A}/\mathcal{K}_\rho$. Even more, it is enough to consider the corresponding restrictions to the isoclinic sub-quotients. Then, it follows from the construction of $\mathcal{K}_\rho$ and the fact that $\rho \in T_b$ that when looking at the isoclinic sub-quotients $\mathcal{G}^i \to (\mathcal{G}/\mathcal{K}_\rho)^i = \mathcal{G}^i/\mathcal{K}_\rho^i$ the induced structures are indeed well defined isogenies.

Finally, it is clear that, for any i, the Barsotti-Tate group $\mathcal{G}^i/\mathcal{K}_\rho^i$ is pointwise isomorphic to Σ^i, and moreover the isomorphism $j_{m,i}$ induces a trivializations of its $[p^{m-e}]$-torsion subgroup. Equivalently, we see that the the quotient abelian variety belongs to the leaf C_b and that it inherits an Igusa structure of level $m - e$. □

It is easy to check that, for all $\rho \in S_b$, the morphisms $\rho : J_{b,m} \to J_{b,m-e}$ are compatible with the projections among Igusa varieties of different levels $m \geq e = e_1(\rho)$, and that they give rise to an action of the monoid S_b on the system of Igusa varieties which extends the action of Γ_b.

In a similar way, it is possible to define a σ-semi-linear action of Frobenius on the Igusa varieties as arising from the Frobenius morphism of the Barsotti-Tate group Σ, $F : \Sigma \to \Sigma^{(p)}$, i.e. to define

$$Frob : J_{b,m} \to J_{b,m-1}$$

to be the σ-semi-linear morphism corresponding to the quotient abelian variety $\mathcal{A}^{(p)} = \mathcal{A}/\mathcal{A}[F]$ endowed with the induced additional structures. Then, $Frob$ is simply the absolute Frobenius of $J_{b,m}$ followed by the projection $q_{b,m,m-1}$.

Let us now reconsider the above constructions as the level away from p varies. We reintroduced in our notation the level U^p away from p, and write

$$J_{b,U^p,m} \to C_{b,U^p(0)} \subset \bar{X}^{(b)}_{U^p(0)} \subset \bar{X}_{U^p(0)}$$

for the Igusa varieties of level U^p away from p and level m at p. We already remarked that, as the level U^p varies, the leaves of Oort's foliation form a system endowed with an action of the group $G(\mathbb{A}^{\infty,p})$ inherited from the action on the Shimura varieties $X_{U^p(0)}$. It follows from the defining universal properties that the same holds for the Igusa varieties. More precisely, for any $b \in B(G)$ and $m \geq 1$, we define an action of the group $G(\mathbb{A}^{\infty,p})$ on the Igusa varieties of level m at p (as the level U^p away from p varies) as

$$\forall g \in G(\mathbb{A}^{\infty,p}) : J_{b,m,U^p} \to J_{b,m,U^p}$$

$$(A, \lambda, i, \overline{\mu^p}; j_{m,1}, \ldots, j_{m,k}) \mapsto (A, \lambda, i, \overline{\mu^p \circ g}; j_{m,1}, \ldots, j_{m,k}).$$

This action is clearly compatible under the projection maps with the action of $G(\mathbb{A}^{\infty,p})$ on the leaves $C_{b,U^p(0)}$. Further more, it commutes with the natural projections among the Igusa varieties, as the level at p varies, and also with the previously defined action of S_b.

Thus, we may consider the Igusa varieties $J_{b,U^p,m}$ as a projective system, indexed by the levels U^p, m, endowed with the action of the monoid $S_b \times G(\mathbb{A}^{\infty,p})$.

We now focus our attention on the l-adic cohomology with compact supports of the Igusa varieties, for $l \neq p$ a prime number. As the Igusa varieties form a projective system endowed with an action of $S_b \times G(\mathbb{A}^{\infty,p})$, their cohomology groups naturally form a direct system

$$H^i_c(J_b, \mathbb{Q}_l) = \varinjlim_{U^p,m} H^i_c(J_{b,m,U^p}, \mathbb{Q}_l) \qquad \forall i$$

also endowed with an action of the sub-monoid $S_b \times G(\mathbb{A}^{\infty,p}) \subset T_b \times G(\mathbb{A}^{\infty,p})$. We show that, while this action cannot be extended to an action of $T_b \times G(\mathbb{A}^{\infty,p})$ on the varieties, it is possible to do so on the cohomology groups. Because of the equality $T_b = \langle S_b, p, fr^B \rangle$, it suffices to prove that the action of $p^{-1}, fr^{-B} \in S_b$ on the cohomology groups of the Igusa varieties is invertible.

Lemma 6. *Maintaining the above notations.*

1. *For any* $m \geq 1$, $(p^{-1}, 1) \in S_b \times G(\mathbb{A}^{\infty,p})$ *acts on the Igusa varieties as* $(p, 1) \circ q_{b,m,m-1}$.
2. *For any* $m \geq \lambda_1 B$, $(fr^{-B}, 1) \in S_b \times G(\mathbb{A}^{\infty,p})$ *acts on the Igusa varieties as* $q_{b,m-\lambda_1 B,m-B} \circ Frob$.

Proof. Part (1) is obvious. Part (2) follows from the definition of slope, i.e. from the equalities $\Sigma^i[p^{\lambda_i B}] = \Sigma^i[F^B]$, for all i. □

Proposition 7. *The action of the sub-monoid $S_b \times G(\mathbb{A}^{\infty,p})$ on the cohomology groups of the Igusa varieties extends uniquely to an action of the group $T_b \times G(\mathbb{A}^{\infty,p})$.*

Moreover, as representations of $T_b \times G(\mathbb{A}^{\infty,p})$, the $H^i_c(J_b, \mathbb{Q}_l)$ are admissible, for all $i \geq 0$.

Proof. The extendibility of the action follows directly from the previous lemma. The admissibility of the representations from the equalities

$$H^i_c(J_b, \mathbb{Q}_l)^{\Gamma^m_b \times U^p} = H^i_c(J_{b,U^p,m}, \mathbb{Q}_l),$$

where Γ^m_b is the subgroup of T_b consisting of the isomorphisms of Σ which restrict to the identity on $\Sigma[p^m]$. □

5. The geometry of the Newton polygon strata

In the previous section, we introduced the Igusa varieties as certain coverings of the leaves of Oort's foliation of the Newton polygon strata of the reduction of the Shimura varieties. The leaves are obtained by isolating the abelian varieties with p-divisible part in a prescribed isomorphism class.

On the other hand, one may consider inside a Newton polygon stratum the isogeny class of a given abelian variety. In [26], Rapoport and Zink associate to each Newton polygon stratum (i.e. to each $b \in B(G)$) a moduli space $\mathcal{M}_b$ of Barsotti-Tate groups with additional structures whose reduced fiber may be regarded as a cover of the isogeny classes of the abelian varieties in the stratum.

In this section, we show how these two constructions can be pieced together to realize the products of the Igusa varieties with the Rapoport-Zink spaces as a system of covers of the Newton polygon strata, and how the Newton polygon strata can then be viewed as the quotients of these spaces under the action of the groups T_b, $b \in B(G)$.

More precisely, we construct a system of finite morphisms

$$\pi_N : J_{m,b} \times \bar{\mathcal{M}}^{n,d}_b \to \bar{X}^{(b)} \times_k \bar{\mathbb{F}}_p,$$

indexed by positive integers m, n, d, N ($m \geq d$ and $N \geq d/\delta f$, where $\delta \in \mathbb{Q}$ is a numeric invariant of $b \in B(G)$), which are compatible under the projections among the Igusa varieties and the inclusion among the truncated Rapoport-Zink spaces $\bar{\mathcal{M}}^{n,d}_b$, and also T_b-invariant and W_{E_v}-equivariant. Moreover, we prove that for any geometric closed point x of $\bar{X}^{(b)}$ the fibers of π_N at x form a T_b-principal homogeneous space. The machinery developed in [20] enable us to deduce from the existence of a system of morphisms with these property the existence of a spectral sequence of representations of the Weil group computing the cohomology with compact supports of the strata $\bar{X}^{(b)}_{U^p(0)}$ in terms of the cohomology with compact supports of the corresponding Igusa varieties and Rapoport-Zink spaces and of the action on them of the group T_b, for all $b \in B(G)$ and $U^p \subset G(\mathbb{A}^{\infty,p})$.

We start by recalling the definition of the formal scheme $\mathcal{M}_b$ over $\hat{\mathbb{Z}}^{nr}_p = W(\bar{\mathbb{F}}_p) \supset \mathcal{O}_{E_v}$ associated by Rapoport and Zink to a decent isogeny class $b \in B(G)$. Let us recall that an isogeny class b is said to be decent if contains a Barsotti-Tate group defined over a finite field. In [23] (Cor. 1.5, p. 187) Oort and Zink prove that this is equivalent to containing a completely slope divisible Barsotti-Tate group. For convenience, we choose a completely slope divisible Barsotti-Tate group $\Sigma = \Sigma_b$ in the isogeny class b.

For any scheme $S/\hat{\mathbb{Z}}_p^{nr}$ where p is locally nilpotent, $\mathcal{M}_b(S)$ is the set of isomorphism classes of Barsotti-Tate groups with additional structures H/S together with a quasi-isogeny $\beta : \Sigma \times_{\bar{\mathbb{F}}_p} \bar{S} \to H \times_S \bar{S}$ defined over $\bar{S} = Z(p) \subset S$.

The group T_b naturally acts on the Rapoport-Zink space $\mathcal{M}_b$ by right translations

$$\forall \rho \in T_b : \quad (H, \beta) \mapsto (H, \beta \circ \rho).$$

Similarly, one can defined a σ-semi-linear isomorphism of $\mathcal{M}_b$ arising from the Frobenius of Σ

$$Frob : \mathcal{M}_b \to \mathcal{M}_b$$

$$(H, \beta) \mapsto (H, F^{-1} \circ \beta),$$

which commutes with the action of T_b.

Let us recall that the formal scheme $\mathcal{M}_b$ is only locally of finite type. Because of this, it is technically convenient to focus our attention on certain subspaces of $\mathcal{M}_b$ (which we call truncated Rapoport-Zink spaces) which are of finite type, and which form a cover of $\mathcal{M}_b$. For any pair of positive integers n, d, we define $\mathcal{M}_b^{n,d} \subset \mathcal{M}_b$ as the locus classifying pairs (H, β) such that the quasi-isogenies $p^n\beta$ and $p^{d-n}\beta^{-1}$ are isogenies (equivalently, such that $p^n\beta$ is an isogeny whose kernel is killed by p^d). The $\mathcal{M}_b^{n,d}$ are closed subspaces of $\mathcal{M}_b$ and, as n, d vary, they form a direct system under the natural inclusions $i_b = i^{n,d}_{n',d'}$, with $\mathcal{M}_b$ as a limit. It is clear from the definition that the truncated Rapoport-Zink spaces are not preserved by the actions of T_b and $Frob$, nevertheless one can anyway think of the action of T_b and $Frob$ on $\mathcal{M}_b$ as arising from an action on the direct system. More precisely, for any $\rho \in T_b$, let $n(\rho)$ and $d(\rho)$ be the smallest integer such that $p^{n(\rho)}\rho$ and $p^{d(\rho)-n(\rho)}\rho^{-1}$ are isogenies. Then, the morphism $\rho : \mathcal{M}_b \to \mathcal{M}_b$ induces by restriction some morphisms

$$\rho : \mathcal{M}_b^{n,d} \to \mathcal{M}_b^{n+n(\rho),d+d(\rho)} \qquad \forall n, d,$$

which obviously commute with the natural inclusions. Analogously, by restricting $Frob$ to $\mathcal{M}_b^{n,d} \subset \mathcal{M}_b$ we obtain some σ-semi-linear morphisms

$$Frob : \mathcal{M}_b^{n,d} \to \mathcal{M}_b^{n+1,d+1} \qquad \forall n, d.$$

Let $\bar{\mathcal{M}}_b$ (resp. $\bar{\mathcal{M}}_b^{n,d}$) over $\bar{\mathbb{F}}_p$ denote the reduced fiber of $\mathcal{M}_b$ (resp. $\mathcal{M}_b^{n,d}$, for any n, d). $\bar{\mathcal{M}}_b$ uniformizes the isogeny classes of the abelian varieties classified by $\bar{X}^{(b)}$. More precisely, let x be a geometric closed point of $\bar{X}^{(b)}$ and $A/\bar{\mathbb{F}}_p$ the corresponding abelian variety. We also assume for simplicity that $A[p^\infty] \simeq \Sigma$. To any isomorphism $\phi : \Sigma \to A[p^\infty]$ one can associate a morphism

$$f_{x,\phi} : \mathcal{M}_b \to \bar{X}^{(b)} \times_k \bar{\mathbb{F}}_p$$

which maps a point (H, β) to the abelian variety $A/\phi(\ker(p^n\beta))$ (for any n sufficiently large) endowed with the additional structures induced by the ones of A. The image of $f_{x,\phi}$ is the isogeny class of x. It is a simple but important remark that the restriction of $f_{x,\phi}$ to the subscheme $\bar{\mathcal{M}}_b^{n,d}$ depends only on the restriction of the isomorphism ϕ to the p^d-torsion subgroups, for any n, d.

We proceed to construct some morphisms

$$\pi_N : J_{m,b} \times \bar{\mathcal{M}}_b^{n,d} \to \bar{X}^{(b)} \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{\mathbb{F}}_p, \qquad m, n, d, N \in \mathbb{N},$$

which can be described as piecing together (up to some power of Frobenius) the above morphisms $f_{x,\phi}$, as x and ϕ vary. A key ingredient in the construction of the maps π_N is the following observation.

Lemma 8. *Let G be a Barsotti-Tate group with additional structures over a k-scheme S. Assume $\mathcal{G}$ has constant isogeny class and is completely slope divisible. Let $(0) \subset \mathcal{G}_1 \subset \cdots \mathcal{G}_k = \mathcal{G}$ denote the slope filtration of $\mathcal{G}$ and write $\mathcal{G}^i$ for its slope divisible isoclinic sub-quotient of slope λ_i, for all i. Let $\delta = \min_{i=1,\dots k-1}(\lambda_i - \lambda_{i+1})$. Then, for any pair of integers $d \geq 0$ and $r \geq d/\delta f$ there exists a canonical isomorphism*

$$\mathcal{G}^{(q^r)}[p^d] \simeq \oplus_{i=1}^{k} \mathcal{G}^{i\,(q^r)}[p^d]$$

compatible with the induced additional structures.

Proof. It follows from the property of complete slope divisibility that, for any $d \geq 0$, the rf-th power of Frobenius of the Barsotti-Tate group underlying $\mathcal{G}$ canonically splits the restriction of the slope filtration to the p^d-torsion subgroups, for $r \geq d/\delta f$ (see [20], Lemma 4.1, p. 251). It is easy to check that the corresponding splitting sections commute with the additional structures of $\mathcal{G}$,$\mathcal{G}_i$ and $\mathcal{G}^i$ (for all i). □

Proposition 9. *For any positive integers m, n, d, N, $m \geq d$ and $N \geq d/\delta f$, there exist some morphisms*

$$\pi_N : J_{m,b} \times \bar{\mathcal{M}}_b^{n,d} \to \bar{X}^{(b)} \times_k \bar{\mathbb{F}}_p$$

such that

1. $\pi_N = (Fr^f \times 1) \circ \pi_N$;
2. $\pi_N \circ q_b = \pi_N$;
3. $\pi_N \circ i_b = \pi_N$;
4. $\pi_N \circ \rho \times \rho = \pi_N$, *for any* $\rho \in S_b$, *and* $m \geq d + d(\rho) + e(\rho)$, $N \geq (d + d(\rho))/\delta f$;
5. $\pi_N \circ (Frob^f \times Frob^f) = (1 \times \sigma^f) \circ \pi_N$, *for* $m \geq d+1$ *and* $N \geq (d+1)/\delta f$.

Proof. Let $\mathcal{A}$ be the universal abelian variety over $J_{m,b}$ and $j_{m,i}$, for $i = 1, \dots, k$ the Igusa structures on $\mathcal{A}/J_{b,m}$. By lemma 8 the Igusa structures on $\mathcal{A}$ give rise to an isomorphism

$$j(N) = \oplus_i j_{m,i}^{(q^N)}{}_{|[p^d]} : \Sigma^{(q^N)}[p^d] = \oplus_i \Sigma^{i\,(q^N)}[p^d] \to \oplus_i \mathcal{G}^{i\,(q^N)}[p^d] \simeq \mathcal{G}^{(q^N)}[p^d],$$

for any $d \leq m$, $d \leq Nf\delta$.

Let $(\mathcal{H}, \beta)$ the universal family over $\bar{\mathcal{M}}_b^{n,d}$. It follows from the definition of $\bar{\mathcal{M}}_b^{n,d}$ that $p^n\beta : \Sigma \to \mathcal{H}$ is an isogeny and $\ker(p^n\beta) \subset \Sigma[p^d]$, and thus also $p^n\beta^{(q^N)} : \Sigma^{(q^N)} \to \mathcal{H}^{(q^N)}$ is an isogeny and $\ker(p^n\beta^{(q^N)}) \subset \Sigma^{(q^N)}[p^d]$.

We define $\pi_N : J_{m,b} \times \bar{\mathcal{M}}_b^{n,d} \to \bar{X}^{(b)} \times_k \bar{\mathbb{F}}_p$ to be the morphism corresponding to the abelian variety $\mathcal{A}/j(N)(\ker(p^n\beta^{(q^N)}))$, endowed with additional structures induced by the ones on $\mathcal{A}$ and with level structure the orbit of the isomorphism

$$V \otimes \mathbb{A}^{\infty,p} \xrightarrow{\mu^p} V^p(\mathcal{A}^{(q^N)}) \xrightarrow{p^{-n}} V^p(\mathcal{A}^{(q^N)}) \longrightarrow V^p\Big(\frac{\mathcal{A}^{(q^N)}}{j(N)(\ker(p^n\beta^{(q^N)}))}\Big).$$

As in the proof of proposition 5, in order to show the existence of induced additional structures on $\mathcal{A}/j(N)(\ker(p^n\beta^{(q^N)}))$, we first remark that it is always possible to define such structures via quasi-isogenies, and that in order to check that such quasi-isogenies are well defined isogenies it suffices to prove it for their restrictions to the p^∞-torsion subgroups. Finally, we observe that this holds because the Barsotti-Tate group $\mathcal{H}$ is endowed with additional structures, compatible under β with the ones on Σ.

It is clear from the construction that the abelian variety $\mathcal{A}/j(N)(\ker(p^n\beta^{(p^{fN})}))$ and its additional structures depend only on the restrictions of the Igusa varieties to the p^d-torsion subgroups. This is equivalent to part (2) of the statement. Analogously, it is not hard to check that part (3) also holds.

Parts (4) and (5) are easy consequences of the definition of the action of S_b and $Frob$ on the product spaces $J_{b,m} \times \bar{\mathcal{M}}_b^{n,d}$, while part (1) follows directly from the construction and the equality $\ker(\beta)^{(p)} = \ker(\beta^{(p)})$. □

We now focus our attention on the fibers of the morphisms π_N. Let x be a point of $\bar{X}^{(b)}(\bar{\mathbb{F}}_p)$, and m, n, d some positive integers with $m \geq d$. For any $N \geq N_0 \geq d/\delta f$, we consider the fibers over x of the maps $\pi_N : J_{m,b} \times \bar{\mathcal{M}}_b^{n,d} \to \bar{X}^{(b)} \times_k \bar{\mathbb{F}}_p$. Using part (1) of proposition 9, we can identify

$$\pi_N^{-1}(Fr^{fN}x) = \pi_{N_0}^{-1}(Fr^{fN_0}x).$$

Moreover, as m varies, the sets $\pi_N^{-1}(Fr^{fN}x)$ form an inverse system under the projections q_b, and the corresponding limits are a direct system under the inclusions i_b, as n, d vary. We call the fiber above x the resulting set

$$\Pi^{-1}(x) = \varinjlim{}_{n,d} \varprojlim{}_m \pi_N^{-1}(Fr^{fN}x),$$

endowed with the topology of direct limit of inverse limits of discrete sets. It follows from part (4) of proposition 9 that $\Pi^{-1}(x)$ is also endowed with a continuous action of S_b, and moreover it is easy to see that this action extends uniquely to a continuous action of the group $T_b = \langle S_b, p, fr^B \rangle$ (this follows from part (5) and lemma 6).

Let us give an alternative and more explicit description of the fibers $\Pi^{-1}(x)$, for all $x \in \bar{X}^{(b)}(\bar{\mathbb{F}}_p)$. Let $J_b(\bar{\mathbb{F}}_p) = \varprojlim{}_m J_{m,b}(\bar{\mathbb{F}}_p)$, or equivalently

$$J_b(\bar{\mathbb{F}}_p) = \{(B, \lambda, i, \bar{\mu}^p; j) | (B, \lambda, i, \bar{\mu}^p) \in \bar{X}^{(b)}(\bar{\mathbb{F}}_p) \text{ and } j : \Sigma \simeq B[p^\infty]\}.$$

It has a natural topology of inverse limit of discrete sets, which is defined by a basis of opens consisting of the subsets

$$V_{j_m,x} = \{(B, \lambda, i, \bar{\mu}^p; j) | (B, \lambda, i, \bar{\mu}^p) = x \in \bar{X}^{(b)}(\bar{\mathbb{F}}_p) \text{ and } j_{|[p^m]} = j_m\} \subset J_b(\bar{\mathbb{F}}_p),$$

for all $x \in \bar{X}^{(b)}(\bar{\mathbb{F}}_p)$ and $j_m : \Sigma[p^m] \simeq B[p^m]$.

Let us consider the $\bar{\mathcal{M}}_b(\bar{\mathbb{F}}_p)$ as endowed with the discrete topology and the product $J_b(\bar{\mathbb{F}}_p) \times \bar{\mathcal{M}}_b(\bar{\mathbb{F}}_p)$ with the product topology. Then, the action of S_b on $J_b(\bar{\mathbb{F}}_p) \times \bar{\mathcal{M}}_b(\bar{\mathbb{F}}_p)$, which arises from the action on the corresponding varieties, can be explicitly described as

$$\forall \rho \in S_b : ((B, \lambda, i, \bar{\mu}^p; j), (H, \beta)) \mapsto \left((B/j(\ker \rho^{-1}), \lambda', i, \bar{\mu}^p; j \circ \rho), (H, \beta \circ \rho)\right).$$

It is clearly that this action is continuous for the product topology, and also that it has an obvious extension to the sub-monoid of T_b consisting of all quasi-isogenies whose inverses are isogenies. Further more, in order to extend it to a continuous action of T_b, it is enough to check that p^{-1} acts invertibly, and this follows from the observation

$$\left((B/j(\ker p), \lambda', i, \bar{\mu}^p; j \circ p^{-1}), (H, \beta \circ p^{-1})\right) = \left((B, \lambda, i, \bar{\mu}^p; j), (H, \beta \circ p^{-1})\right).$$

Let us now define a map $\Pi : J_b(\bar{\mathbb{F}}_p) \times \bar{\mathcal{M}}_b(\bar{\mathbb{F}}_p) \to \bar{X}^{(b)}(\bar{\mathbb{F}}_p)$ as

$$((B, \lambda, i, \bar{\mu}^p; j), (H, \beta)) \mapsto (B/j(\ker(p^n\beta)), \lambda', i, \bar{\mu'}^p),$$

where the additional structures on $B/j(\ker(p^n\beta))$ are induced by the ones on B via the isogeny $\nu : B \to B/j(\ker(p^n\beta))$, and $\mu' = \nu p^{-n}\mu$. It is easy to see that the above map is continuous for the discrete topology on $\bar{X}^{(b)}(\bar{\mathbb{F}}_p)$, and moreover that for any $x \in \bar{X}^{(b)}(\bar{\mathbb{F}}_p)$, one can identify $\Pi^{-1}(x)$ with the fiber of Π above x (not just as sets but also as topological spaces).

Proposition 10. *For any $x \in \bar{X}^{(b)}(\bar{\mathbb{F}}_p)$, its fiber $\Pi^{-1}(x)$ is a principle homogeneous space for the continuous action of T_b.*

Proof. Let us first remark that if $\phi : A \to B$ isogeny between abelian varieties and A (resp. B) is endowed with additional structures, then if they exist the induced additional structures on B (resp. A) via ϕ are unique and moreover they always exist as structures defined via quasi-isogenies. With this in mind, one can then apply the same argument used in [20] (Prop. 4.4, p. 259). □

We observe that the previous proposition implies in particular that $\Pi^{-1}(x)$ is non-empty, for any $x \in \bar{X}^{(b)}(\bar{\mathbb{F}}_p)$, and thus equivalently that the morphisms π_N are surjective on geometric points, for m, n, d, N large.

Proposition 11. *For any positive integers m, n, d, N, $m \geq d$ and $N \geq d/\delta f$, the morphism $\pi_N : J_{m,b} \times \bar{\mathcal{M}}_b^{n,d} \to \bar{X}^{(b)} \times \bar{\mathbb{F}}_p$ is finite.*

Proof. First, let us remark that the morphism π_N is quasi-finite. Let x be a geometric closed point of $\bar{X}^{(b)}$, and assume $\pi_N^{-1}(x)$ non empty. Then, any isomorphism $T_b \simeq \Pi^{-1}(x)$ gives rise to a surjection $G\backslash K \to \pi_N^{-1}(x)$, for some compact subset $K \subset T_b$ and open subgroup $G \subset \Gamma_b$. This implies, in particular, that $\pi_N^{-1}(x)$ is finite.

Knowing that the morphism π_N is quasi-finite, in order to conclude it suffices to show that it also satisfies the Valuative Criterium of Properness. The same proof used in [20] (Prop. 4.8, p. 266) applies to this case. □

The existence of some morphisms $\pi_N : J_{m,b} \times \bar{\mathcal{M}}_b^{n,d} \to \bar{X}^{(b)} \times \bar{\mathbb{F}}_p$ with the properties we described is all we need to apply the constructions and results of section 5 in [20] to compute the cohomology of the stratum $\bar{X}^{(b)}$ in terms of the cohomology of the Igusa varieties and the Rapoport-Zink spaces, and of the action of them of T_b.

We first recall some definitions and notations. We denote by $U_b^{n,d}$ the maximal open subspaces of $\mathcal{M}_b$ contained in $\mathcal{M}_b^{n,d}$, and by $\bar{U}_b^{n,d}$ their reduced fibers. The interiors of the truncated Rapoport-Zink spaces form an open cover of $\bar{\mathcal{M}}$ and thus they can be used to compute its cohomology with compact supports, i.e. for any sheaf $\mathcal{F}/\bar{\mathcal{M}}_b$

$$H_c^t(\bar{\mathcal{M}}_b, \mathcal{F}) = \varinjlim{}_{n,d} H_c^t(\bar{U}_b^{n,d}, \mathcal{F}),\ t \geq 0.$$

We write $\dot{\pi}_N$ for the restrictions of the morphisms π_N to the subschemes $J_{b,m} \times \bar{U}_b^{n,d} \subset J_{b,m} \times \bar{\mathcal{M}}_b^{n,d}$ and $\bar{pr} : J_{b,m} \times \bar{U}_b^{n,d} \to \bar{\mathcal{M}}_b$ for the projections to the second factor.

Let $\tau \mapsto \bar{\tau} = (\sigma^f)^{r(\tau)}$ denote the projection $W_{E_v} \to Gal(\bar{\mathbb{F}}_p/k)$. For any abelian torsion étale sheaf $\mathcal{L}$ over $\bar{X}^{(b)} \times \bar{\mathbb{F}}_p$, we define an action of the Weil group W_{E_v} on $\mathcal{L}$ to be the data of isomorphisms

$$\tau : (1 \times \bar{\tau})^* \mathcal{L} \simeq \mathcal{L} \quad \forall \tau \in W_{E_v}$$

such that $\tau \circ \tau' = \tau'\tau$. (This definition is modeled on the natural action of W_{E_v} on the vanishing cycles of a sheaf over the generic fiber $X \times_{\mathcal{O}_{E_v}} E_v$.) Analogously, we call an action of the Weil group W_{E_v} on a sheaf $\mathcal{G}$ over $\bar{\mathcal{M}}$ the data of isomorphisms $\tau : (Frob^f)^{r(\tau)\,*}\mathcal{G} \simeq \mathcal{G}$, for all $\tau \in W_{E_v}$, $r(\tau) \geq 0$, such that $\tau \circ \tau' = \tau'\tau$.

Theorem 12. *(see [20] 5.11, p.280) For any $b \in B(G)$ and $U^p \subset G(\mathbb{A}^{\infty,p})$, let $\mathcal{L}$ be an abelian torsion étale sheaf over $\bar{X}^{(b)} \times \bar{\mathbb{F}}_p$, with torsion orders prime to p, endowed with an action of the Weil group W_{E_v}. Then, there exists a W_{E_v}-equivariant spectral sequence*

$$E_2^{p,q} = H_q(T_b, \varinjlim{}_{m,n,d} H_c(J_{b,m} \times \bar{U}_b^{n,d}, \dot{\pi}_N^* \mathcal{L})) \Rightarrow H_c^{p+q}(\bar{X}^{(b)}_{U^p(0)} \times \bar{\mathbb{F}}_p, \mathcal{L}).$$

We now restrict our focus to the case of l^r-torsion sheaves, for $l \neq p$ a prime and any integer $r \geq 1$. We denote by $\mathcal{H}_r(T_b)$ the Hecke algebra of T_b with coefficients in $\mathbb{Z}/l^r\mathbb{Z}$. In the cases which allow it, one can use the Künneth formula for étale cohomology with compact supports to rewrite the previous result as follows.

Theorem 13. *(see [20], 5.13, p. 283) For any $b \in B(G)$, $U^p \subset G(\mathbb{A}^{\infty,p})$, let $\mathcal{L}$ (resp. $\mathcal{G}$) be an étale sheaf of $\mathbb{Z}/l^r\mathbb{Z}$-modules over $\bar{X}^{(b)} \times \bar{\mathbb{F}}_p$ (resp. $\bar{\mathcal{M}}$), endowed with an action of the Weil group W_{E_v}.*

Suppose there exists a system of W_{E_v}-equivariant isomorphisms of étale sheaves

$$\{\dot{\pi}_N^* \mathcal{L} \simeq \bar{pr}^* \mathcal{G}\}_{m,n,d,N,m \geq d, N \geq d/\delta f}$$

which are compatible under the natural transaction maps induced by q_b, i_b and $\rho \in S_b$.

Then, there exists a W_{E_v}-equivariant spectral sequence of $\mathbb{Z}/l^r\mathbb{Z}$-modules

$$\oplus_{t+s=q} Tor^p_{\mathcal{H}_r(T_b)} \left(H_c^s(\bar{\mathcal{M}}, \mathcal{G}), \varinjlim{}_m H_c^t(J_{b,U^p,m}, \mathbb{Z}/l^r\mathbb{Z}) \right) \Rightarrow H_c^{p+q}(\bar{X}^{(b)}_{U^p(0)} \times \bar{\mathbb{F}}_p, \mathcal{L}).$$

It is easy to see that when $\mathcal{L}$ (resp. $\mathcal{G}$) is the constant étale sheaf over $\bar{X}^{(b)}$ (resp. over $\bar{\mathcal{M}}$) associated to $\mathbb{Z}/l^r\mathbb{Z}$ the hypothesis of the previous theorem are satisfied. We thus deduce the following corollary.

Corollary 14. *For any $b \in B(G)$, $U^p \subset G(\mathbb{A}^{\infty,p})$. There exists a W_{E_v}-equivariant spectral sequence of $\mathbb{Z}/l^r\mathbb{Z}$-modules*

$$\oplus_{t+s=q} Tor^p_{\mathcal{H}_r(T_b)} \left(H_c^s(\bar{\mathcal{M}}, \mathbb{Z}/l^r\mathbb{Z}), H_c^t(J_{b,U^p}, \mathbb{Z}/l^r\mathbb{Z}) \right) \Rightarrow H_c^{p+q}(\bar{X}^{(b)}_{U^p(0)} \times \bar{\mathbb{F}}_p, \mathbb{Z}/l^r\mathbb{Z}),$$

where we write $H_c^t(J_{b,U^p}, \mathbb{Z}/l^r\mathbb{Z}) = \varinjlim{}_m H_c^t(J_{b,U^p,m}, \mathbb{Z}/l^r\mathbb{Z})$, for all $t \geq 0$.

6. Some integral models for spaces with level structure at p

This and the next sections have the ultimate goal of proving that there exists certain integral models for the Shimura varieties and the Rapoport-Zink spaces with level structure at p, whose vanishing cycles sheaves satisfy the hypothesis of theorem 13. The key ingredient of the construction is Katz's and Mazur's notion of full set of sections of a finite flat scheme ([16], section 1.8.2, p. 33).

We construct integral models for the Shimura varieties and the Rapoport-Zink spaces with level structure at p as some finite covers of the corresponding schemes with no level structure at p. Let us remark that our goal is to give integral models not only of the moduli spaces but also of the action on them of $G(\mathbb{A}^\infty)$ and $G(\mathbb{Q}_p)$, respectively.

In order to adapt Katz's and Mazur's definition to our context we first introduce some new notations. Let $\Lambda \subset V_{\mathbb{Q}_p}$ be a self-dual lattice preserved by the action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$. For any integer $m \geq 1$, let us consider the lattices $\Lambda \subset p^{-m}\Lambda \subset V_{\mathbb{Q}_p}$. The quotient $p^{-m}\Lambda/\Lambda$ is naturally a $\mathcal{O}_{B_{\mathbb{Q}_p}}$-module endowed with a non degenerate $*$-hermitian alternating pairing induced by the pairing $\langle,\rangle$ on $V_{\mathbb{Q}_p}$

$$\langle,\rangle_m : p^{-m}\Lambda/\Lambda \times p^{-m}\Lambda/\Lambda \to p^{-m}\mathbb{Z}_p/\mathbb{Z}_p$$
$$(x+\Lambda, y+\Lambda) \mapsto \langle p^m x, y\rangle,$$

for all $x, y \in p^{-m}\Lambda$. As m vary, the groups $p^{-m}\Lambda/\Lambda$ form a direct system under the natural inclusions; we regard its limit V/Λ as an étale Barsotti-Tate group endowed with the compatible action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ and polarization arising from the additional structures on the $p^{-m}\Lambda/\Lambda$, for all $m \geq 1$. We identify Λ with its Tate space. Thus, under the latter identification, any $g \in G(\mathbb{Q}_p)$ may be regarded as a quasi-self-isogeny of V/Λ commuting with its additional structures.

For any integer $m \geq 1$, we define

$$K_m = \{g \in G(\mathbb{Q}_p) |\, g(\Lambda) \subset \Lambda, gg^\# \in \mathbb{Z}_p^\times, g_{|\Lambda} \equiv 1 \mod p^m\Lambda\}.$$

Then, the quotient K_0/K_m can be identified with a group of $\mathcal{O}_{B_{\mathbb{Q}_p}}$-equivariant automorphisms of $p^{-m}\Lambda/\Lambda$ preserving the pairing $\langle,\rangle_m$ up to scalar multiple in $(\mathbb{Z}/p^m\mathbb{Z})^\times$.

For any level U^p away from p and integer $m \geq 0$, we define

$$U^p(m) = U^p \times K_m \subset G(\mathbb{A}^\infty).$$

As U^p, m vary, the $U^p(m)$ naturally form a direct system of sufficiently small open compact subgroups of $G(\mathbb{A}^\infty)$, cofinal to the system of all open compact subgroups.

For all U of the form $U = U^p(m)$, we define some integral models $\mathcal{X}_m = \mathcal{X}_{U^p(m)}$ over $\mathcal{O}_{E_v}$ of the corresponding Shimura varieties.

We first consider the following general situation. Let S be a $\mathcal{O}_{E_v}$-scheme and $\mathcal{G}/S$ a polarized Barsotti-Tate group endowed with a compatible action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$. For any integer $m \geq 1$, we define a contravariant set-valued functor $S_m = S(p^m\Lambda/\Lambda, \mathcal{G}/S)$ on the category of S-schemes. To an S-scheme T the functor S_m associates the set of group morphisms

$$\alpha : p^{-m}\Lambda/\Lambda \to \mathcal{G}[p^m](T)$$

satisfying the conditions:

(1) $\{\alpha(x) |\, x \in p^{-m}\Lambda/\Lambda\}$ is a full set of sections of $\mathcal{G}[p^m]_T/T$;
(2) α is $\mathcal{O}_{B_{\mathbb{Q}_p}}$-equivariant;
(3) α maps the pairing $\langle,\rangle$ to the ℓ-Weil pairing on $\mathcal{G}[p^m](T)$, up to a scalar multiple in $(\mathbb{Z}/p^m\mathbb{Z})^\times$.

Proposition 15. *For any $\mathcal{O}_{E_v}$-scheme S, polarized Barsotti-Tate group $\mathcal{G}/S$ endowed with a compatible action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$, and integer $m \geq 1$, the contravariant set-valued functor*

$$S_m = S(p^m\Lambda/\Lambda, \mathcal{G}/S)$$

is represented by a finite S-scheme.

Proof. It follows from proposition 1.9.1 of [16] (p. 38) that the functor which associates to a S-scheme T the set of group morphisms $\alpha : p^{-m}\Lambda/\Lambda \to \mathcal{G}[p^m](T)$ satisfying the condition (1) is represented by a closed subscheme of $\mathcal{G}[p^m]\times_S \cdots \times_S \mathcal{G}[p^m]$. Thus, in order to conclude, it suffices to remark that both conditions (2) and (3) are closed. □

It follows from the definition that the group K_0/K_m acts on the space S_m by composition on the right. It is not hard to see that the spaces S_m form a projective system indexed by the positive integers m, and that this system can be endowed with an action of K_0 arising from the action of K_0 on each quotient S_m via the projections $K_0 \to K_0/K_m$.

For any level $U^p \subset G(\mathbb{A}^{\infty,p})$ and $m \geq 1$, we define

$$f_{U^p,m} : \mathcal{X}_{U^p(m)} = S(p^m\Lambda/\Lambda, \mathcal{G}/\mathcal{X}_{U^p(0)}) \to \mathcal{X}_{U^p(0)},$$

for $\mathcal{G} = \mathcal{A}[p^\infty]$ the Barsotti-Tate group associated to the universal abelian variety over $\mathcal{X}_{U^p(0)}$. (When there is no ambiguity, we remove the level U^p from the notations and simply write $f_m : \mathcal{X}_m \to \mathcal{X}$.) As both U^p, m vary, the schemes $\mathcal{X}_{U^p(m)}$ form a projective system which can be endowed with an action of the subgroup $G(\mathbb{A}^{\infty,p}) \times K_0 \subset G(\mathbb{A}^\infty)$.

We remark that over the generic fiber $X = X_{U^p(0)}/E_v$ the Barsotti-Tate group $\mathcal{G}$ is étale in which case the notion of full set of sections recovers the usual notion of level structure. Thus we can identify the generic fibers $\mathcal{X}_{U^p(m)} \times_{\mathcal{O}_{E_v}} E_v$ of the above schemes with the Shimura varieties $X_{U^p(m)}$ of the same level. Further more, under these identifications the action of $G(\mathbb{A}^{\infty,p}) \times K_0$ on the generic fibers of the schemes $\mathcal{X}_{U^p(m)}$ agrees with the action of $G(\mathbb{A}^\infty)$ on the Shimura varieties.

Extending the action of $G(\mathbb{A}^{\infty,p}) \times K_0$ on these integral models of the Shimura varieties to an action of $G(\mathbb{A}^\infty)$ is not possible in general. (It is possible when the Barsotti-Tate group $\mathcal{A}[p^\infty]$ can replace by a one-dimensional one, as for example in [10].) To overcome this obstacle we consider a larger class of integral models for the Shimura varieties on which the action of $G(\mathbb{A}^{\infty,p}) \times K_0$ extends to the action of a certain sub-monoid $G(\mathbb{A}^\infty)^+$ of $G(\mathbb{A}^\infty)$, with the property that $G(\mathbb{A}^\infty) = \langle G(\mathbb{A}^\infty)^+, p\rangle$.

We return to the general situation of a $\mathcal{O}_{E_v}$-scheme S and a polarized Barsotti-Tate group $\mathcal{G}/S$ endowed with a compatible action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$. Maintaining the previous notations, we write $S_m = S(p^m\Lambda/\Lambda, \mathcal{G}/S)$, for all $m \geq 1$, and

$$\alpha : p^m\Lambda/\Lambda \to \mathcal{G}[p^m](S_m)$$

for the universal full set of sections of $\mathcal{G}[p^m]$ over S_m.

Let $G(\mathbb{Q}_p)^+$ denote the sub-monoid of the inverses of self-isogenies of V/Λ,

$$G(\mathbb{Q}_p)^+ = \{g \in G(\mathbb{Q}_p) |\, g^{-1}(\Lambda) \subset \Lambda\} \subset G(\mathbb{Q}_p).$$

For any $g \in G(\mathbb{Q}_p)^+$ and integer $m \geq e = e(g)$ (i.e. such that $\ker(g^{-1}) \subset p^{-m}\Lambda/\Lambda$), we define

$$S_{m,g} = S(p^{-m}\Lambda/\Lambda, \mathcal{G}/S, g) \to S_m$$

as the universal space for the existence of a finite flat subgroup $\mathcal{E} \subset \mathcal{G}[p^m]$ of order $p^{\deg(g^{-1})}$ such that

(1) $\mathcal{E}$ is self-dual under the ℓ-Weil pairing on $\mathcal{G}[p^m]$ and $\mathcal{O}_{B_{\mathbb{Q}_p}}$-invariant;
(2) $\alpha(\ker g^{-1}) \subset \mathcal{E}(S_{m,g})$;

(3) the induced morphism $\bar{\alpha} : p^{-m+e}\Lambda/\Lambda \to (\mathcal{G}/\mathcal{E})[p^{m-e}](S_{m,g})$ is a full set of sections of $(\mathcal{G}/\mathcal{E})[p^{m-e}]/S_{m,g}$.

The scheme $S_{m,g} \to S_m$ can be realized as a closed subscheme of the Grassmanian variety classifying all finite flat subschemes of $\mathcal{G}[p^m]/S_m$ of order $p^{\deg(g^{-1})}$.

Proposition 16. *(see* [20]*, Prop. 7.3, p. 298) Maintaining the above notations.*

1. *The morphism $S_{m,g} \to S_m$ is proper and induces an isomorphism on the generic fibers over E_v.*
2. *For $g \in K_0 \subset G(\mathbb{Q}_p)^+$, $S_{m,g} = S_m$.*
3. *For $n \leq m$, $S_{m,p^{-n}} = S_m$.*
4. *For $\gamma \in K_0$, $g \in G(\mathbb{Q}_p)^+$, there are natural isomorphisms*
$$\gamma : S_{m,g} \to S_{m,\gamma^{-1}\circ g}$$
compatible with the action of γ on S_m.

Proof. The arguments used in [20] apply also in this context. □

For any level $U = U^p(m)$ and $g \in G(\mathbb{Q}_p)^+$, $e(g) \leq m$, we write
$$\mathcal{X}_{m,g} = \mathcal{X}_{U^p(m),g} = S(p^m\Lambda/\Lambda, \mathcal{G}/\mathcal{X}_{U^p(0)}, g),$$
and $f_{m,g} = f_{U^p,m,g} : \mathcal{X}_{U^p(m),g} \to \mathcal{X}_{U^p(0)}$. Thus the $f_{m,g}$ are proper morphisms which factor via $f_m : \mathcal{X}_m \to \mathcal{X}$. Let $U = U^p(m)$ be any level, then, for all $g \in G(\mathbb{Q}_p)^+$ such that $e(g) \leq m$, the schemes $\mathcal{X}_{U^p(m),g}/\mathcal{O}_{E_v}$ can be regarded as models for the Shimura variety of level U.

Proposition 17. *Let $U^p(m) \subset G(\mathbb{A}^\infty)$, $g \in G(\mathbb{Q}_p)^+$, $m \geq e = e(g)$. There exists a morphism*
$$g : \mathcal{X}_{U^p(m),g} \to \mathcal{X}_{U^p(m-e)}$$
associated to data of the abelian variety $\mathcal{A}/\mathcal{E}$, endowed with the additional structures induced by the ones of $\mathcal{A}$, and of the full set of sections $\bar{\alpha}$ of $(\mathcal{A}/\mathcal{E})[p^{m-e}]$.

Further more,

1. *g is proper;*
2. *the generic fiber of g can be identify with the action of g on the Shimura varieties;*
3. *for $n \leq m$, $p^{-n} : \mathcal{X}_{U^p(m),p^{-n}} = \mathcal{X}_{U^p(m)} \to \mathcal{X}_{U^p(m-n)}$ is the natural projection;*
4. *as the level $U^p(m)$ varies, the morphisms associated to $g \in G(\mathbb{Q}_p)^+$ are compatible under the natural projections.*

Proof. It follows from the defining property of $\mathcal{E}$ that the abelian variety $\mathcal{A}/\mathcal{E}$ satisfies the properties required to define a morphism $g : \mathcal{X}_{U^p(m),g} \to \mathcal{X}_{U^p(m-e)}$. The proof of the listed properties of g is the same as in [20] (Prop. 7.3, p. 298). □

Analogously, by considering the universal Barsotti-Tate group $\mathcal{H}$ over $\mathcal{M}_b$, one can construct some formal $\hat{\mathbb{Z}}_p^{nr}$-schemes $\mathcal{M}_{b,m,g}$ together with some morphisms $\phi_{b,m,g} : \mathcal{M}_{b,m,g} \to \mathcal{M}_b$, for $g \in G(\mathbb{Q}_p)^+$ and $m \geq e(g)$, whose associated rigid analytic spaces can be identify with the Rapoport-Zink spaces $\mathcal{M}_{b,K_m}^{\mathrm{rig}}/\mathbb{Q}_p^{nr}$, for all $m \geq 0$, and which are endowed with an action of $G(\mathbb{Q}_p)^+$ extending the action on the generic fibers.

7. Formal completions along the Newton polygon strata

Let $b \in B(G)$, and fix a level $U = U^p(M) = U^p \times K_M \subset G(\mathbb{A}^{\infty,p}) \times G(\mathbb{Q}_p)$, $M \geq 0$. For any $g \in G(\mathbb{Q}_p)^+$, $e(g) \leq M$, we denote by $\mathfrak{X}_{U^p(M),g}$ the formal completion of $\mathcal{X}_{U^p(M),g}$ along its fiber mod p.

Let us suppose for a moment that it were possible to define some formal schemes $\mathcal{J}_{b,U^p,m}$ over $\hat{\mathbb{Z}}_p^{nr}$, lifting the Igusa varieties to characteristic zero, together with a system of morphisms of formal schemes

$$\mathcal{J}_{b,U^p,m} \times \mathcal{M}_b^{n,d} \to \mathfrak{X}_{U^p(0)} \times \hat{\mathbb{Z}}_p^{nr}$$

lifting the morphisms π_N defined in the section 5. Then, to study the pull-backs of the vanishing cycle sheaves of the Shimura varieties and of the Rapoport-Zink spaces with structure of level M at p would be equivalent to studying the pull-backs over $\mathcal{J}_{b,m} \times \mathcal{M}^{n,d}$ of the formal schemes $\mathfrak{X}_{U^p(M),g} \to \mathfrak{X}_{U^p(0)}$ and $\mathcal{M}_{b,M,g} \to \mathcal{M}_b$, respectively.

We start by constructing some formal liftings of the Igusa varieties over $\hat{\mathbb{Z}}_p^{nr}$. Let $\mathfrak{C}_{b,U^p(0)}$ denote the completion of the Shimura variety of level U^p along the leaf $C_{b,U^p(0)}$ (thus $\mathfrak{C}_{b,U^p(0)} \subset \mathfrak{X}_{U^p(0),g}$). Then, a theorem of Grothendieck (see [8], Exp. I, 8.4) establishes that the natural functor from the category of finite étale covers of the formal scheme $\mathfrak{C}_{b,U^p(0)}/\hat{\mathbb{Z}}_p^{nr}$ to the category of finite étale covers of $C_{b,U^p(0)}/\bar{\mathbb{F}}_p$, which to any cover $\mathcal{S} \to \mathfrak{C}_{b,U^p(0)}$ associates the reduced fiber $S = \mathcal{S} \times_{\hat{\mathbb{Z}}_p^{nr}} \bar{\mathbb{F}}_p$, is an equivalence of categories.

For any $m \geq 1$, we define $\mathcal{J}_{b,U^p,m}$ to be the finite étale cover of $\mathfrak{C}_{b,U^p(0)}$ corresponding to $J_{b,U^p,m} \to C_{b,U^p(0)}$, under Grothendieck's equivalence. Thus, as the levels U^p, m vary, the formal schemes $\mathcal{J}_{b,U^p,m}$ form a projective system endowed with an action of $G(\mathbb{A}^{\infty,p}) \times \Gamma_b$, extending the action on the reduced fibers. By definition, $\mathcal{J}_{b,U^p,m}$ is a smooth formal scheme over $\hat{\mathbb{Z}}_p^{nr}$ whose reduced fiber is the Igusa variety $J_{b,U^p,m}/k$. (The smoothness of the formal schemes $\mathcal{J}_{b,U^p,m}$ is important for our constructions.)

Before investigating the possibility of lifting the morphisms π_N, for this choice of formal liftings of the Igusa varieties, we replace the π_N with some other morphisms $\pi_N(1)$, also compatible with the transaction maps among the product of Igusa varieties and truncated Rapoport-Zink spaces and with the action of S_b on them. The $\pi_N(1)$ differ from the morphisms π_N by some purely inseparable finite maps, i.e. $\pi_N(1) \circ (1 \times \phi_N) = \pi_N$ for ϕ_N a purely inseparable finite endomorphism of $\bar{\mathcal{M}}_b$. Because of the latter property, it is possible to reformulate theorems 12 and 13 in terms of the $\pi_N(1)$ in place of the π_N. On the other hand, the advantage of working with the new morphism $\pi_N(1)$ comes from the property

$$\pi_N(1)^* \mathcal{G}[p^M] \simeq \bar{pr}^* \mathcal{H}[p^M],$$

for any m, N sufficiently large with respect to M.

Let $\Sigma = \Sigma_b/\bar{\mathbb{F}}_p \supset k$ be our choice of a completely slope divisible Barsotti-Tate group with additional structures in the isogeny class determined by $b \in B(G)$. Let λ_i denote the slopes of Σ, then we recall that there exists a positive integer B $(f|B)$ such that

$$\nu = \oplus_i F^B p^{\lambda_i B} : \Sigma \to \Sigma^{(p^B)}$$

is an isomorphism commuting with the additional structures on the Barsotti-Tate groups. To the isomorphism $\nu : \Sigma \simeq \Sigma^{(p^B)}$ corresponds a purely inseparable finite map

$$\Upsilon : \bar{\mathcal{M}}_b \to \bar{\mathcal{M}}_b$$

defined as $(H, \beta) \mapsto (H^{(p^B)}, \beta^{(p^B)} \circ \nu)$.

Let m, n, d, N be some positive integers such that $m \geq d$, $N \geq d/\delta f$, and consider the scheme $J_{b,m} \times \bar{\mathcal{M}}_b^{n,d}/\bar{\mathbb{F}}_p$. As in the proof of proposition 9 we denote by

$$j(N) : \Sigma^{(q^N)}[p^d] \to \mathcal{G}^{(q^N)}[p^d]$$

the isomorphism of truncated Barsotti-Tate groups with additional structures induced by the Igusa structure on the universal Barsotti-Tate group $\mathcal{G} = \mathcal{A}[p^\infty]/J_{b,m}$. We also write $(\mathcal{H}, \beta)$ for the universal family over $\bar{\mathcal{M}}_b^{n,d}$.

For all N such that $N \equiv 0 \mod (B/f)$, we define

$$\pi_N(1) : J_{b,m} \times \bar{\mathcal{M}}_b^{n,d} \to \bar{X}^{(b)} \times_k \bar{\mathbb{F}}_p$$

to be the morphism corresponding to the abelian variety $\mathcal{A}/j(N)(\nu^{Nf/B}(\ker p^n\beta))$, endowed with the additional structures induced by the ones on $\mathcal{A}$ via the projection $\mathcal{A} \to \mathcal{A}/j(N)(\nu^{Nf/B}(\ker p^n\beta))$, and with level structure $\overline{p^{-n}\mu^p}$.

It follows from the definition that

$$\pi_N(1)(1 \times \Upsilon^{Nf/B}) = \pi_N$$

and that $\pi_N(1)$ commutes with the morphisms $q_b \times 1$, $1 \times i_b$ and with the action of S_b. Moreover, let us fix an integer $M \geq 0$ and assume $m \geq d+M$ and $N \geq (d+M)/\delta f$. Then, the isomorphism $j(N)$ canonically extends to an isomorphism $\Sigma^{(q^N)}[p^{d+M}] \simeq \mathcal{G}^{(q^N)}[p^{d+M}]$ and this induces an isomorphism between the p^M-torsion subgroups of $\mathcal{A}/j(N)(\nu^{Nf/B}(\ker p^n\beta))$ and $\mathcal{H}$, which is compatible with the induced additional structures. Equivalently, there exists a canonical isomorphism

$$\pi_N(1)^*\mathcal{G}[p^M] \simeq \bar{pr}^*\mathcal{H}[p^M].$$

We show that it is possible to étale locally lift the morphisms $\pi_N(1)$ to characteristic zero, and that moreover the morphisms $\pi_N(1)$ also extend to any subscheme of $\mathcal{J}_{b,U^p,m} \times \mathcal{M}_b^{n,d}$ cut by a power of the ideal of definition. Further more, in both cases, it is possible to choose these extensions of $\pi_N(1)$ such that the corresponding pullback of $\mathfrak{X}_{U^p(M),g}$ over $\mathcal{J}_{b,U^p,m} \times \mathcal{M}_b^{n,d}$ is isomorphic to the pullback of $\mathcal{M}_{b,M,g}$ by the projection $\mathcal{J}_{b,U^p,m} \times \mathcal{M}_b^{n,d} \to \mathcal{M}_b$, for any positive integers m, n, d.

Let t be a positive integer. For any formal scheme $Y/\hat{\mathbb{Z}}_p^{nr}$, we choose an ideal of definition $\mathcal{I}$, $p \in \mathcal{I}$, and denote by $Y(t)$ the $\hat{\mathbb{Z}}_p^{nr}/(p^t)$-scheme cut by the t-th power of $\mathcal{I}$ inside Y. Analogously, for any morphisms of $\hat{\mathbb{Z}}_p^{nr}$-formal schemes $f : Y \to Z$, we write $f(t) : Y(t) \to Z(t)$ for the morphisms of $\hat{\mathbb{Z}}_p^{nr}/(p^t)$-schemes induced by the restriction of f to $Y(t)$.

Proposition 18. *Let $t \geq 1$ be an integer. For any m, n, d, N such that $m \geq d+t/2$, $N \geq (d+t/2)/\delta f$ and $N \equiv 0 \mod (B/f)$, there exist some morphisms*

$$\pi_N(t) : (\mathcal{J}_{b,m} \times \mathcal{M}_b^{n,d})(t) \to (\mathfrak{X} \times_{W(k)} \hat{\mathbb{Z}}_p^{nr})(t)$$

compatible with the projection among formal Igusa varieties and with the inclusion among truncated Rapoport-Zink spaces such that

(1) $(\pi_N(t))(t-1) = \pi_N(t-1)$*;*

(2) *there exists an isomorphism*

$$\pi_N(t)^*\mathcal{G}[p^{[t/2]}] \simeq pr(t)^*\mathcal{H}[p^{[t/2]}]$$

compatible with the induced additional structures.

Proof. We remark that it follows from Serre-Tate theorem that defining some morphisms $\pi_N(t)$ lifting $\pi_N(1)$ is equivalent to defining a Barsotti-Tate group with additional structures lifting $\pi_N(1)^*\mathcal{G}$ over $(\mathcal{J}_{b,m} \times \mathcal{M}_b^{n,d})(t)$.

For the moment we focus our attention on the underlying Barsotti-Tate groups. A theorem of Grothendieck (see [12], Thm. 4.4, p. 171 and Cor. 4.7, p. 178) shows that any deformation of the Barsotti-Tate group $\pi_N(1)^*\mathcal{G}/(\mathcal{J}_{b,m}\times\mathcal{M}_b^{n,d})(1)$ defined over $(\mathcal{J}_{b,m} \times \mathcal{M}_b^{n,d})(t)$ is uniquely determined by its $[p^{[t/2]}]$-torsion subgroup and moreover any deformation of its truncation $\pi_N(1)^*\mathcal{G}[p^{[t/2]}]$ extends (uniquely) to a deformation of the entire Barsotti-Tate group.

Under the assumptions $m \geq d + t/2$ and $N \geq (d + t/2)/\delta f$, there exists an isomorphism

$$\pi_N(1)^*\mathcal{G}[p^{[t/2]}] \simeq pr(1)^*\mathcal{H}[p^{[t/2]}],$$

and moreover there is a canonical deformation of $pr(1)^*\mathcal{H}[p^{[t/2]}]$ defined over $(\mathcal{J}_{b,m}\times \mathcal{M}_b^{n,d})(t)$, namely $pr(t)^*\mathcal{H}[p^{[t/2]}]$.

Thus, there exists a unique deformation $\hat{\mathcal{G}}/(\mathcal{J}_{b,m}\times\mathcal{M}_b^{n,d})(t)$ of the Barsotti-Tate group underlying $\pi_N(1)^*\mathcal{G}$, which satisfy the condition $\hat{\mathcal{G}}[p^{[t/2]}] \simeq pr(t)^*\mathcal{H}[p^{[t/2]}]$. We show that $\hat{\mathcal{G}}$ can be endowed with additional structures lifting the ones on $\pi_N(1)^*\mathcal{G}$.

A theorem of Drinfeld ([14], Lemma 1.1.3, part 3) shows that there exist unique quasi-isogenies on $\hat{\mathcal{G}}$ which lift the quasi-polarization and the action of $\mathcal{O}_{B_{\mathbb{Q}_p}}$ on $\pi_N(1)^*\mathcal{G}$. Thus, it suffices to prove that these quasi-isogenies are indeed well-defined isogenies. This is a property which may be checked on their restrictions to $\hat{\mathcal{G}}[p^{[t/2]}]$. The existence of an isomorphism $\hat{\mathcal{G}}[p^{[t/2]}] \simeq pr(t)^*\mathcal{H}[p^{[t/2]}]$ lifting $\pi_N(1)^*\mathcal{G}[p^{[t/2]}] \simeq pr(1)^*\mathcal{H}[p^{[t/2]}]$, implies that the truncated Barsotti-Tate group $\hat{\mathcal{G}}[p^{[t/2]}]$ can be endowed with additional structures extending the ones on $\pi_N(1)^*\mathcal{G}[p^{[t/2]}]$. By uniqueness these have to agree with the restrictions of the quasi-isogenies on $\hat{\mathcal{G}}$ (lifting the additional structures of $\pi_N(1)^*\mathcal{G}$), which are therefore well-defined morphisms.

Finally, it follows immediately from the construction that the morphisms $\pi_N(t)$ satisfy properties (1) and (2) of the statement. □

Corollary 19. *Let t be positive integer. For any $M \geq 1$ and $g \in G(\mathbb{Q}_p)^+$ such that $e(g) \leq M \leq t/2$, there exists a system of isomorphisms*

$$\mathfrak{X}_{M,g} \times_{\mathfrak{X}\times\hat{\mathbb{Z}}_p^{nr},\pi_N(t)} (\mathcal{J}_{b,m} \times \mathcal{M}_b^{n,d})(t) \simeq \mathcal{M}_{b,M,g} \times_{\mathcal{M}_b,pr(t)} (\mathcal{J}_{b,m} \times \mathcal{M}_b^{n,d})(t),$$

indexed by all m, n, d, N such that $m \geq d + t/2$, $N \geq (d + t/2)/\delta f$, $N \equiv 0 \mod (B/f)$, which are compatible under the natural transaction maps and commute with the action of $G(\mathbb{Q}_p)^+$, as M, g vary.

Proof. The corollary follows from the existence of compatible isomorphisms

$$\pi_N(t)^*\mathcal{G}[p^M] \simeq pr(t)^*\mathcal{H}[p^M]$$

over $(\mathcal{J}_{b,m} \times \mathcal{M}_b^{n,d})(t)$, for all m, n, d, N. □

Corollary 20. *Maintaining the above notations. For any m, n, d, t as above, there exists an open cover $\mathcal{V} = \mathcal{V}_{m,n,d}$ of $(\mathcal{J}_{b,m} \times \mathcal{M}_b^{n,d})$ such that for any $V \in \mathcal{V}$ there exists a morphisms*

$$\pi_N[t, V] : V \to \mathfrak{X} \times \hat{\mathbb{Z}}_p^{nr}$$

such that $\pi_N[t, V](t) = \pi_N(t)_{|V}$ and also

$$\mathfrak{X}_{M,g} \times_{\mathfrak{X} \times \hat{\mathbb{Z}}_p^{nr}, \pi_N[t,V]} V \simeq \mathcal{M}_{b,M,g} \times_{\mathcal{M}_b, pr_{|V}} V,$$

for all M, g, $e(g) \leq M \leq t/2$, compatibly with the action of $G(\mathbb{Q}_p)^+$.

Moreover, the $\pi_N[t, V]$ are formally smooth.

Proof. The existence of morphisms $\pi_N[t, V]$ as in the statement is equivalent to the existence of a Barsotti-Tate group $\hat{\mathcal{G}}/V$ lifting $\pi_N(t)^*_{|V}\mathcal{G}$ and such that $\hat{\mathcal{G}}[p^{[t/2]}] \simeq pr^*_{|V}\mathcal{H}[p^{[t/2]}]$. The smoothness of the Shimura varieties $\mathcal{X} = \mathcal{X}_{U^p(0)}/\mathcal{O}_{E_v}$ implies that there are no local obstructions to these deformations.

In order to prove that the morphisms $\pi_N[t, V]$ are formally smooth, it is enough to show that the morphisms

$$p_1 \times \pi_N[t, V] : V \to \mathcal{J}_{b,m} \times \mathfrak{X} \times \hat{\mathbb{Z}}_p^{nr},$$

where $p_1 : V \to \mathcal{J}_{b,m}$ is the projection on the first factor, are étale. Property (2) of proposition 18 implies that the restriction $(p_1 \times \pi_N[t, V])(t) = p_1(t) \times \pi_N(t)$ induces isomorphisms on the completed local rings, and this suffices to conclude. □

The existence of a system of morphisms $\pi_N[t, V]$ with the properties listed in proposition 20 enable us to apply the constructions of section 7.4 in [20] to deduce that the push-forwards of the vanishing cycles of the formal schemes $\mathfrak{X}_{M,g}$ and $\mathcal{M}_{b,M,g}$, for all M, g, satisfy the hypothesis of theorem 13. More precisely, one can prove the following statement (cfr. [20], Thm. 7.13, p. 313).

Proposition 21. *Let U^p be a level away from p and $b \in B(G)$.*

For any $M \geq 1$ and $g \in G(\mathbb{Q}_p)^+$, there exist a system of quasi-isomorphisms of complexes

$$\pi_N(1)^* R\Psi_\eta R(f_{M,g})_*(\mathbb{Z}/l^r\mathbb{Z}) \simeq \bar{pr}^* R\Psi_\eta R(\phi_{b,M,g})_*(\mathbb{Z}/l^r\mathbb{Z}) \quad \forall m, n, d, N,$$

compatible with the action of $W_{E_v} \times T_b$.

Moreover, they are also compatible with the action of $G(\mathbb{Q}_p)^+$, as M, g vary.

The main ingredient behind this result is a theorem of Berkovich's which enable us to use the equalities $\pi_N[t, V](t) = \pi_N(t)_{|V}$ to show that the morphisms

$$\pi_N[t, V]^* : \pi_N(1)^* R\Psi_\eta R(f_{M,g})_*(\mathbb{Z}/l^r\mathbb{Z})_{|V} \to R\Psi_\eta R(1 \times \phi_{b,M,g})_*(\mathbb{Z}/l^r\mathbb{Z})_{|V}$$

glue together, as V varies in $\mathcal{V}$, to a morphism ζ over $\mathcal{J}_{b,m} \times \mathcal{M}_b^{n,d}$, for any m, n, d. Further more, the morphisms $\pi_N[t, V]^*$ (and thus the resulting morphism ζ)) are isomorphisms. This follows from the fact that the formal schemes $\mathfrak{X}_{M,g} \times_{\mathfrak{X} \times \hat{\mathbb{Z}}_p^{nr}, \pi_N[t,V]} V$ and $\mathcal{M}_{b,M,g} \times_{\mathcal{M}_b, pr_{|V}} V$, are isomorphic over V, for all M, g, and that the morphisms $\pi_N[t, V]$ are smooth.

On the other hand, the projections $pr : \mathcal{J}_{b,m} \times \mathcal{M}_{b,M,g}^{n,d} \to \mathcal{M}_{b,M,g}$ induce some morphisms

$$pr^* : \bar{pr}^* R\Psi_\eta R(\phi_{b,M,g})_*(\mathbb{Z}/l^r\mathbb{Z}) \to R\Psi_\eta R(1 \times \phi_{b,M,g})_*(\mathbb{Z}/l^r\mathbb{Z}),$$

which are also isomorphisms since the projections pr are smooth (because the liftings $\mathcal{J}_{b,m}$ of the Igusa varieties are formally smooth).

Finally, it follows from the constructions that the corresponding isomorphisms

$$\pi_N(1)^* R\Psi_\eta R(f_{M,g})_*(\mathbb{Z}/l^r\mathbb{Z}) \simeq \bar{pr}^* R\Psi_\eta R(\phi_{b,M,g})_*(\mathbb{Z}/l^r\mathbb{Z})$$

form a system, as m, n, d, N vary, which is compatible with the action of $W_{E_v} \times T_b$; and also they commute with the action of $G(\mathbb{Q}_p)^+$, as M and g vary.

8. A final formula for proper Shimura varieties

In this last section, we focus our attention on the Shimura varieties in our class which are proper. E.g, in [18] (p. 392), Kottwitz shows that PEL type Shimura varieties attached to some data $(B, *, V, \langle,\rangle, h)$ are proper if V is a simple B-module.

In the case of proper Shimura varieties, the theory of vanishing cycles enable us to deduce from the previous results regarding the geometry in positive characteristic and, in the case of bad reduction, the vanishing cycles sheaves of the Shimura varieties a formula which describes their l-adic cohomology in terms of the l-adic cohomology of the corresponding Rapoport-Zink spaces and Igusa varieties.

For expository reasons, we choose to formulate the main theorem of this section as an equality in the Grothendieck group of the representations attached to the l-adic cohomology groups. We prefer this over a statement in terms of quasi-isomorphisms in the derived category, even if the existence of such a compatible system of quasi-isomorphisms (which is established in the proof) is a stronger result than the theorem as stated. Further more, we should remark that the underlying quasi-isomorphisms exist also for $\mathbb{Z}_l$-coefficients and $\mathbb{Z}/l^r\mathbb{Z}$-coefficients (this is also established in our proof), while the corresponding equalities do not make sense in the pertinent Grothendieck's groups as the associated representations are not (at least not *a priori*) admissible.

We denote by D the dimension of the Shimura varieties.

Theorem 22. *There is an equality of virtual representations of the group $G(\mathbb{A}^\infty) \times W_{E_v}$*

$$\sum_{t\geq 0}(-1)^t \varinjlim_{U\subset G(\mathbb{A}^\infty)} H^t(X_U \times_{E_v} \bar{E}_v, \mathbb{Q}_l) =$$
$$= \sum_{b\in B(G)} \sum_{i,j,k\geq 0} (-1)^{i+j+k} \varinjlim_{K\subset G(\mathbb{Q}_p)} \mathcal{E}^{i,j,k}_{b,K},$$

where for all $b \in B(G)$ and $i, j, k \geq 0$ we write

$$\mathcal{E}^{i,j,k}_{b,K} = Ext^i_{T_b-smooth}(H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,K} \times_{E_v^{nr}} \bar{E}_v, \mathbb{Q}_l(-D)), H^k_c(J_b, \mathbb{Q}_l)).$$

In proposition 21 we established that the vanishing cycles sheaves, associated to Shimura varieties and Rapoport-Zink spaces with level structure at p, satisfy the hypothesis of theorem 13. A formal argument combines the corresponding results, as the level at p varies, in the previous formula. For a detailed proof of theorem 22, we refer to section 8 in [20]. Here we simply outline the main steps.

Let us first focus on the term on the left hand side of the above formula. We fix a level $U = U^p(M) \subset G(\mathbb{A}^\infty)$. Then, for each integral model $\mathcal{X}_{U^p,M,g}$ of the Shimura varieties X_U, i.e. for each $g \in G(\mathbb{Q}_p)^+$ such that $e(g) \leq M$, there is a quasi-isomorphism of complexes in the derived category

$$R\Gamma(X_U \times_{E_v} \bar{E}_v, \mathbb{Z}/l^r\mathbb{Z}) \simeq R\Gamma(\bar{X}_{U^p,M,g} \times_k \bar{k}, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})).$$

As we let U, g vary, the above complexes form two systems, each one endowed with an action of $G(\mathbb{A}^\infty) \times W_{E_v}$, and these actions are compatible under the above quasi-isomorphisms. (Let us remark that a priori the system on right hand side is only endowed with an action of the sub-monoid $G(\mathbb{A}^{\infty,p}) \times G(\mathbb{Q}_p)^+ \times W_{E_v} \subset G(\mathbb{A}^\infty) \times W_{E_v}$. It is by observing that the above quasi-isomorphisms are equivariant for the actions on the two sides and that on the left hand side the action extends to an action of the whole group that we are able conclude.)

On the other hand, there exist natural quasi-isomorphisms

$$R\Gamma(\bar{X}_{U^p,M,g} \times_k \bar{k}, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})) \simeq R\Gamma(\bar{X}_{U^p(0)} \times_k \bar{k}, R\Psi_\eta R(f_{M,g})_*(\mathbb{Z}/l^r\mathbb{Z})),$$

also compatible with the action of $G(\mathbb{A}^\infty) \times W_{E_v}$, as U^p, M, g vary.

We thus may focus our attention on the latter complexes. By iteratively applying the Mayer-Vietoris long exact sequence for an open subscheme (see [7], Theorem I.8.7(3)) to the complements of the closed strata, one can show that the Newton polygon stratification of the reductions of the Shimura varieties give rise to sequences of exact triangles in the derived category, which relates the complexes $R\Gamma(\bar{X}_{U^p(0)} \times_k \bar{k}, R\Psi_\eta R(f_{M,g})_*(\mathbb{Z}/l^r\mathbb{Z}))$ to the complexes

$$R\Gamma_c(\bar{X}^{(b)}_{U^p(0)} \times_k \bar{k}, R\Psi_\eta R(f_{M,g})_*(\mathbb{Z}/l^r\mathbb{Z})_{|\bar{X}^{(b)}_{U^p(0)}}) \quad \forall b \in B(G),$$

for all U^p, M, g, and which are equivariant for the action of $G(\mathbb{A}^\infty) \times W_{E_v}$, as U^p, M, g vary. Thus, we can reduced ourself to consider the latter complexes for each $b \in B(G)$ separately.

Before proceeding let us observe that, for $g = 1$, the above constructions in the derived category translates into an equality of virtual representations of $G(\mathbb{A}^{\infty,p}) \times K_0 \times W_{E_v} \subset G(\mathbb{A}^\infty) \times W_{E_v}$

$$\sum_{t\geq 0}(-1)^t \varinjlim{}_U H^t(X_U \times_{E_v} \bar{E}_v, \mathbb{Q}_l) =$$

$$= \sum_{b\in B(G)} \sum_{s,p,q\geq 0} (-1)^{s+p+q} \varinjlim{}_{U^p,M} H^s_c(\bar{X}^{(b)}_{U^p(0)} \times_k \bar{k}, R^q\Psi_\eta R^p(f_M)_*(\mathbb{Q}_l)).$$

We remark that the right hand side does not make sense in the Grothendieck group of $G(\mathbb{A}^\infty) \times W_{E_v}$, although the corresponding complex in the derived category can be endowed with an action of $G(\mathbb{A}^\infty) \times W_{E_v}$.

For simplifying the exposition, we reduce our discussion here to the case $g = 1$ and speak in terms of equalities of virtual representations of $G(\mathbb{A}^{\infty,p}) \times K_0 \times W_{E_v}$. This approach proves the formula of theorem 22 when restricted the Grothendieck group of $G(\mathbb{A}^{\infty,p}) \times K_0 \times W_{E_v}$. In order to prove theorem 22, one should consider the corresponding quasi-isomorphisms in the derived category, as both the level $U^p(M)$ and g vary.

For any $b \in B(G)$, we consider

$$\sum_{i,j,k\geq 0} (-1)^{i+j+k} \varinjlim{}_M Ext^i_{T_b-smooth}(H^j_c(\mathcal{M}^{\mathrm{rig}}_{b,K_M} \times_{E_v^{nr}} \bar{E}_v, \mathbb{Q}_l(-D)), H^k_c(J_b, \mathbb{Q}_l)).$$

A first step is to prove that these are well defined elements in the Grothendieck group of $G(\mathbb{A}^\infty) \times W_{E_v}$ (see section 4.3 of [6]).

Then, for each $b \in B(G)$, we compare the above representation with

$$\sum_{s,p,q\geq 0} (-1)^{s+p+q} \varinjlim{}_{U^p,M} H^s_c(\bar{X}^{(b)}_{U^p(0)}, R^q\Psi_\eta R^p(f_M)_*(\mathbb{Q}_l)).$$

For any level $U = U^p(M)$, the spectral sequence of theorem 13, applied to the vanishing cycles sheaves, translates into an equality between the U-invariants of the virtual representations. The fact that the Tor-groups appearing in the statement of theorem 13 are substituted in the final formula by Ext-groups reflects the fact that the cohomology groups with compact supports of the reduced fiber of the Rapoport-Zink spaces with coefficient in the vanishing cycles sheaves compute not the cohomology groups with compact supports of the corresponding rigid analytic spaces, but its contragradient dual, up to Tate twist (see Theorem 90 in [20]).

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