Supplemental Material to "Shortcuts to nonabelian braiding"

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S1: DERIVATION OF EQ. (14) IN THE MAIN TEXT

Here we derive general expressions for the counterdiabatic correction for non-interacting systems in terms of the (instantaneous) eigenmodes $\gamma_{n,\alpha}$ of the original (time-dependent) Hamiltonian in a Majorana representation,

$$H_0 = i\sum_{n\alpha} \epsilon_n \gamma_{n,2\alpha-1}\gamma_{n,2\alpha}. \tag{S1}$$

Note that both the eigen-Majoranas $\gamma_{n,\alpha}$ and the eigenvalues ϵ_n are time dependent. Degeneracies of the many-body spectrum can arise when one or more of the eigenvalues ϵ_n vanish or when some nonzero ϵ_n is degenerate, independent of time. The various Majorana operators associated with each single-particle eigenvalue ϵ_n are labeled by the index α. If the single-particle eigenvalue ϵ_n is N-fold degenerate, α takes on $2N$ different values, $\alpha = 1, \ldots, 2N$.

A direct derivation of the counterdiabatic terms based on the general Eq. (12) in the main text is cumbersome. Here, we choose to proceed as follows. The counterdiabatic terms suppress transitions out of the degenerate subspace but leave the dynamics within the degenerate subspace as governed by the nonabelian Berry connection of the original time evolution unchanged. Thus, we can determine the counterdiabatic terms H_1 uniquely from the following constraints:

(a) H_1 has no matrix elements which act within the degenerate eigenspaces of H_0. This ensures that H_1 affects only transitions between states with different energies.

(b) The time evolution of the $\gamma_{n,\alpha}$ with respect to the full shortcut Hamiltonian $H = H_0 + H_1$ does not include any transitions between different degenerate subspaces.

To implement these constraints, we note that the time evolution of the Majorana operators is governed by the Heisenberg equation of motion

$$\frac{d\gamma_{n,\alpha}}{dt} = \frac{\partial\gamma_{n,\alpha}}{\partial t} + i[H_0 + H_1, \gamma_{n,\alpha}], \tag{S2}$$

where the first term on the right-hand side accounts for the explicit time dependence of the Majorana operators. This term allows for the expansion

$$\frac{\partial\gamma_{n,\alpha}}{\partial t} = \sum_{m,\beta} C^{\alpha\beta}_{nm}\gamma_{m,\beta}. \tag{S3}$$

Here, the coefficients $C^{\alpha\beta}_{nm}$ can in principle be expressed in terms of the instantaneous eigenfunctions and their time derivatives. It turns out, however, that we do not need these explicit expressions for the present purpose. To proceed, we also write the counterdiabatic terms in the general form

$$H_1 = i\sum_{n,m}\sum_{\alpha,\beta} h^{\alpha,\beta}_{n,m}\gamma_{n,\alpha}\gamma_{m,\beta}. \tag{S4}$$

Thus, it is our goal to derive the coefficients $h^{\alpha,\beta}_{n,m}$ which satisfy the antisymmetry relation $h^{\alpha,\beta}_{n,m} = -h^{\beta,\alpha}_{m,n}$.

Implementing the contraints (a) and (b), we demand that the $\gamma_{n,\alpha}$ satisfy the time evolution

$$\frac{d\gamma_{n,\alpha}}{dt} = \sum_{\beta} C^{\alpha\beta}_{nn}\gamma_{n,\beta} + i[H_0, \gamma_{n,\alpha}]. \tag{S5}$$

Here, the first term of the right-hand side contains only those terms of $\frac{\partial\gamma_{n,\alpha}}{\partial t}$ that belong to the same subspace n. All terms in $\frac{\partial\gamma_{n,\alpha}}{\partial t}$ which belong to different subspaces must be cancelled by the counterdiabatic terms H_1.

The desired shortcut time evolution in Eq. (S5) satisfies

$$\left\{\frac{d\gamma_{n,\alpha}}{dt}, \gamma_{m,\beta}\right\} = 0 \tag{S6}$$

for $m \neq n$. Inserting the Heisenberg equation of motion (S2) into this condition, we obtain

$$4h_{nm}^{\alpha\beta} - 4h_{mn}^{\beta\alpha} + \left\{\frac{\partial\gamma_{n,\alpha}}{\partial t}, \gamma_{m,\beta}\right\} = 0 \tag{S7}$$

for $m \neq n$. Using the antisymmetry property of the $h_{nm}^{\alpha\beta}$ yields

$$H_1 = -\frac{i}{8}\sum_{\substack{n,m\\(n\neq m)}}\sum_{\alpha\beta}\left\{\frac{\partial\gamma_{n,\alpha}}{\partial t}, \gamma_{m,\beta}\right\}\gamma_{n,\alpha}\gamma_{m,\beta}. \tag{S8}$$

Finally, we write this as

$$H_1 = \frac{i}{8}\sum_{n,m}\sum_{\alpha\beta}\gamma_{m,\beta}\left\{\gamma_{m,\beta}, \frac{\partial\gamma_{n,\alpha}}{\partial t}\right\}\gamma_{n,\alpha} - \frac{i}{8}\sum_{n}\sum_{\alpha\beta}\gamma_{n,\beta}\left\{\gamma_{n,\beta}, \frac{\partial\gamma_{n,\alpha}}{\partial t}\right\}\gamma_{n,\alpha}. \tag{S9}$$

and use the relation

$$\sum_{m}\sum_{\beta}\gamma_{m,\beta}\left\{\gamma_{m,\beta}, \frac{\partial\gamma_{n,\alpha}}{\partial t}\right\} = 2\frac{\partial\gamma_{n,\alpha}}{\partial t} \tag{S10}$$

to obtain Eq. (14) of the main text.

In the following, we derive the counterdiabatic terms for the braiding procedure given in Eq. (17) of the main text using this general result as well as a more basic approach starting with Eq. (12).

S2: DERIVATION OF EQ. (17) USING THE GENERAL MAJORANA COUNTERDIABATIC TERMS

Here, we derive Eq. (17) using H_1 in the Majorana operator representation as given in Eq. (14). The braiding Hamiltonian in Eq. (16) is

$$H_0 = ih_\Delta(t)\gamma_0\gamma_\Delta(t). \tag{S11}$$

The system comprises a mode of energy $h_\Delta(t)$ associated with the two Majorana operators γ_0 and $\gamma_\Delta(t)$ as defined in the main text. In addition there are two Majoranas $\gamma_A(t)$ and $\gamma_B(t)$ which remain uncoupled by Eq. (S11) and form a zero-energy mode. Using the identity

$$\{\dot\gamma_\alpha, \gamma_\beta\} = -\{\dot\gamma_\beta, \gamma_\alpha\} \tag{S12}$$

(with the shorthand $\dot\gamma = \frac{\partial\gamma}{\partial t}$) we can write the time derivatives (suppressing time arguments) in the most generic form as

$$\begin{aligned}\dot\gamma_A &= \eta_1\gamma_B + \eta_2\gamma_\Delta,\\ \dot\gamma_B &= -\eta_1\gamma_A + \eta_3\gamma_\Delta,\\ \dot\gamma_\Delta &= -\eta_2\gamma_A - \eta_3\gamma_B.\end{aligned} \tag{S13}$$

with real coefficients η_i. The first term in H_1 can be written as

$$\sum_{n\alpha}\dot\gamma_{n,\alpha}\gamma_{n,\alpha} = \dot\gamma_A\gamma_A + \dot\gamma_B\gamma_B + \dot\gamma_\Delta\gamma_\Delta = 2\dot\gamma_\Delta\gamma_\Delta + 2\eta_1\gamma_B\gamma_A \tag{S14}$$

The second contribution to H_1 subtracts all terms within a degenerate subspace and thus eliminates the term $\sim \gamma_B\gamma_A$. Thus we obtain

$$H_1 = \frac{i}{2}\dot\gamma_\Delta\gamma_\Delta = \frac{i}{2h_\Delta^2}\sum_{\alpha\beta}\dot\Delta_\alpha\Delta_\beta\gamma_\alpha\gamma_\beta \tag{S15}$$

as given in Eq. (17).

S3: DERIVATION OF EQ. (17) USING THE SPIN CONSTRUCTION

We can alternatively derive Eq. (17) using the general formulation of the counterdiabatic terms in Eq. (12). To this end, we introduce conventional fermionic operators through

$$c_1 = \frac{1}{2}(\gamma_1 - i\gamma_2) \;\; ; \;\; c_2 = \frac{1}{2}(\gamma_0 - i\gamma_3). \tag{S16}$$

Using the inverse relations

$$\gamma_1 = c_1 + c_1^\dagger \; ; \; \gamma_2 = i(c_1 - c_1^\dagger) \; ; \; \gamma_3 = i(c_2 - c_2^\dagger) \; ; \; \gamma_0 = c_2 + c_2^\dagger, \tag{S17}$$

we can write H_0 in terms of c_1 and c_2

$$\begin{aligned} H_0 &= i\sum_{j=1}^{3} \Delta_j \gamma_0 \gamma_j \\ &= i\Delta_1(c_2^\dagger c_1 - c_1^\dagger c_2 + c_2 c_1 - c_1^\dagger c_2^\dagger) - \Delta_2(c_2^\dagger c_1 + c_1^\dagger c_2 + c_2 c_1 + c_1^\dagger c_2^\dagger) - \Delta_3(2c_2^\dagger c_2 - 1). \end{aligned} \tag{S18}$$

Specifically, we write the Hamiltonian in the basis $\{|00\rangle, |11\rangle, |10\rangle, |01\rangle\}$, where the basis states are defined as

$$|11\rangle = c_1^\dagger c_2^\dagger |00\rangle \; , \; |10\rangle = c_1^\dagger |00\rangle \; , \; |01\rangle = c_2^\dagger |00\rangle \tag{S19}$$

with $c_1|00\rangle = c_2|00\rangle = 0$. This yields

$$H_0 = \begin{pmatrix} \Delta_3 & i\Delta_1 - \Delta_2 & 0 & 0 \\ -i\Delta_1 - \Delta_2 & -\Delta_3 & 0 & 0 \\ 0 & 0 & \Delta_3 & -i\Delta_1 - \Delta_2 \\ 0 & 0 & i\Delta_1 - \Delta_2 & -\Delta_3 \end{pmatrix}. \tag{S20}$$

The block-diagonal structure originates from the conservation of fermion-number parity. In fact, it is easy to show that the Hamiltonian H commutes with the parity operator

$$P = \gamma_0\gamma_1\gamma_2\gamma_3. \tag{S21}$$

The top-left block $H_{\text{even}} = \Delta_3\tau_z - \Delta_1\tau_y - \Delta_2\tau_x$ corresponds to even fermion parity, while the bottom-right block $H_{\text{odd}} = \Delta_3\tau_z + \Delta_1\tau_y - \Delta_2\tau_x$ has odd fermion parity. Here we have defined Pauli matrices τ_i within the even and odd subspaces. If we also define Pauli matrices π_j in the even-odd subspace, then we can write

$$H_0 = \Delta_3\tau_z - \Delta_1\tau_y\pi_z - \Delta_2\tau_x \tag{S22}$$

for the overall Hamiltonian H. Expressing H_{even} and H_{odd} in terms of Pauli matrices makes it obvious that these Hamiltonians take the form of a spin Hamiltonian in magnetic fields $B_{\text{even}} = (-\Delta_2, -\Delta_1, \Delta_3)$ and $B_{\text{odd}} = (-\Delta_2, \Delta_1, \Delta_3)$, respectively. The degeneracy due to the presence of the Majorana modes implies that the two subspaces have the same eigenvalues. At the same time, the spectrum for each subspace by itself is non-degenerate.

In order to evaluate the counterdiabatic terms, it is useful to eliminate the time derivatives of the states from Eq. (12). To achieve this, we first multiply the first term on the right-hand side of Eq. (12) by $\mathbf{1} = \sum_m \sum_\beta |\psi_\beta^m\rangle\langle\psi_\beta^m|$ from the left and obtain

$$H_1 = i\sum_{m\neq n}\sum_{\alpha\beta} |\psi_\beta^m\rangle\langle\psi_\beta^m|\partial_t\psi_\alpha^n\rangle\langle\psi_\alpha^n|. \tag{S23}$$

Taking the time derivative of Eq. (5) and multiplying from the left by $\langle\psi_\beta^m|$, one finds

$$\langle\psi_\beta^m|\partial_t\psi_\alpha^n\rangle = \frac{\langle\psi_\beta^m|\partial_t H_0|\psi_\alpha^n\rangle}{E_n - E_m} \tag{S24}$$

for $n \neq m$. Inserting this into Eq. (S23) yields

$$H_1 = i\sum_{m\neq n}\sum_{\alpha\beta} |\psi_\beta^m\rangle \frac{\langle\psi_\beta^m|\partial_t H_0|\psi_\alpha^n\rangle}{E_n - E_m} \langle\psi_\alpha^n|. \tag{S25}$$

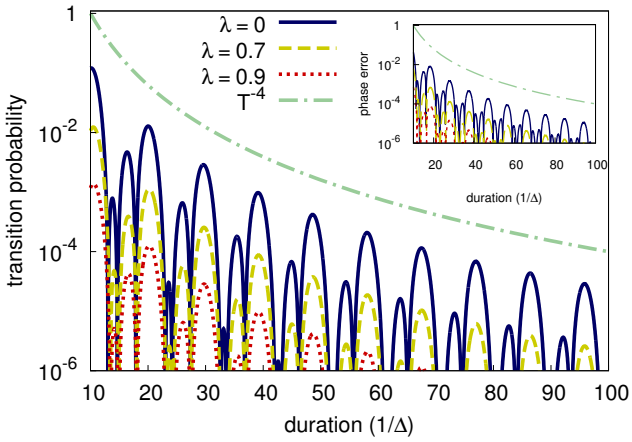

FIG. S1. Diabatic errors *vs* duration of the braiding protocol defined by $\Delta_1(t) = \Delta \sin\varphi(t)$ and $\Delta_3(t) = \Delta\cos\varphi(t)$ for the transition probability out of the degenerate subspace of the initial state. The inset shows the phase error relative to the nonabelian Berry phase. The excitation gap remains unchanged during the entire braiding protocol, which results in a power-law dependence on the duration. The function $\varphi(t)$ is chosen to have zero derivative at both endpoints. For both transition probability and phase error, curves are shown in the absence of counterdiabatic terms and with counterdiabatic terms with 10% and 30% relative error. There would be no diabatic error if the counterdiabatic errors were implemented exactly.

Using this expression, the counterdiabatic terms H_1 can be conveniently derived.

To do so, we temporarily perform rotations within the even and odd subspaces such that $H_0(t)$ in Eq. (S22) involves only the τ_z term. Then, the eigenstates in the even and odd subspaces are simply the "spin-up" and the "spin-down" states. Using

$$\partial_t H_0 = \dot{\Delta}_3\tau_z - \dot{\Delta}_1\tau_y\pi_z - \dot{\Delta}_2\tau_x, \tag{S26}$$

we then find that

$$H_1 = \frac{1}{2(\Delta_1^2+\Delta_2^2+\Delta_3^2)}\left\{(\Delta_3\dot{\Delta}_1 - \Delta_1\dot{\Delta}_3)\tau_x\pi_z + (\Delta_2\dot{\Delta}_3 - \Delta_3\dot{\Delta}_2)\tau_y - (\Delta_1\dot{\Delta}_2 - \Delta_2\dot{\Delta}_1)\tau_z\pi_z\right\}. \tag{S27}$$

This can be readily expressed in terms of the original Majorana operators. Indeed, we have the identities

$$\begin{aligned} i\Delta_{12}\gamma_1\gamma_2 &= -\Delta_{12}(2c_1^\dagger c_1 - 1) \\ i\Delta_{13}\gamma_1\gamma_3 &= \Delta_{13}(c_2c_1 + c_1^\dagger c_2^\dagger - c_2^\dagger c_1 - c_1^\dagger c_2) \\ i\Delta_{23}\gamma_2\gamma_3 &= i\Delta_{23}(c_2c_1 - c_1^\dagger c_2^\dagger + c_1^\dagger c_2 - c_2^\dagger c_1), \end{aligned} \tag{S28}$$

or, in the basis specified above,

$$\begin{aligned} i\Delta_{12}\gamma_1\gamma_2 &= \Delta_{12}\tau_z\pi_z \\ i\Delta_{13}\gamma_1\gamma_3 &= \Delta_{13}\tau_x\pi_z \\ i\Delta_{23}\gamma_2\gamma_3 &= -\Delta_{23}\tau_y. \end{aligned} \tag{S29}$$

Thus, we finally find

$$H_1 = \frac{i}{2(\Delta_1^2+\Delta_2^2+\Delta_3^2)}\left\{(\Delta_2\dot{\Delta}_1 - \Delta_1\dot{\Delta}_2)\gamma_1\gamma_2 + (\Delta_3\dot{\Delta}_1 - \Delta_1\dot{\Delta}_3)\gamma_1\gamma_3 + (\Delta_3\dot{\Delta}_2 - \Delta_2\dot{\Delta}_3)\gamma_2\gamma_3\right\} \tag{S30}$$

in terms of the original Majorana operators.

S4: NUMERICAL CALCULATION OF THE ROBUSTNESS

In this section we provide details of the numerical calculations. For completeness we also include numerical results for the transition probability and Berry phase errors of the non-exponential protocols mentioned in the main text. Due to the

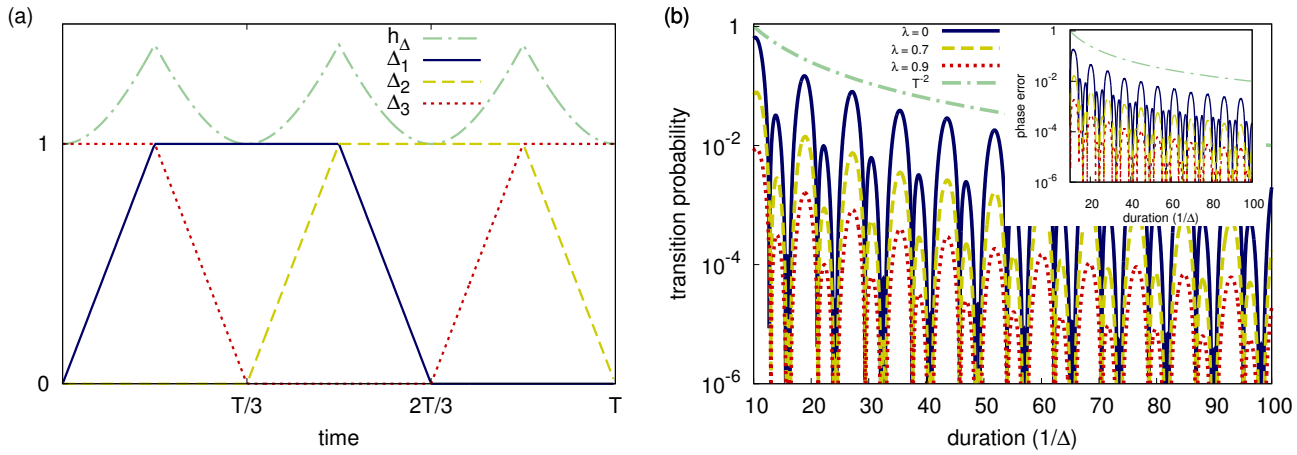

FIG. S2. (a) Couplings Δ_j and gap h_Δ during the braiding process for the protocol in Ref. [30]. (b) Diabatic errors *vs* duration of braiding protocol. Similar to the protocol in Fig. S1 the error has a power-law dependence on the duration. The derivative of $\phi(t)$ jumps at the end points and therefore the error scales as T^{-2}.

conservation of fermion-number parity the nonabelian Berry phase takes the form $\exp(i\gamma\tau_z)$, where τ_z is a Pauli matrix in parity space. Performing the braiding protocol adiabatically yields $\gamma = \pi/4$. For finite durations we numerically compute the Berry phase as $\gamma = \arg[\langle\Psi_e(T)\Psi_e(0)\rangle/\langle\Psi_o(T)\Psi_o(0)\rangle]/2$, where $\Psi_{e/o}(t)$ denotes the ground state wavefunction at time t with even (odd) parity. The diabatic phase error is $|\gamma - \pi/4|$.

We first consider the protocol with the basic step $\Delta_1(t) = \Delta\sin\varphi(t)$ and $\Delta_3(t) = \Delta\cos\varphi(t)$. When $\varphi(t)$ has zero derivative at both endpoints ($\varphi = 0$ for $t = 0$ and $\varphi = \pi/2$ for $t = T/3$), the transition probability scales as T^{-4} with the protocol duration T. This is shown in Fig. S1. Specifically, we have chosen $\varphi(t) = (\pi/2)[3(3t/T)^2 - 2(3t/T)^3]$ between 0 and $t = T/3$ for this calculation (as well as for the one in the main text). When the derivative jumps at either (or both) end points (as for the simplest choice $\varphi(t) = 3\pi t/2T$), we find the errors to decay even more slowly, namely as T^{-2}. Note that both the transition probability out of the degenerate subspace and the phase error of the topological qubit scale in the same manner with T, see inset of Fig. S1.

Interestingly, the same dependences are found for the protocol given in Ref. [30] and displayed in Fig. S2(a). The initial step of the braiding operation is effected by increasing Δ_1 first at constant Δ_3. The latter is reduced to zero only subsequently. In this protocol, the gap increases and takes on a maximum halfway through this basic step. Nevertheless, the diabatic errors still vary as a power law of T. Fig. S2(b) shows corresponding numerical results. Here we chose a linear protocol, $\varphi(t) = 3\pi t/2T$, in which the derivatives of $\varphi(t)$ do not vanish at the end points.