Mobility of Bodies in Contact—Part I: A 2nd-Order Mobility Index for Multiple-Finger Grasps

Elon Rimon and Joel W. Burdick

***Abstract*— Using a configuration-space approach, this paper develops a novel 2nd-order mobility theory for rigid bodies in contact. A major component of this theory is a coordinate invariant 2nd-order mobility index for a body, $\mathcal{B}$, in frictionless contact with finger bodies $\mathcal{A}_1, \cdots, \mathcal{A}_k$. The index is an integer that captures the inherent mobility of $\mathcal{B}$ in an equilibrium grasp due to second order, or surface curvature, effects. It differentiates between grasps which are deemed equivalent by classical 1st-order theories, but are physically different. We further show that 2nd-order effects can be used to lower the effective mobility of a grasped object, and discuss implications of this result for achieving new lower bounds on the number of contacting finger bodies needed to immobilize an object. Physical interpretation and stability analysis of 2nd-order effects are taken up in the companion paper.**

***Index Terms*—Curvature, fixturing, geometry, grasping, kinematics, mobility.**

I. Introduction

We are concerned with the problem of analyzing the mobility of a body, $\mathcal{B}$, in frictionless contact with finger bodies $\mathcal{A}_1, \cdots, \mathcal{A}_k$. Mobility traditionally measures the intrinsic number of instantaneous kinematic degrees of freedom possessed by a coupled system of rigid bodies [1]. In our case $\mathcal{B}$ is coupled with the finger bodies via a general surface contact, and may possibly be free to break contact with any of the fingers. The mobility of bodies in contact has heretofore been studied using 1st-order theories that are based on notions of instantaneous force and velocity [5], [11], [22], [32]. For example, Ohwovoriole and Roth [22] describe the relative motions of contacting bodies in contact in terms of Screw Theory, which is a 1st-order theory. Using 1st-order notions, Reuleaux [23], Somoff [30], Mishra *et al.* [20], and Markenscoff *et al.* [13], derived bounds on the number of frictionless point contacts required for force closure, which is one means to immobilize an object.

However, 1st-order theories are often inadequate in practice. For example, consider the "maximal" and "minimal" three-fingered planar frictionless grasps shown in Fig. 1. First-order theories, such as Screw Theory, indicate that in both

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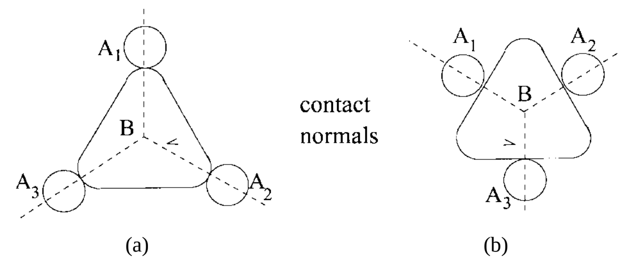

Fig. 1. (a) Maximal three-finger equilibrium grasp. (b) Minimal three-finger grasp.

examples the object being grasped possesses one degree of mobility, which is instantaneous rotation about a vertical axis passing through the point where the contact normals intersect. However, intuition dictates that if the disc fingers are rigidly immobile, the object is completely immobilized in the minimal grasp. This immobility can be rigorously determined using the 2nd-order theory introduced in this paper. We show that the freedom of the grasped object to move in a frictionless equilibrium grasp is not only a function of the surface normals (the basis of the 1st-order mobility theories), but is also a function of the relative curvature of the fingers and the grasped object at the contact points.

The source of deficiency of 1st-order theories is that the relative mobility of an object in contact with finger bodies is *not an infinitesimal notion, but a local one*. One must consider the local motions of the object, not only the tangential aspects of the motions, as employed by the 1st-order theories. In [24], [25] we describe a novel configuration-space based approach for analyzing the ith-order mobility of bodies in contact. This analysis is summarized here for the reader's convenience. The work in [24], [25] also introduces a preliminary notion of a coordinate invariant 2nd-order mobility index, which was defined only for the most trivial case of two-finger grasps. The index measures the effective mobility of a grasped object due to second order, or surface curvature, effects.

The goal of this paper is to provide a complete 2nd-order analysis which is valid for k-finger grasps. First we introduce a coordinate invariant 1st-order mobility index, which captures the effect of the contact points' location and contact normals' orientation on the mobility of the object. The 1st-order index is shown to be solely a function of the number of contact points. (Thus the two grasps in Fig. 1 have the same 1st-order index.) Then we introduce a coordinate invariant 2nd-order mobility index for k-finger grasps. The 2nd-order index differentiates between alternative grasps having the same number of fingers, which are deemed equivalent by the 1st-order theories. For example, it conveys the information that the object in Fig. 1 is not immobilized in the maximal grasp, while it is completely immobilized in the minimal grasp. Furthermore, this new analysis tool suggests that 2nd-order effects can be used to immobilize a grasped object with fewer fingers (or fixtures) than predicted by the 1st-order theories. This insight has important implications for multi-fingered grasp planning and workpiece fixturing applications.

The second goal of this work, taken up in the companion paper, is to investigate the physical forces generated by 2nd-order effects. It is shown in the companion paper that forces which arise from curvature effects cannot be accounted for in a strictly rigid body paradigm. Thus, we introduce a class of elastic deformation contact models to explain the forces that arise from 2nd-order effects. Using these models, we show that first- and 2nd-order immobility implies dynamic stability with elastic contacts. This result provides physical justification and computational modeling tools for applications of 2nd-order immobilization, some of which are listed at the end of the companion paper.

The 1st and 2nd-order indices provide a complete mobility analysis in the following sense. Let *essential equilibrium grasps* be those grasps in which each finger is essential for generating a zero net force and torque on the grasped object. Then the mobility of an object held in an essential equilibrium grasp *can be generically determined from its 1st and 2nd-order indices*. That is, in the space of all possible objects, all objects except those in a set of measure zero have their mobility completely determined by first and 2nd-order effects. Thus there is no need to consider 3rd-order effects, except in special nongeneric cases.

This paper focuses on frictionless contacts. Yet, the tools developed in this paper are also useful when friction is present. Although friction always acts to enhance the immobilization of the grasped object, it may be arbitrarily small or poorly modeled. In contrast, the immobilization based on contact-geometry considerations is always guaranteed to work, no matter what is the particular friction at the contacts[1]. Our goal in developing the mobility indices is to provide a method for computing the effective number of degrees of freedom of a grasped object in a way that is analogous to the mobility theory of closed-loop linkages [1]. (Although closed-loop linkage mobility theories consider only what we would term 1st-order effects.) These analyses *never* include joint friction in the analysis of mobility. If one were to include friction in these analyzes, then the linkage mobility would depend on the current linkage configuration, the amount of friction at the joints, and the amount of driving torque applied to the linkage. Practically speaking, this becomes more of an analysis of jamming due to frictional effects, rather than a theory of mobility. In an analogous manner, we base the mobility indices for a grasped object solely on geometrical effects at the contacts.

In the related literature, the use of 2nd-order effects in the analysis of grasping first appeared in a work by Hanafusa and Asada [9], where planar objects are grasped with three elastic rods. Cai and Roth [2] and Montana [21] developed an expression for the velocity of the point of contact between two rigid bodies that includes the curvature of the contacting bodies. We use their results in [24], [25] to develop the c-space curvature form, which characterizes the 2nd-order geometry of configuration space. This curvature is one component of the 2nd-order effects considered here, but not the only one. Sarkar *et al.* [28] extended the work of [2], [21], and developed an expression for the acceleration of the contact point between two contacting bodies. However, their analysis is not relevant to the issues of mobility considered here. Howard and Kumar [10] developed a stability test for compliant grasps which includes the effects of contact curvature. The relations of our work to that of Howard and Kumar are discussed in the companion paper, where we show that kinematic immobility automatically implies stability when compliance effects are taken into account. Second-order considerations have also appeared in work by Trinkle *et al.* [34], [33] in the study of the stability of frictionless polyhedral objects in the presence of gravity. However, a notion of mobility was not considered in that work. In [16] and [15] we have extended the methods presented in this paper to the analysis of gravitational stability of curved objects.

The paper is organized as follows. Configuration space terminology is reviewed in Section II. The basics of our rigid-body mobility theory are reviewed in Section III. Required facts concerning rigid-body dynamics are reviewed in Section IV. The new results concerning the 2nd-order index of k-finger grasps are described in Section V. Finally, the impact of 2nd-order mobility on the number of frictionless fingers necessary for immobilizing an object is considered.

II. Configuration Space Terminology

In this section, we briefly review the geometrical setting of our mobility analysis. Our analysis is concerned with a rigid object, $\mathcal{B}$, which is located in physical space $\mathcal{W} = I\!R^n$ where $n = 2$ or 3. The object is in contact with rigid "finger bodies" $\mathcal{A}_1, \cdots, \mathcal{A}_k$ which are considered to be stationary. Hence, our analysis is immediately applicable when these finger bodies are interpreted as fixtures, and is also appropriate for the study of prehensile grasps where the mobility of the grasped object relative to the finger tips is of concern. The fingers contact $\mathcal{B}$ with *frictionless point contact* and can deliver any force in the direction normal to the boundary of $\mathcal{B}$. We assume that the boundaries of $\mathcal{B}$ and $\mathcal{A}_1, \cdots, \mathcal{A}_k$ are *smooth*, so that the surface normals are well defined.

For the problems addressed in this paper we can focus on the configuration space of $\mathcal{B}$, rather than the composite configuration space of the $k+1$ rigid bodies. The configuration space of $\mathcal{B}$, termed the *c-space*, is the smooth manifold $\mathcal{C} = I\!R^n \times SO(n)$, where $I\!R^n$ parametrizes the position of $\mathcal{B}$, and $SO(n)$, the group of rotations of $I\!R^n$, parametrizes the orientation of $\mathcal{B}$. We parametrize $SO(3)$ by $\theta \in I\!R^3$, via the usual exponential map parametrization [17], where $\hat{\theta} = \theta / \|\theta\|$ is the axis of rotation and $\|\theta\|$ is the angle of rotation. We regard $SO(2)$ as a subgroup of $SO(3)$ with rotation axis $\hat{\theta}$

[1]Furthermore, results by Mirtich and Canny [19] suggest that frictional grasps chosen on the basis of frictionless analysis yield optimal disturbance rejection.

normal to the plane. Thus we parametrize the manifold $\mathcal{C}$ by a single copy of $\mathbb{R}^m$, where $m = \frac{1}{2}n(n+1)$ ($m = 3$ or 6). We call this parametrization the *hybrid coordinate* parametrization of $\mathcal{C}$, in order to differentiate it from the commonly used exponential coordinates. Points in $\mathbb{R}^m$ are denoted $q = (d, \theta)$. For example, the c-space of a planar object is parametrized by $(d_x, d_y, \theta) \in \mathbb{R}^3$, where $d = (d_x, d_y)$ and the orientation angle θ is periodic in 2π.

Next we review the notion of an "obstacle" in c-space. Let $\mathcal{B}(q)$ denote the subset of $\mathcal{W}$ occupied by $\mathcal{B}$ when $\mathcal{B}$ is at a configuration q. Each finger $\mathcal{A}_i$ has a corresponding *c-space obstacle* in $\mathcal{C}$, denoted $\mathcal{CA}_i$, which is the set of all configurations q such that $\mathcal{B}(q)$ intersects $\mathcal{A}_i$. The boundary of $\mathcal{CA}_i$, denoted $\mathcal{S}_i$, consists of those configurations q where the surfaces of $\mathcal{B}(q)$ and $\mathcal{A}_i$ touch each other, while their interiors are disjoint. It can be verified that $\mathcal{S}_i$ is smooth when $\mathcal{B}$ and $\mathcal{A}_i$ have smooth boundary and maintain point contact. The free configuration space, termed the *freespace* $\mathcal{F}$, is the complement of the c-obstacles' interior. Thus the motions of $\mathcal{B}$ that are free from intersection with the fingers correspond to curves in $\mathcal{F}$. Last, if $\mathcal{B}$ is at a configuration q_0 in contact with k fingers, the point q_0 lies on the boundary of $\mathcal{F}$, at the intersection of the c-obstacle boundaries, $q_0 \in \cap_{i=1}^{k} \mathcal{S}_i$. Fig. 2 schematically illustrates the c-space obstacle of an elliptical planar object due to a disc-shaped finger. Actual examples will be seen in the sequel.

We also review the mapping of $\mathcal{B}$'s body points to their world coordinates. Let r denote points in $\mathcal{B}$'s reference frame and let x denote points in some fixed world reference frame. Given that $\mathcal{B}$ is at a configuration $q = (d, \theta)$, the world coordinates of r are given by the *rigid body transformation*

$$x = X(r, q) \triangleq R(\theta)r + d \tag{1}$$

where $R(\theta)$ is the $n \times n$ orientation matrix of $\mathcal{B}$.

III. A C-Space Approach to Rigid Body Mobility

In this section, we review and formalize some of the essential components of the mobility theory introduced in [25]. This mobility analysis is based on the concept of the *free motions* of $\mathcal{B}$. Let $\mathcal{B}$ be held by k stationary and frictionless fingers $\mathcal{A}_1, \cdots, \mathcal{A}_k$ in an equilibrium grasp. The free motions of $\mathcal{B}$ are those local motions of $\mathcal{B}$ along which it either breaks away from or roll-slides[2] on the surface of the finger bodies. More precisely, let q_0 be $\mathcal{B}$'s configuration, and let $\mathcal{D}$ be a small m-dimensional ball centered at q_0. The local free motions are the c-space paths which emanate from q_0 and lie in $\mathcal{D} \cap \mathcal{F}$. As we shall see, the 1st-order (i.e., tangents and tangent hyperplanes) and 2nd-order (i.e., curvatures and curvature forms) properties of these paths and the c-obstacle boundaries can be directly related to the mobility of $\mathcal{B}$ at the equilibrium grasp. The 1st-order properties of these curves can be equated to other well known 1st-order theories such as Screw Theory [25]. We cast these notions in the configuration-space framework, as this new interpretation is the basis for the novel consideration of second and higher order aspects of mobility.

[2] By a "roll-slide" motion, we mean a general displacement between two bodies which maintains surface contact.

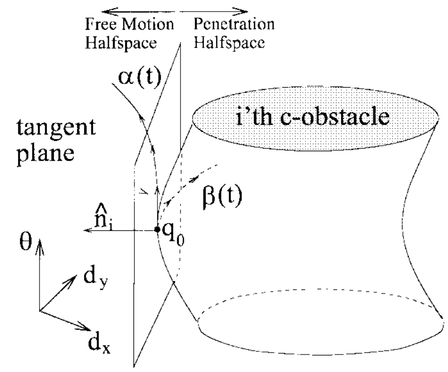

Fig. 2. The 1st-order approximation to the free motions of $\mathcal{B}$ at q_0. $\dot{\alpha}(0)$ and $\dot{\beta}(0)$ are 1st-order roll-slide motions. $\alpha(t)$ locally lies in $\mathcal{F}$, $\beta(t)$ locally lies in the c-obstacle.

To make this *key concept* precise and to aid in the modeling of the rigidity of the contacting bodies, we introduce the following signed c-space distance function. It measures the distance of a configuration point q from $\mathcal{S}_i$ as follows:

$$d_i(q) \triangleq \begin{cases} \mathtt{dst}(q, \mathcal{S}_i) & \text{if } q \text{ is outside of } \mathcal{CA}_i \\ 0 & \text{if } q \text{ is on } \mathcal{S}_i \\ -\mathtt{dst}(q, \mathcal{S}_i) & \text{if } q \text{ is in the interior of } \mathcal{CA}_i \end{cases}$$

where $\mathtt{dst}(q, \mathcal{S}_i)$ is the minimal Euclidean distance of q from $\mathcal{S}_i$ in the c-space parametrization. Note that $d_i(q) = 0$ when $\mathcal{B}(q)$ is in contact with $\mathcal{A}_i$, since $q \in \mathcal{S}_i$. The unit normal to $\mathcal{S}_i$ at q, pointing outward with respect to $\mathcal{CA}_i$, is denoted $\hat{n}_i(q)$ (Fig. 2).

A. 1st-Order Free Motions

We now review the 1st-order properties of the free-motion curves, which lead to a *1st-order mobility theory*. Let $\alpha(t)$ be a smooth c-space path such that $\alpha(0) = q_0$, $q_0 \in \mathcal{S}_i$, and let $\dot{\alpha}(0) = \dot{q}$. The set of 1st-order free motions of $\mathcal{B}$ at q_0 is related to the following 1st-order Taylor expansion of d_i along $\alpha(t)$

$$(d_i \circ \alpha)(t) = d_i(q_0) + (\nabla d_i(q_0) \cdot \dot{q})t + O(t^2)$$

where $d_i \circ \alpha$ denotes function composition and ∇d_i denotes the gradient of d_i. First note that $d_i(q_0) = 0$, since $q_0 \in \mathcal{S}_i$. Second, it can be shown that $\|\nabla d_i(q_0)\| = 1$ for the Euclidean distance [3, p. 66]. It can also be verified that $\nabla d_i(q_0)$ is equal to $\hat{n}_i(q_0)$, the outward pointing unit normal to $\mathcal{S}_i$ [25]. Thus we get in a neighborhood of q_0

$$(d_i \circ \alpha)(t) = (\hat{n}_i(q_0) \cdot \dot{q})t + O(t^2). \tag{2}$$

The following definition characterizes the free-motion curves in terms of their tangents. We use the notation $T_{q_0}\mathbb{R}^m$ for the tangent space at q_0 of the parametrization of $\mathcal{C}$ by $\mathbb{R}^m$ ($T_{q_0}\mathbb{R}^m \cong \mathbb{R}^m$), and $T_{q_0}\mathcal{S}_i$ for the tangent space of $\mathcal{S}_i$ at q_0 ($T_{q_0}\mathcal{S}_i \cong \mathbb{R}^{m-1}$).

Definition 1 [25]: Let $\mathcal{B}$ be at a configuration q_0, in contact with $\mathcal{A}_i$. The **1st-order free motions** of $\mathcal{B}$ at q_0 is the set (halfspace of $T_{q_0}\mathbb{R}^m$)

$$M^1_{q_0}(\mathcal{CA}_i) \triangleq \{\dot{q} \in T_{q_0}\mathbb{R}^m : \hat{n}_i(q_0) \cdot \dot{q} \geq 0\}.$$

The halfspace's boundary, $T_{q_0}\mathcal{S}_i = \{\dot{q} \in T_{q_0}\mathbb{R}^m : \hat{n}_i(q_0)\cdot\dot{q} = 0\}$, is called the set of **1st-order roll-slide motions**. Its interior, $\{\dot{q} \in T_{q_0}\mathbb{R}^m : \hat{n}_i(q_0)\cdot\dot{q} > 0\}$, is termed the set of **1st-order escape motions**. For k fingers, the set of 1st-order free motions is

$$M^1_{q_0}(\mathcal{CA}_1,\cdots,\mathcal{CA}_k) \triangleq \bigcap_{i=1}^{k} M^1_{q_0}(\mathcal{CA}_i)$$

(we shall hereafter use $M^1_{q_0}$ as shorthand notation for $M^1_{q_0}(\mathcal{CA}_1,\cdots,\mathcal{CA}_k)$).

In other words, along escape motions the distance d_i is increasing to first order, while d_i is zero to first order along 1st-order roll-slide motions. In the classical Screw Theory [22], 1st-order roll-slide motions are represented by reciprocal screws, while the 1st-order escape motions are represented by repelling screws. In Desai's work [5], the set $M^1_{q_0}(\mathcal{CA}_1,\cdots,\mathcal{CA}_k)$ is called the separation cone associated with the k contacts. We also note that the finger c-obstacle normal $\hat{n}_i$ used in the definition has a physical interpretation as a generalized force or wrench. This interpretation is discussed in Theorem 1 below.

The following are two important properties of the 1st-order free motions. The first is their *coordinate invariance*. Given two parametrizations of the same c-space, q and $\bar{q}$, let $\bar{q}_0$ be the parametrization of q_0 in the $\bar{q}$ coordinates, and let $\overline{(\cdot)}$ denote objects in the $\bar{q}$ coordinates. Then it can be shown that [25]:

$$\dot{q} \in M^1_{q_0}(\mathcal{CA}_1,\cdots,\mathcal{CA}_k) \quad \text{iff } \dot{\bar{q}} \in M^1_{\bar{q}_0}(\overline{\mathcal{CA}}_1,\cdots,\overline{\mathcal{CA}}_k).$$

Coordinate invariance is an important property in the development of our theory. The structures we define, such as $M^1_{q_0}(\mathcal{CA}_i)$, involve an inner product, which is not necessarily preserved by coordinate transformations [6]. Thus we must explicitly check for coordinate invariance. Practically speaking, coordinate invariance implies that the results will be the same regardless of the choice of the world reference frame or $\mathcal{B}$'s body fixed frame.

Second, the 1st-order free motions admit the following geometrical interpretation. If $\dot{q} \in M^1_{q_0}(\mathcal{CA}_i)$ is a 1st-order escape motion, its corresponding path, $\alpha(t)$ with $\alpha(0) = q_0$ and $\dot{\alpha}(0) = \dot{q}$, *locally lies in the freespace*, for all $t \in [0,\epsilon]$, for some $\epsilon > 0$. That is, $\mathcal{B}(\alpha(t))$ locally breaks away from $\mathcal{A}_i$, no matter the value of the higher derivatives of $\alpha(t)$. If $\dot{q} \in M^1_{q_0}(\mathcal{CA}_i)$ is a 1st-order roll-slide motion, *it is not possible* to determine from (2) if $\alpha(t)$ locally lies in the freespace or if it enters $\mathcal{CA}_i$. For example, the curves $\alpha(t)$ and $\beta(t)$ in Fig. 2 have the same tangent vector at q_0, and thus are equivalent to first order. Yet $\alpha(t)$ locally lies in the freespace, while $\beta(t)$ does not. As we shall see, *all the free motions of an object held in an equilibrium grasp are roll-slide to first order*. This key fact implies that 1st-order properties of the free motion curves (which are the basis for classical mobility theories) do not suffice for determining the mobility of an object held in an equilibrium grasp. This leads us to consider the 2nd-order properties of the free motion curves.

B. 2nd-Order Free Motions

We now consider the 2nd-order characteristics of the free-motion curves, as these lead to our novel 2nd-order mobility theory. Let $\alpha(t)$ be a smooth c-space path such that $\alpha(0) = q_0$, $\dot{\alpha}(0) = \dot{q}$, and $\ddot{\alpha}(0) = \ddot{q}$. Analogous to the 1st-order free motions, the *2nd-order free motions* are related to the following 2nd-order Taylor expansion of d_i along $\alpha(t)$:

$$\begin{aligned}(d_i \circ \alpha)(t) &= d_i(q_0) + (\nabla d_i(q_0)\cdot\dot{q})t + \frac{1}{2}(\dot{q}^T D^2 d_i(q_0)\dot{q} \\ &\quad + \nabla d_i(q_0)\cdot\ddot{q})t^2 + O(t^3) \\ &= (\hat{n}_i(q_0)\cdot\dot{q})t + \frac{1}{2}(\dot{q}^T D^2 d_i(q_0)\dot{q} \\ &\quad + \hat{n}_i(q_0)\cdot\ddot{q})t^2 + O(t^3) \qquad (3)\end{aligned}$$

since $d_i(q_0) = 0$ and $\nabla d_i(q_0) = \hat{n}_i(q_0)$. In general, if $\hat{n}_i(q_0)\cdot\dot{q} > 0$ i.e., when $\dot{q}$ is a 1st-order escape motion, the linear term in (3) locally determines the sign of $(d_i \circ \alpha)(t)$ in a neighborhood of q_0. The 2nd-order term affects the sign of $(d_i \circ \alpha)(t)$ only when $\dot{\alpha}(0) = \dot{q}$ lies in $T_{q_0}\mathcal{S}_i$, i.e. when $\dot{q}$ is a 1st-order roll-slide motion. We thus limit our definition of 2nd-order free motions (given below) to those motions which are 1st-order roll-slide motions.

Since $\nabla d_i(q) = \hat{n}_i(q)$ for all points $q \in \mathcal{S}_i$, we have that $\dot{q}^T D^2 d_i(q_0)\dot{q} = \dot{q}^T[D\hat{n}_i(q_0)]\dot{q}$ for all $\dot{q} \in T_{q_0}\mathcal{S}_i$. The quadratic form $\dot{q}^T[D\hat{n}_i(q_0)]\dot{q}$ is the *curvature form* of $\mathcal{S}_i$ at q_0, and it expresses the curvature of the c-space obstacle boundary at q_0. An expression for the curvature form in terms of the object and finger geometries at the contact points is derived in [25]. This expression is repeated in the appendix for the reader's convenience.

It is clear from (3) that the free-motion curves are determined to 2nd-order by their *velocity and acceleration* at q_0. The collection $(q_0, \dot{q}, \ddot{q})$ of all velocities and accelerations of paths $\alpha(t)$ such that $\alpha(0) = q_0$ is called the *2nd jet space* at q_0 of the parametrization of $\mathcal{C}$ by $\mathbb{R}^m$, and is denoted $J^2_{q_0}\mathbb{R}^m$ [8].

Definition 2 [25]: The 2nd-order free motions of $\mathcal{B}$ at q_0 is the subset of $(\dot{q},\ddot{q})$ in $J^2_{q_0}\mathbb{R}^m$ satisfying

$$\begin{aligned}M^2_{q_0}(\mathcal{CA}_i) \triangleq \big\{(\dot{q},\ddot{q}) \in J^2_{q_0}\mathbb{R}^m &: \hat{n}_i(q_0)\cdot\dot{q} = 0 \\ &\text{and } \dot{q}^T[D\hat{n}_i(q_0)]\dot{q} + \hat{n}_i(q_0)\cdot\ddot{q} \geq 0\big\}.\end{aligned}$$

Analogous to the first order case, pairs $(\dot{q},\ddot{q})$ which satisfy $\hat{n}_i(q_0)\cdot\dot{q} = 0$ and $\dot{q}^T[D\hat{n}_i(q_0)]\dot{q} + \hat{n}_i(q_0)\cdot\ddot{q} = 0$ are called **2nd order roll-slide motions**, and the other pairs in $M^2_{q_0}(\mathcal{CA}_i)$ are termed **2nd order escape motions**. For k fingers

$$M^2_{q_0}(\mathcal{CA}_1,\cdots,\mathcal{CA}_k) \triangleq \bigcap_{i=1}^{k} M^2_{q_0}(\mathcal{CA}_i).$$

Note that our definition of 2nd-order free motions focuses on those curves which are 1st-order roll-slide motions, but might not correspond to free-motion curves. The 2nd-order free motions possess the following two important properties (which are proved in [25]). First, they are *coordinate invariant*. Given two parametrizations of the same c-space, q and $\bar{q}$ (as mentioned above)

$$\begin{aligned}&(\dot{q},\ddot{q}) \in M^2_{q_0}(\mathcal{CA}_1,\cdots,\mathcal{CA}_k) \\ &\qquad \text{iff } (\dot{\bar{q}},\ddot{\bar{q}}) \in M^2_{\bar{q}_0}(\overline{\mathcal{CA}}_1,\cdots,\overline{\mathcal{CA}}_k).\end{aligned}$$

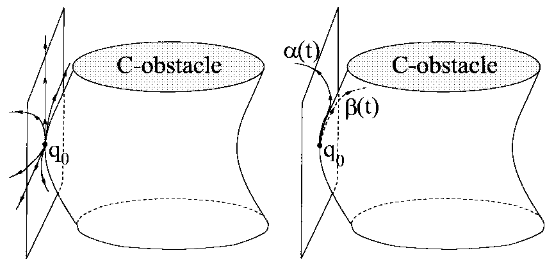

Fig. 3. (a) Second-order escape motions. (b) 2nd-order roll-slide motions.

Second, the 2nd-order free motions admit the following geometrical interpretation. If $(\dot{q}, \ddot{q}) \in M^2_{q_0}(\mathcal{CA}_i)$ is a 2nd-order escape motion, its corresponding path, $\alpha(t)$ with $\alpha(0) = q_0$, $\dot{\alpha}(0) = \dot{q}$, $\ddot{\alpha}(0) = \ddot{q}$, *locally lies in the freespace,* for all $t \in [0, \epsilon]$, for some $\epsilon > 0$. If $(\dot{q}, \ddot{q}) \in M^2_{q_0}(\mathcal{CA}_i)$ is a 2nd order roll-slide motion, *it is not possible* to determine from (3) if $\alpha(t)$ locally lies in the free space or if it enters $\mathcal{CA}_i$. The two types of motions are illustrated in Fig. 3.

IV. Review of Relevant Rigid Body Dynamics

This section reviews some required facts concerning rigid-body dynamics, leading to the characterization of an equilibrium grasp in c-space. Let the real-world finger forces acting on $\mathcal{B}$ be $F_1(x_1), \cdots, F_k(x_k)$, where x_i is the contact point between $\mathcal{A}_i$ and $\mathcal{B}$, for $i = 1, \cdots, k$. These forces give rise to a net generalized force, w, in $\mathcal{B}$'s c-space. It is called a *wrench* when c-space is parametrized by exponential coordinates [17], but we shall call it a wrench in any c-space parametrization. The Lagrangian equation of motion for $\mathcal{B}$ is:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}K - \frac{\partial}{\partial q}K = w, \tag{4}$$

where the kinetic energy K is given by $K(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}$, where $M(q)$ is the $m \times m$ inertia matrix of $\mathcal{B}$. We neglect potential energy effects, such as gravity, and assume that the wrench w is generated solely by the finger forces. Gravitational effects have been considered in [16]. The wrench generated by the finger forces $F_1(x_1), \cdots, F_k(x_k)$ is denoted $w(q; F_1, \cdots, F_k)$. The change in the kinetic energy of $\mathcal{B}$ along motions of (4) is

$$\frac{d}{dt}K(q, \dot{q}) = w(t) \cdot \dot{q}. \tag{5}$$

Equation (5) leads to the interpretation of wrenches as *covectors*, i.e., linear functions $w : T_q\mathbb{R}^m \to \mathbb{R}$ mapping the velocity vectors $\dot{q}$ to the scalars $\dot{K} = w \cdot \dot{q}$, representing the instantaneous *work* done by the force. We represent a covector, $\tilde{c}$, as a tangent vector, c, via the relationship $\tilde{c}(\dot{q}) = c \cdot \dot{q}$.

Theorem 1 below relates the net wrench on $\mathcal{B}$ to the 1st-order geometry of the fingers' c-obstacles. The theorem is well known and is based on the *virtual work principal* [29]. Let $X_r(q) = X(r, q)$ denote the rigid-body transformation when the object point r is kept fixed on $\mathcal{B}$, while only q varies. Further, let r_i be the description of the contact point x_i in $\mathcal{B}$'s body coordinates.

Theorem 1 [25]: The **wrench** due to a single-finger contact force $F_i(x_i)$ acting on $\mathcal{B}(q)$ is

$$w(q, F_i) = \left[DX_{r_i}(q)\right]^T F_i(x_i) \tag{6}$$

where $DX_{r_i}(q) = \frac{d}{dq}X_{r_i}(q)$, and $w(q, F_i)$ is written as a column vector (i.e. the covector $w(q, F_i)$ is represented as a tangent vector). If $F_i(x_i)$ is **normal** to the surface of $\mathcal{B}(q)$ at x_i and is pushing into $\mathcal{B}(q)$, then w is **normal** to $\mathcal{S}_i$, pointing outward with respect to $\mathcal{CA}_i$ (Fig. 2)

$$w(q, F_i) = \lambda_i \hat{n}_i(q) \quad \text{for some } \lambda_i \geq 0.$$

More generally, if $k \geq 1$ fingers push on $\mathcal{B}(q)$ with normal forces $F_1(x_1), \cdots, F_k(x_k)$, the **net wrench** w is **orthogonal** to the subspace generated by intersection of the tangent hyperplanes to the individual finger c-obstacles, and is given by

$$w(q; F_1, \cdots, F_k) = \sum_{i=1}^{k} w(q, F_i) = \sum_{i=1}^{k} \lambda_i \hat{n}_i(q) \quad \lambda_i \geq 0.$$

Remark: The orthogonality between w and a subspace V of tangent vectors is actually a representation of the fact that the action of the covector $\tilde{w}$ on $\dot{q} \in V$ yields zero power, i.e., $\tilde{w}(\dot{q}) = 0$ for all $\dot{q} \in V$. It can be shown that this notion is independent of the specific metric used to represent the action of $\tilde{w}$ on $\dot{q}$ in the form $w \cdot \dot{q}$, and is thus coordinate invariant.

We wish to determine the mobility of $\mathcal{B}$ when it is held in a k-finger *equilibrium grasp*. Since the net wrench acting on $\mathcal{B}$ at an equilibrium grasp must be zero, we get from Theorem 1 the following c-space geometrical characterization of an equilibrium grasp.

Corollary 4.1 [25]: Let $\mathcal{B}$ be at a configuration q_0, and let k fingers push on $\mathcal{B}$ with normal forces $F_1(x_1), \cdots, F_k(x_k)$. Then q_0 can be made an **equilibrium grasp** by a suitable choice of the finger force magnitudes iff zero lies in the **convex hull** of the c-obstacle normals $\hat{n}_1(q_0), \cdots, \hat{n}_k(q_0)$

$$0 = \lambda_1 \hat{n}_1(q_0) + \cdots + \lambda_k \hat{n}_k(q_0) \tag{7}$$

for some scalars $\lambda_1, \cdots, \lambda_k$ such that $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$.

V. The Mobility Index of k-Finger Grasps

A *mobility index* is an integer-valued function that measures the mobility, or effective number of degrees of freedom, of $\mathcal{B}$ when it is held in an equilibrium-grasp configuration. In this section, we derive 1st and 2nd-order mobility indices based on the 1st and 2nd-order free motions of $\mathcal{B}$. First, in Section V-A, we limit our analysis to nonredundant finger arrangements. This restriction simplifies the analysis, and the ensuing results can be extended to include redundant finger arrangements. In Section V-B, we discuss the fact that the 1st-order mobility index is *identical* for all k-fingered equilibrium grasps. This inability of 1st-order theories to differentiate between grasps which use the same number of fingers motivates our development of the 2nd-order index in Section V-C. Throughout this section $\mathcal{B}$ is held at an equilibrium configuration q_0 by frictionless fingers $\mathcal{A}_1, \cdots, \mathcal{A}_k$.

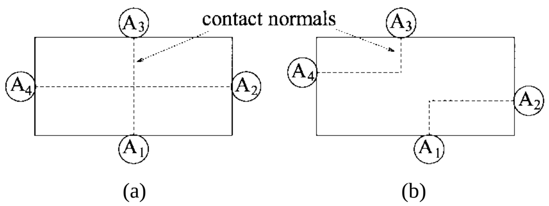

Fig. 4. (a) A nonessential equilibrium arrangement. (b) An essential one.

A. Essential Finger Arrangements

We restrict our attention to the following generic type of equilibrium grasps, called *essential* equilibrium grasps.

Definition 3: A finger $\mathcal{A}_i$ is **essential** to the grasp if its force is necessary for maintaining the equilibrium. Equivalently, the finger c-obstacle normal $\hat{n}_i(q_0)$ is necessary for spanning the origin in (7).

An essential grasp is not necessarily an immobilizing grasp. As shown in Proposition 5.3 below, the essential grasp restricts the 1st-order motions of the object to a subspace.

The following lemma characterizes the coefficients $\lambda_1, \cdots, \lambda_k$ which appear in the equilibrium equation (7). A proof of the lemma appears in the appendix.

Lemma 5.1: The coefficients $\lambda_1, \cdots, \lambda_k$ in (7) are nonzero and unique iff the fingers $\mathcal{A}_1, \cdots, \mathcal{A}_k$ are all **essential**.

Thus, once the fingers are positioned around $\mathcal{B}$ in an essential equilibrium arrangement, the λ_i's are fixed. In other words, the λ_i's are uniquely determined (up to a scaling factor which we take to be unity in our normalizations) for a given collection of contact normals $\{\hat{n}_1(q_0), \cdots, \hat{n}_k(q_0)\}$. We also need the following corollary to the lemma, whose proof appears in the appendix.

Corollary 5.2: If the k fingers participating in the equilibrium grasp are **essential**, then any $k-1$ finger c-obstacle normals from $\{\hat{n}_1(q_0), \cdots, \hat{n}_k(q_0)\}$ are **linearly independent**.

Remark: It follows from the corollary that k fingers can possibly be essential only for $k \leq m+1$ fingers, where $k \leq 4$ for two-dimensional (2-D) grasps and $k \leq 7$ for three-dimensional (3-D) grasps. There is no problem with this upper bound, since we are about to see that generic equilibrium grasps involving $k \geq m+1$ fingers completely immobilize $\mathcal{B}$ due to 1st-order effects. In Section V-D, we generalize the definition of essential grasps to an arbitrary number of fingers.

The equilibrium grasp in Fig. 4(a) is an example of a nonessential finger arrangement, since the equilibrium can be maintained with a single antipodal pair. Fig. 4(b) illustrates the notion that essential finger arrangements are *generic*. Consider a given nonessential equilibrium arrangement with $k \leq m+1$ fingers. Then almost any equilibrium arrangement obtained by small *generic* movement of the fingers along the boundary of $\mathcal{B}$ is essential[3]. Although we do not actually prove this property, it is clearly forthcoming. We henceforth consider only essential equilibrium grasps. (See Section V-D for further discussion of nonessential grasps.)

[3] More precisely, let $\mathcal{U} = \mathbb{R}^4$ be the space parametrizing the position of the four contact points, and let $\mathcal{E} \subset \mathcal{U}$ be the set of equilibrium grasps. Then $\mathcal{E}$ is an open dense subset of $\mathcal{U}$, and the nonessential grasps form a codimension one (or measure zero) subset of $\mathcal{E}$.

B. All k-Finger Grasps have the Same 1st-Order Index

Suppose that $\mathcal{B}$ contacts k fingers. According to Definition 1, the 1st-order free motions of $\mathcal{B}$ lie in the intersection of the 1st-order free motion halfspaces associated with the individual fingers. At an equilibrium grasp the intersection forms a subspace, as is made precise in the following proposition.

Proposition 5.3 [25]: Let $\mathcal{B}$ be held in a k-finger equilibrium grasp at a configuration q_0. Then the 1st-order free motions of $\mathcal{B}$, $M^1_{q_0}$, form a **subspace** of $T_{q_0}\mathbb{R}^m$, given by the intersection of the tangent hyperplanes to the finger c-obstacles, $M^1_{q_0} = \cap_{i=1}^k T_{q_0}\mathcal{S}_i$.

Thus, the only possible 1st-order free motions of $\mathcal{B}$ at an equilibrium grasp are 1st-order roll-slide motions with respect to each of the fingers. The following proposition is the basis for our definition of the 1st-order mobility index.

Proposition 5.4: If all the fingers are **essential** for the equilibrium, the dimension of $M^1_{q_0}$ is $m-k+1$, where $m=3$ or 6. (By definition $k \leq m+1$, hence $m-k+1 \geq 0$.)

The **1st-order mobility index** of an equilibrium grasp is defined as the dimension of $M^1_{q_0}$

$$m^1_{q_0} \triangleq m-k+1. \tag{8}$$

The 1st-order mobility index is *coordinate invariant*, a fact which follows from the coordinate invariance of the 1st-order free motions associated with the individual fingers. A key fact expressed by (8) is that *$m^1_{q_0}$ is identical for all k-fingered grasps*. For example, the three-finger grasps in Fig. 1 have the same 1st-order mobility index of unity (since $m=k=3$). Moreover, *any* 1st-order theory, such as Screw Theory, will be similarly unable to discriminate between equilibrium grasps having the same number of fingers. Only our novel 2nd-order index introduced in the next section can differentiate between grasps involving the same number of fingers.

C. The 2nd-Order Mobility Index

We now present a coordinate invariant 2nd-order mobility index for k-finger equilibrium grasps. This new result is a natural extension of the mobility index introduced in [25] for two fingers. At the equilibrium configuration, q_0, the c-obstacle boundaries intersect each other, $q_0 \in \cap_{i=1}^k \mathcal{S}_i$, and the 1st-order free motions of $\mathcal{B}$ are $M^1_{q_0} = \cap_{i=1}^k T_{q_0}\mathcal{S}_i$. Recall that $d_i(q)$ is the signed distance of a configuration point q from $\mathcal{S}_i$. Also recall that the 2nd-order geometry of each $\mathcal{S}_i$ is captured by its curvature form, denoted $\kappa_i(q_0, \dot{q})$, where

$$\kappa_i(q_0, \dot{q}) = \dot{q}^T D^2 d_i(q_0)\dot{q} = \dot{q}^T D\hat{n}_i(q_0)\dot{q} \quad \dot{q} \in T_{q_0}\mathcal{S}_i.$$

We now use the λ_i's in the equilibrium equation (7) to define a weighted sum of the fingers' c-space curvature forms.

Definition 4: Let $\lambda_1, \cdots, \lambda_k$ be the coefficients of the equilibrium equation (7). The **c-space relative distance** of an equilibrium grasp is the real-valued function d_{rel} defined by

$$d_{\text{rel}}(q) \triangleq \lambda_1 d_1(q) + \cdots + \lambda_k d_k(q).$$

The **c**-space relative curvature form for the equilibrium grasp is the quadratic form

$$\kappa_{\rm rel}(q_0, \dot{q}) \triangleq \dot{q}^T D^2 d_{\rm rel}(q_0)\dot{q} = \dot{q}^T \left[\sum_{i=1}^{k} \lambda_i D^2 d_i(q_0)\right] \dot{q} = \sum_{i=1}^{k} \lambda_i \kappa_i(q_0, \dot{q}) \quad \text{such that } \dot{q} \in M^1_{q_0}. \tag{9}$$

According to Lemma 5.1, the λ_i's are unique for an essential equilibrium grasp. Hence $d_{\rm rel}$ and $\kappa_{\rm rel}$ are well defined.

In Appendix A it is shown that an individual c-obstacle curvature form is in general not coordinate invariant. However, the relative curvature form measures the *relative* curvature between the c-obstacle boundaries that meet at an equilibrium configuration. Since $\hat{n}_i(q_0) = \nabla d_i(q_0)$ for $i = 1, \cdots, k,$ we have from (7) that

$$\nabla d_{\rm rel}(q_0) = \lambda_1 \nabla d_1(q_0) + \cdots + \lambda_k \nabla d_k(q_0) = 0. \tag{10}$$

Hence $d_{\rm rel}$ has a *critical point* at q_0[4]. We shall see that this fact guarantees the desired coordinate invariance of the 2nd-order index. Let us first define the 2nd-order mobility index.

Definition 5: The **2nd-order mobility index** of an equilibrium grasp configuration, denoted $m^2_{q_0}$, is the **number of nonnegative eigenvalues** of the matrix associated with the **c-space relative curvature**, $\sum_{i=1}^k \lambda_i D^2 d_i(q_0)$, at q_0.

Remark: The relative curvature form is defined for $\dot{q} \in M^1_{q_0}$. Hence the dimension of $M^1_{q_0}$, $m^1_{q_0}$, is an upper bound on the possible values of $m^2_{q_0}$ i.e., $0 \le m^2_{q_0} \le m^1_{q_0}$. In particular, if $m^1_{q_0} = 0$, $\mathcal{B}$ is completely immobilized to first order, and the 2nd-order index carries no immediately useful information. The 2nd-order index is always useful for 2-D grasps involving $k = 2, 3$ fingers, and for 3-D grasps involving $k = 2, \cdots, 6$ fingers. (Any equilibrium grasp must have at least two fingers.) In these cases $\mathcal{B}$ is not immobilized to first order $(m^1_{q_0} > 0)$, but may be immobilized to second order $(m^2_{q_0} = 0)$. Of course, there can be degenerate equilibrium grasps, such as 4-fingered planar grasps whose fingers contact normals intersect at a single point [Fig. 4(a)], for which the subspace $M^1_{q_0}$ is still nontrivial, and the 2nd-order index is useful there too.

The following proposition and theorem discuss important properties of the 2nd-order index. The first is its coordinate invariance. Let $\sigma : I\!\!R^m \to I\!\!R$ be a smooth real-valued function with a critical point at q_0, $\nabla\sigma(q_0) = 0$. The *Morse index* of σ at q_0 is the number of negative eigenvalues of its second derivative matrix $D^2\sigma(q_0)$ [18]. It can be verified by application of the chain rule that the Morse index is coordinate invariant. This is stated in the following lemma in our slightly more general context:

Lemma 5.5: Let $f : I\!\!R^m \to I\!\!R^m$ be a smooth coordinate transformation (a diffeomorphism) such that $q = f(\bar{q})$, and let $\tau(\bar{q}) = (\sigma \circ f)(\bar{q})$. Then there is **one-to-one correspondence** between the critical points of σ and τ

$$\nabla\tau(\bar{q}_0) = 0 \quad \text{iff } \nabla\sigma(q_0) = 0.$$

[4]Let $f : I\!\!R^m \to I\!\!R$ be a smooth real-valued function. Then a point $x \in I\!\!R^m$ is a *critical point* of f if the derivative of f vanishes there.

Additionally, the sign of the eigenvalues of $D^2\sigma$ is **preserved** when evaluated at a critical point. That is, for every $\dot{\bar{q}} \in T_{\bar{q}_0} I\!\!R^m$ there exists $\dot{q} \in T_{q_0} I\!\!R^m$, $\dot{q} = Df(\bar{q}_0)\dot{\bar{q}}$, such that

$$\dot{q}^T[D^2\sigma(q_0)]\dot{q} = \dot{\bar{q}}^T[D^2\tau(\bar{q}_0)]\dot{\bar{q}}. \tag{11}$$

In our case $\sigma(q) = d_{\rm rel}(q)$ is the relative distance function, and $\nabla d_{\rm rel}(q_0) = 0$ at an equilibrium grasp q_0. The lemma, together with the coordinate invariance of the subspace of 1st-order free motions (Section III-A), implies the following invariance property:

Proposition 5.6: Let q and $\bar{q}$ be two parametrizations of the same c-space, related by $q = f(\bar{q})$. Let q_0 and $\bar{q}_0$ be the equilibrium configuration in the respective parametrization. Then the **2nd order mobility index** is **preserved** under the coordinate transformation i.e., $m^2_{q_0} = m^2_{\bar{q}_0}$.

The following theorem provides a geometrical interpretation for the 2nd-order index, and is a key contribution of this paper.

Theorem 2: Let $\dot{q} \in M^1_{q_0}$ be a 1st-order free motion of $\mathcal{B}$ at the equilibrium. If $\kappa_{\rm rel}(q_0, \dot{q}) > 0$, there **exists** $\ddot{q}$ such that the path $\alpha(t)$ with $\alpha(0) = q_0$, $\dot{\alpha}(0) = \dot{q}$, $\ddot{\alpha}(0) = \ddot{q}$ locally lies **in the freespace**

$$\alpha(t) \in \mathcal{F} \quad \text{for all } t \in [0, \epsilon], \text{ for some } \epsilon > 0.$$

If $\kappa(q_0, \dot{q}) < 0$, **any** $\ddot{q}$ yields a path $\alpha(t)$ that locally lies **outside the freespace**

$$\alpha(t) \in I\!\!R^m - \mathcal{F} \quad \text{for all } t \in (0, \epsilon], \text{ for some } \epsilon > 0.$$

The theorem provides the following interpretation for the 2nd-order index. Let K be the matrix associated with the c-space relative curvature form. If K has at least one positive eigenvalue then there exists a vector $\dot{q}$ such that $\kappa_{\rm rel}(q_0, \dot{q}) > 0$. If all the eigenvalues of K are negative then $\kappa_{\rm rel}(q_0, \dot{q}) < 0$ for all $\dot{q} \in M^1_{q_0}$. In other words, if K has positive eigenvalues then $\mathcal{B}$ is not immobilized, since there exist 2nd-order escape motions. However, if K has all negative eigenvalues i.e., $m^2_{q_0} = 0$, then $\mathcal{B}$ is in fact completely immobilized, even though it is mobile to first order. In the special case where K has a zero eigenvalue the mobility of $\mathcal{B}$ must be determined from 3rd-order geometrical effects, which are not considered here. However, the 1st and 2nd-order indices completely determine the mobility of *generic* objects. The set of objects for which 3rd-order considerations are necessary is a set of zero measure in the space of all objects.

Proof: By hypothesis, $q_0 \in \cap_{i=1}^k \mathcal{S}_i$ and $\dot{q} \in M^1_{q_0} = \cap_{i=1}^k T_{q_0}\mathcal{S}_i$. Hence $d_i(q_0) = 0$ and $\dot{q} \perp \nabla d_i(q_0)$ for $i = 1, \cdots, k$. The 2nd-order expansion of $(d_i \circ \alpha)(t)$ is thus given by

$$(d_i \circ \alpha)(t) = \frac{1}{2}\{\dot{q}^T[D^2 d_i(q_0)]\dot{q} + \nabla d_i(q_0) \cdot \ddot{q}\}t^2 \quad \text{for } i = 1, \cdots, k.$$

We shall use the notation $a_i = \dot{q}^T[D^2 d_i(q_0)]\dot{q}$ for $i = 1, \cdots, k$.

In the case $\kappa_{\rm rel}(q_0, \dot{q}) > 0$, we have that $\lambda_1 a_1 + \cdots + \lambda_k a_k > 0$, and we have to show that there exists $\ddot{q}$ such that $a_i + \nabla d_i(q_0) \cdot \ddot{q} > 0$ for all i. Consider the following set of linear equations:

$$a_2 + \nabla d_2 \cdot \ddot{q} = \epsilon_2, \cdots, a_k + \nabla d_k \cdot \ddot{q} = \epsilon_k$$

in which $\ddot{q}$ is the variable and $\epsilon_2, \cdots, \epsilon_k$ are (as yet undetermined) parameters. Writing the equations in matrix form gives

$$[\nabla d_2 \cdots \nabla d_k]^T \ddot{q} = \vec{\epsilon} - \vec{a} \tag{12}$$

where $\vec{\epsilon} = (\epsilon_2, \cdots, \epsilon_k)^T$ and $\vec{a} = (a_2, \cdots, a_k)^T$. Let $\ddot{q}$ be arbitrarily chosen as a linear combination of $\nabla d_2, \cdots, \nabla d_k$

$$\ddot{q} = \xi_2 \nabla d_2 + \cdots + \xi_k \nabla d_k = [\nabla d_2 \cdots \nabla d_k]\vec{\xi}$$

where $\vec{\xi} = (\xi_2, \cdots, \xi_k)^T$ are parameters yet to be determined. Substituting for $\ddot{q}$ in (12) gives

$$[\nabla d_2 \cdots \nabla d_k]^T [\nabla d_2 \cdots \nabla d_k]\vec{\xi} = \vec{\epsilon} - \vec{a}.$$

All the fingers are essential, hence, according to Lemma 5.2, $\{\nabla d_2, \cdots, \nabla d_k\}$ are linearly independent. But in general the rank of a matrix A is equal to the rank of $A^T A$. Hence the $(k-1) \times (k-1)$ matrix $[\nabla d_2 \cdots \nabla d_k]^T [\nabla d_2 \cdots \nabla d_k]$ is invertible. Thus we may choose a vector ξ (and therefore $\ddot{q}$) such that the resulting $\epsilon_2, \cdots, \epsilon_k$ have any desired value. In particular, they may have any desired positive value, which implies that a $\ddot{q}$ exists such that

$$a_i + \nabla d_i \cdot \ddot{q} = \epsilon_i > 0 \quad \text{for } i = 2, \cdots, k. \tag{13}$$

Last we show that for ϵ_i sufficiently small, $a_1 + \nabla d_1 \cdot \ddot{q}$ is positive as well. Since $\lambda_1 > 0$, we may equivalently show that $\lambda_1(a_1 + \nabla d_1 \cdot \ddot{q}) > 0$. Using (7), we substitute $-\sum_{i=2}^{k} \lambda_i \nabla d_i$ for $\lambda_1 \nabla d_1$ in the inequality $\lambda_1(a_1 + \nabla d_1 \cdot \ddot{q}) > 0$

$$\lambda_1(a_1 + \nabla d_1 \cdot \ddot{q}) = \lambda_1 a_1 - \lambda_2 \nabla d_2 \cdot \ddot{q} - \cdots - \lambda_k \nabla d_k \cdot \ddot{q}.$$

Substituting for the terms $\lambda_i \nabla d_i \cdot \ddot{q}$ for $i = 2, \cdots, k$ according to (13) gives

$$\begin{aligned}\lambda_1(a_1 + \nabla d_1 \cdot \ddot{q}) &= \lambda_1 a_1 - \lambda_2(\epsilon_2 - a_2) - \cdots - \lambda_k(\epsilon_k - a_k) \\ &= \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k \\ &\quad - (\lambda_2 \epsilon_2 + \cdots + \lambda_k \epsilon_k) > 0\end{aligned}$$

for $\epsilon_2, \cdots, \epsilon_k$ sufficiently small, since $\lambda_1 a_1 + \cdots + \lambda_k a_k > 0$ by hypothesis.

Consider now the case where $\kappa_{\mathrm{rel}}(q_0, \dot{q}) = \lambda_1 a_1 + \cdots + \lambda_k a_k < 0$. We have to show that for any $\ddot{q}$, $a_i + \nabla d_i(q_0) \cdot \ddot{q} < 0$ for some i, $1 \leq i \leq k$. Adding $\sum_{i=1}^{k} \lambda_i \nabla d_i \cdot \ddot{q} = 0$ to the left side of $\lambda_1 a_1 + \cdots + \lambda_k a_k < 0$ gives

$$\lambda_1(a_1 + \nabla d_1 \cdot \ddot{q}) + \cdots + \lambda_k(a_k + \nabla d_k \cdot \ddot{q}) < 0.$$

The λ_i's are all positive, hence one of the summands must be negative, and the result follows. □

Fig. 5 shows a conceptual sketch of the c-space corresponding to the three-finger grasps of Fig. 1. In both cases $M^1_{q_0}$ is a one-dimensional (1-D) subspace, so that $0 \leq m^2_{q_0} \leq 1$. A graphical technique introduced in [25], or direct computation of the relative curvature form (using the curvature formula in the appendix), can be used to show that $m^2_{q_0} = 1$ for the maximal three-finger grasp, while $m^2_{q_0} = 0$ for the minimal grasp. Indeed, Fig. 5(a) depicts a path $\alpha(t)$ that starts at q_0 and locally lies in the freespace. No such path emanating from q_0 exists in Fig. 5(b). The exact c-space picture of Fig. 1 can be seen in Figs. 6 and 8. For clarity, Figs. 7 and 9 show constant orientation slices of these configuration spaces.

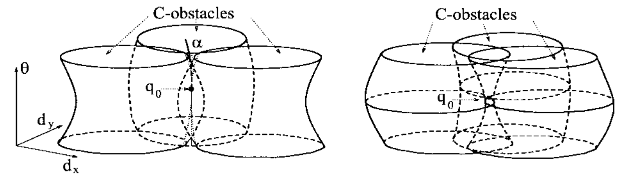

Fig. 5. (a) Three-finger grasp with $m^2_{q_0} = 1$. (b) three-finger grasp with $m^2_{q_0} = 0$.

D. Redundant and Non-Essential Grasps

In practice, the essential grasp assumption is not restrictive, as essential grasps (involving four or fewer planar contacts and seven or fewer spatial contacts) include nearly all situations of practical relevance. If desired, this theory can be extended to grasps involving an arbitrary number of fingers. We shall use the term *redundant* to refer to grasps involving $k > m+1$ fingers, i.e., more than four planar or seven spatial fingers.

Generically, objects held in redundant equilibrium grasps are 1st-order immobile, and hence 2nd-order effects need not be considered. In particular, an object held in a redundant equilibrium grasp is 1st-order immobile when there exists a sub-collection of $m+1$ fingers which forms an essential grasp. We call such redundant grasps *essential in a generalized sense*, and note that this is a generic property of redundant grasps. We define the 1st-order mobility index of essential equilibrium grasps with an arbitrary number of fingers as $m^1_{q_0} = \max\{m - k + 1, 0\}$, where k is the number of fingers and m the dimension of c-space. Fig. 10 shows a redundant 5-finger equilibrium grasp. The grasp has two essential 4-finger subgrasps, hence it is essential in the generalized sense. The 5-finger grasp consequently has a vanishing 1st-order mobility index, and the rectangular object is 1st-order immobile.

Next we consider nonessential grasps. Recall that these are special nongeneric grasps, where the set of 1st-order free motions of $\mathcal{B}$, $M^1_{q_0}$, forms a cone in $T_{q_0}\mathbb{R}^m$ rather than a subspace. If a given nonessential grasp is not 1st-order immobile, a consideration of 2nd-order effects is necessary. In the definition of the c-space relative curvature form in (9), the coefficients $\{\lambda_i\}$ are unique for essential grasps, and therefore the relative curvature form is uniquely defined only for essential grasps. In the case of additional fingers, there is no longer a unique set of finger force reaction coefficients, $\{\lambda_i\}$, that result in an equilibrium grasp. Consequently, the relative curvature form is not uniquely defined. However, the relative curvature form can still be used to determine immobility in these nongeneric cases. The details of this extension are quite lengthy, and here we only sketch the main result.

Let Λ be the set of all feasible equilibrium finger reaction force coefficients

$$\Lambda = \{\lambda_i \mid \lambda_1 \hat{n}_1(q_0) + \cdots + \lambda_k \hat{n}_k(q_0) = 0,\ \lambda_i > 0\ \forall i\}.$$

One then analyzes the behavior of the c-space relative curvature form on this set. For example, a straight-forward analysis shows that:

Proposition 5.7 ([14]): Let $\mathcal{B}$ be held in a k-finger nonessential equilibrium grasp at a configuration q_0. If for

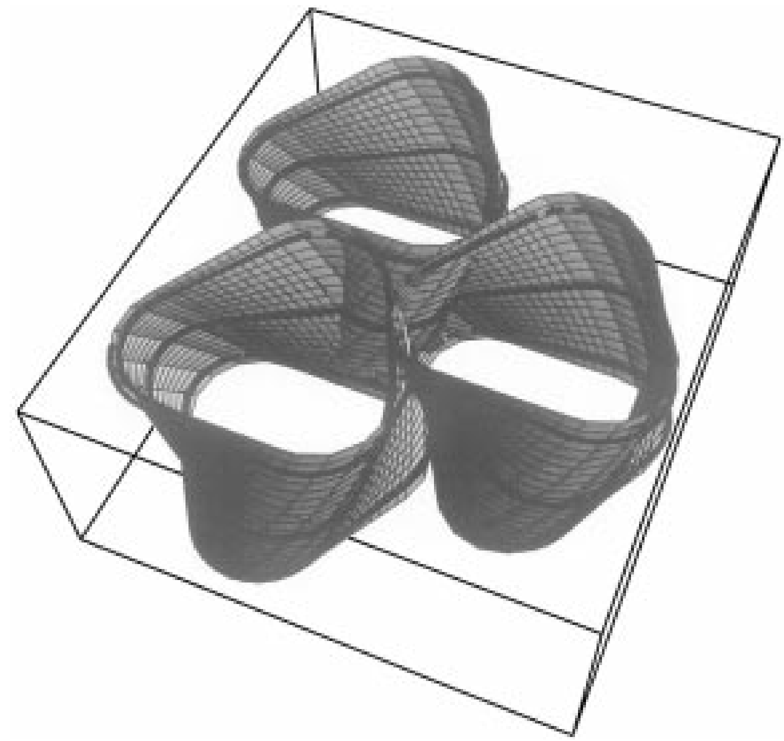

Fig. 6. Portion of the c-space of the maximal three-fingered grasp. The dark lines indicate the location of the fixed-orientation slices shown in the figure below.

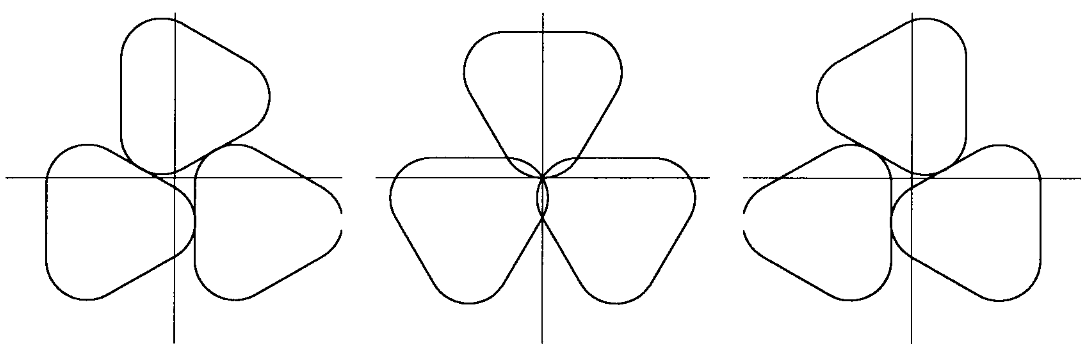

Fig. 7. Constant orientation c-space slices of the maximal three-fingered grasp. The slices are taken at $\theta = -25^\circ$, 0°, 25°.

some $\vec{\lambda} \in \Lambda$, $\kappa_{\rm rel}(q_0, \dot{q}) < 0$ for all $\dot{q} \in M^1_{q_0}$, then $\mathcal{B}$ is **immobilized**. Conversely, if for all $\vec{\lambda} \in \Lambda$, $\kappa_{\rm rel}(q_0, \dot{q}) > 0$ for all $\dot{q} \in M^1_{q_0}$, then $\mathcal{B}$ is **not immobilized**.

VI. Application and Discussion

This paper introduced a configuration-space based methodology for analyzing the mobility of bodies in contact. It is an appealing tool for analyzing mobility, since it naturally leads to the notion of free motions of $\mathcal{B}$, or curves lying in the free configuration space. The ith-order characteristics of these curves lead to a precise notion of ith-order mobility. When $\mathcal{B}$ is held at an equilibrium grasp, q_0, the set of 1st-order free motions becomes a subspace whose dimension is captured by the coordinate invariant 1st-order mobility index, $m^1_{q_0}$. This index classifies in turn the inherent 1st-order mobility of the object at the equilibrium, and is analogous to the definition of mobility traditionally applied to closed-loop linkages.

Since the 1st-order mobility index is generically determined solely by the number of fingers, k, all generic k-finger equilibrium grasps look alike to first order. However, Fig. 1 clearly shows that this is not true. Consequently, *reasoning about the mobility of an object at an equilibrium grasp using forces and velocities implies too crude an approximation*. This deficiency motivated the 2nd-order mobility theory developed in this work. One might consider 2nd-order immobility as a type of "higher order" form closure. In [27], we discuss the relationship between immobility, form-closure, and force-closure. Our notion of 1st-order immobility is shown to be equivalent to frictionless *force-closure*. Further, we introduce in [27] a new notion of "2nd-order force closure," and show that it is equivalent to the definition of 2nd-order immobility given in this paper. In general, our concepts of first and 2nd-order immobility are more precise characterizations of the concept of "form-closure" that is often used in the literature.

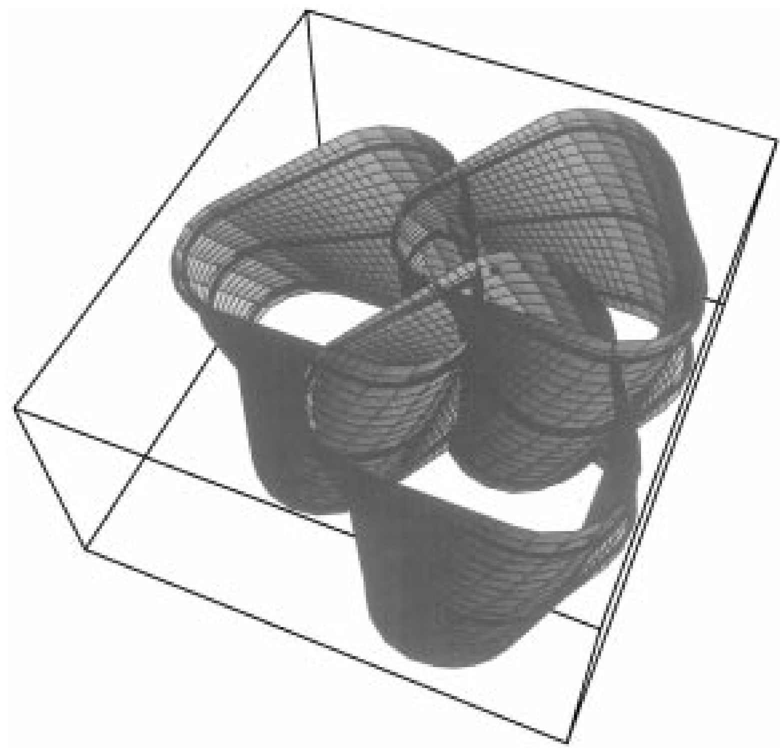

Fig. 8. A portion of the c-space of the minimal three-fingered grasp. The dark lines indicate the location of the fixed-orientation slices shown in the figure below.

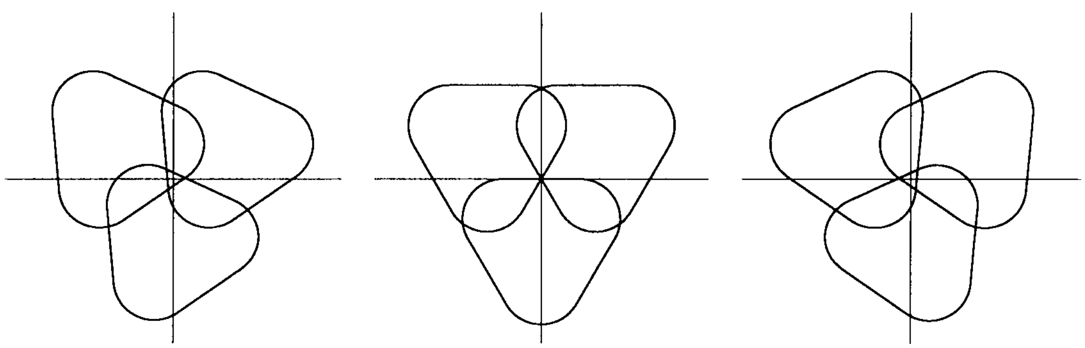

Fig. 9. Constant orientation c-space slices of the minimal three-fingered grasp. The slices are taken at $\theta = -25°$, $0°$, $25°$.

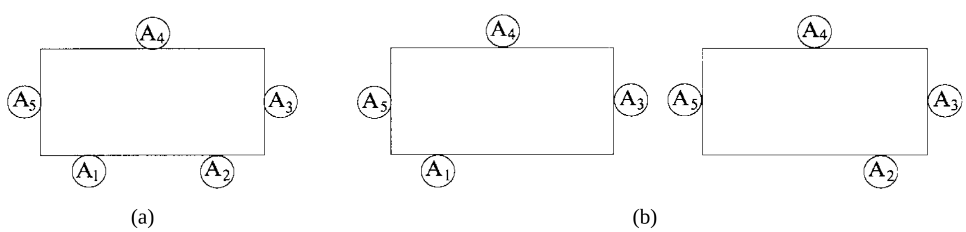

Fig. 10. (a) A generic five-finger grasp. (b) Two four-finger essential subgrasps.

The 2nd-order theory has important theoretical and practical applications. We consider one ramification of this work here, while others are discussed in the companion paper. We have shown that 2nd-order, or curvature, effects of contact can effectively lower the mobility (as predicted by 1st-order theories) of the object. That is, $0 \leq m^2_{q_0} \leq m^1_{q_0}$. We say that an object held in an equilibrium grasp is *completely immobile* if its configuration q_0 is completely isolated from the freespace by c-obstacles. $\mathcal{B}$ is *immobile to first order* if $m^1_{q_0} = 0$. First-order immobility is sufficient for complete immobility of $\mathcal{B}$. It is known that almost all 2-D or 3-D objects can be held in a frictionless force-closure grasp (which is equivalent to 1st-order immobility) by $m+$ 1 contact points, where m is the c-space dimension. Thus, four contacts suffice for 2-D grasps and seven for 3-D grasps [13], [23], [30]. Moreover, $m + 1$ is the smallest possible number when only the point-contact aspect of the fingers is considered.

However, our 2nd-order mobility theory suggests that objects can be immobilized with less than $m+1$ frictionless contacts if curvature effects are taken into account. We say that an object held in an equilibrium grasp is *immobile to second order* if $m^2_{q_0}=0$. Physically, this means that all instantaneous motions at q_0 give rise to local c-space motions that cause the object to penetrate the fingers. Thus, 2nd-order immobility is sufficient to guarantee complete immobility. Czyzowicz *et al.* [4] have shown that $n+1$ frictionless contact points suffice to completely immobilize almost any n-dimensional *polygonal* shape. We can extend this result to a larger class of objects using the theory outlined in this paper.

Research in progress supports the following two conjectures. *If one is free to choose the fingers' point of contact, then generic piecewise smooth n-dimensional objects can be completely immobilized by $n+1$ frictionless fingers*, each maintaining a point contact with $\mathcal{B}$. That is, three contact points (instead of four for force closure) for 2-D grasps and four contact points (instead of seven for force closure) for 3-D grasps. Furthermore, in many fixturing or work holding applications it is conceivable that the surface geometry of the fingers (or work holding fixtures) can be chosen. In that case our theory indicates: *if one is allowed to choose suitably concave finger tips, then generic piecewise smooth n-dimensional objects can be completely immobilized by n frictionless fingers*. In [26] (written after this paper was submitted but accepted before this paper was reported on) we use the mobility theory developed in this paper to prove the $n+1$ and n bounds for 2-D objects: any planar piecewise smooth object can be immobilized by three convex (possibly flat) fixtures, while any similar object can be immobilized by two suitable curved (and possibly nonconvex) fixtures.

Why would one want to use fewer fixtures or fingers than is required for 1st-order closure? For lightly loaded grasps or for applications of this theory to quasistatic locomotion planning, the 2nd-order effects will be sufficiently large to offset expected disturbance forces. However, it may be desirable to use these effects for fixturing as well. In [12], we show by example that 2nd-order stiffness effects can be comparable to 1st-order effects in compliant fixtures. Furthermore, it is often true that machining forces are restricted to a subspace or subset of the wrench space. Hence, fixtures need not be uniformly stiff in all directions, and the reduction in number of fixtures afforded by 2nd-order effects may lead to simpler fixture planning algorithms and more useful fixturing arrangements.

The mobility theory outlined in this paper is geometric/kinematic in its nature. It is based on the rigid body idealization, whereupon bodies in contact cannot deform or interpenetrate. But in order to justify practical applications of the theory, we must investigate how forces of restraint are generated by second order, or surface curvature, effects, and if the analysis based upon rigid body effects is still useful when compliance is taken into account. This subject is taken up in the companion paper, where we show that the stiffness matrix associated with any kinematically immobilizing grasp is automatically positive definite. Hence, the geometric notion of rigid body immobility presented in this paper automatically guarantees stability when compliance effects are considered.

Appendix A
The C-Obstacle Curvature Form

This appendix contains expressions for the curvature form of $\mathcal{S}_i$, the boundary of the c-obstacle $\mathcal{CA}_i$. For a detailed derivation, see [25]. By definition, the curvature form of $\mathcal{S}_i$ at a point $q\in\mathcal{S}_i$ is the change of the unit normal $\hat{n}_i$

$$\dot{q}^T[D\hat{n}_i(q)]\dot{q}=\dot{q}\cdot\left.\frac{d}{dt}\right|_{t=0}\hat{n}_i(\alpha(t))\quad \dot{q}\in T_q\mathcal{S}_i \tag{14}$$

where $\alpha(t)$ is a c-space path which lies in $\mathcal{S}_i$, such that $\alpha(0)=q\in\mathcal{S}_i$. Note that the trajectory $\mathcal{B}(\alpha(t))$ corresponds to a roll-slide motion of $\mathcal{B}$ on the surface of $\mathcal{A}_i$.

The c-obstacle curvature formula depends on the curvature of the two bodies, for which some notation is now introduced. Let $\mathcal{B}(q)$ and $\mathcal{A}_i$ be described by $\mathcal{B}(q)=\{x\in I\!\!R^n:\beta_q(x)\le 0\}$ and $\mathcal{A}_i=\{x\in I\!\!R^n:\gamma_i(x)\le 0\}$, where β_q and γ_i are smooth real-valued functions defining the boundaries of $\mathcal{B}(q)$ and $\mathcal{A}_i$, which are respectively denoted $\partial\mathcal{B}(q)$ and $\partial\mathcal{A}_i$. By definition, the curvature of $\mathcal{B}(q)$ at $x_i\in\partial\mathcal{B}(q)$ and of $\mathcal{A}_i$ at $x_i\in\partial\mathcal{A}_i$ measures the change in the respective unit normal: $\widehat{\nabla\beta_q}(x)=\nabla\beta_q(x)/\|\nabla\beta_q(x)\|$ and $\widehat{\nabla\gamma_i}(x)=\nabla\gamma_i(x)/\|\nabla\gamma_i(x)\|$, along various tangent directions. If x_i is the contact point between $\mathcal{B}(q)$ and $\mathcal{A}_i$, their respective curvature at x_i is determined by the following linear maps (the Weingarten map [31])

$$L_{\mathcal{B}(q)}(x_i)\triangleq\left.\frac{d}{dx}\right|_{x=x_i}\widehat{\nabla\beta_q}(x):T_{x_i}\partial\mathcal{B}(q)\to T_{x_i}\partial\mathcal{B}(q)$$

$$L_{\mathcal{A}_i}(x_i)\triangleq\left.\frac{d}{dx}\right|_{x=x_i}\widehat{\nabla\gamma_i}(x):T_{x_i}\partial\mathcal{A}_i\to T_{x_i}\partial\mathcal{A}_i.$$

For notational simplicity we write $L_{\mathcal{B}_i}$ for $L_{\mathcal{B}(q)}(x_i)$, and $L_{\mathcal{A}_i}$ for $L_{\mathcal{A}_i}(x_i)$. In the planar case the tangent spaces $T_{x_i}\partial\mathcal{B}(q)$ and $T_{x_i}\partial\mathcal{A}_i$ are 1-D. The action of $L_{\mathcal{B}(q)}(x_i)$ and $L_{\mathcal{A}_i}(x_i)$ is simply a multiplication by scalars $\kappa_{\mathcal{B}_i}$ and $\kappa_{\mathcal{A}_i}$, which are the curvatures at x_i of the curves $\partial\mathcal{B}(q)$ and $\partial\mathcal{A}_i$.

Note that the curvature form depends on the choice of the object frame. We first give a curvature form formula when $\mathcal{B}$'s reference frame is located at the contact point x_i

$$\begin{aligned}\dot{q}\cdot\left.\frac{d}{dt}\right|_{t=0}\hat{n}_i(\alpha(t))&=\frac{1}{\|n_i\|}(v^T\quad\omega^T)\begin{bmatrix}I&0\\0&[\hat{N}_i\times]^T\end{bmatrix}\\&\quad\times\begin{bmatrix}L_{\mathcal{B}_i}[L_{\mathcal{A}_i}+L_{\mathcal{B}_i}]^{-1}L_{\mathcal{A}_i}&-L_{\mathcal{A}_i}[L_{\mathcal{A}_i}+L_{\mathcal{B}_i}]^{-1}\\-[L_{\mathcal{A}_i}+L_{\mathcal{B}_i}]^{-1}L_{\mathcal{A}_i}&-[L_{\mathcal{A}_i}+L_{\mathcal{B}_i}]^{-1}\end{bmatrix}\\&\quad\times\begin{bmatrix}I&0\\0&[\hat{N}_i\times]\end{bmatrix}\begin{pmatrix}v\\\omega\end{pmatrix}\end{aligned}\tag{15}$$

where $\dot{q}=(v,\omega)$ is a tangent vector in $T_q\mathcal{S}_i$, and $\hat{N}_i$ is the unit-magnitude contact normal common to $\mathcal{A}_i$ and $\mathcal{B}$, pointing into $\mathcal{B}$, and $[\hat{N}_i\times]$ is the skew-symmetric matrix associated with the vector $\hat{N}_i$ i.e., $[\hat{N}_i\times]\omega=\hat{N}_i\times\omega$. In the planar case ω is always perpendicular to the plane. Hence $\omega\perp\hat{N}_i$, and the matrix $\left[\begin{smallmatrix}I&0\\0&[\hat{N}_i\times]\end{smallmatrix}\right]$ in (15) is nonsingular on the tangent space $T_q\mathcal{S}_i$. In the 3-D case, however, the matrix is singular along the tangent direction $(v,\omega)=(0,\hat{N}_i)$. This corresponds to instantaneous rotation of $\mathcal{B}$ about its common normal with $\mathcal{A}_i$.

We now consider the more general case, where $\mathcal{B}$'s reference frame is located at a fixed translation, r_i, from the contact point x_i. Object points whose coordinates used to be r, are now expressed in terms of new coordinates, denoted $\bar{r}$, as $r = \bar{r} + r_i$. Suppose that $R_0 \in SO(3)$ is the orientation of $\mathcal{B}$ at its contact configuration. We parametrize $SO(3)$ as

$$\theta \in I\!R^3 \mapsto R(\theta) = e^{\Omega(\theta)} R_0 \tag{16}$$

where $\Omega(\theta)$ is a 3×3 skew-symmetric matrix. This is a parametrization of $SO(3)$ centered at R_0, such that the contact configuration is parametrized by $q_0 = (d_0, 0)$. There is no loss of generality in making such a specific choice. The coordinate transformation between the two parametrizations, $f : (\bar{d}, \bar{\theta}) \in I\!R^m \to (d, \theta) \in I\!R^m$, is given by

$$\begin{pmatrix} d \\ \theta \end{pmatrix} = f(\bar{d}, \bar{\theta}) = \begin{pmatrix} \bar{d} + R(\bar{\theta}) r_i \\ \bar{\theta} \end{pmatrix}. \tag{17}$$

Let $\mathcal{CA}_i$ and $\overline{\mathcal{CA}_i}$ be the c-obstacles corresponding to $\mathcal{A}_i$ in q-space and $\bar{q}$-space, respectively. And let $\mathcal{S}_i$ and $\bar{\mathcal{S}}_i$ be their respective boundary. The real-valued function on $\bar{q}$-space

$$\tilde{d}_i(\bar{q}) = (d_i \circ f)(\bar{q}) \tag{18}$$

is typically not the Euclidean distance function, but it is zero on $\bar{\mathcal{S}}_i$, negative in the interior of $\overline{\mathcal{CA}_i}$, and positive outside it. Let $\bar{q}_0 \in \bar{\mathcal{S}}_i$ be the point such that $q_0 = f(\bar{q}_0)$. The curvature-form of $\bar{\mathcal{S}}_i$ at $\bar{q}_0$ is given by

$$\dot{\bar{q}}^T [D^2 \bar{d}_i(\bar{q}_0)] \dot{\bar{q}} = \frac{1}{\|\nabla \tilde{d}_i(\bar{q}_0)\|} \dot{\bar{q}}^T [D^2 \tilde{d}_i(\bar{q}_0)] \dot{\bar{q}}$$

where $\bar{d}_i(q)$ is the exact Euclidean distance function in the $\bar{q}$-parametrization of c-space. Taking the second derivative of $\tilde{d}_i$ in (18) and substituting for $D^2 \tilde{d}_i(\bar{q}_0)$ in the last equation gives

$$\dot{\bar{q}}^T [D^2 \bar{d}_i(\bar{q}_0)] \dot{\bar{q}} = \frac{1}{\|\nabla \tilde{d}_i(\bar{q}_0)\|} \left\{ \overbrace{\dot{q}^T [D^2 d_i(q_0)] \dot{q}}^{\dagger} + \overbrace{\nabla d_i(q_0)^T [D^2 f(\bar{q}_0)](\dot{\bar{q}}, \dot{\bar{q}})}^{\ddagger} \right\} \tag{19}$$

where we have substituted $\dot{q} = Df(\bar{q})\dot{\bar{q}}$. The (†) term is the curvature form of $\mathcal{S}_i$ at q_0 i.e., the curvature form when the object frame is located at the contact point. The presence of the (‡) term implies that *in general, the curvature is not invariant under coordinate transformation*.

Substituting for the (‡) term in (19), and substituting for $\dot{q}^T [D^2 d_i(q_0)] \dot{q}$ according to (15), gives the desired curvature form

$$\begin{aligned} &\dot{\bar{q}}^T D^2 \bar{d}_i(\bar{q}_0) \dot{\bar{q}} \\ &= \frac{1}{\|n_i(q_0)\|} \frac{1}{\|\nabla \tilde{d}_i(\bar{q}_0)\|} (\bar{v}^T \quad \bar{\omega}^T) \left\{ \begin{bmatrix} I & -[R_0 r_i \times] \\ 0 & [\hat{N}_i \times]^T \end{bmatrix}^T \right. \\ &\quad \times \begin{bmatrix} L_{\mathcal{B}} [L_{\mathcal{A}_i} + L_{\mathcal{B}}]^{-1} L_{\mathcal{A}_i} & -L_{\mathcal{A}_i} [L_{\mathcal{A}_i} + L_{\mathcal{B}}]^{-1} \\ -[L_{\mathcal{A}_i} + L_{\mathcal{B}}]^{-1} L_{\mathcal{A}_i} & -[L_{\mathcal{A}_i} + L_{\mathcal{B}}]^{-1} \end{bmatrix} \\ &\quad \left. \times \begin{bmatrix} I & -[R_0 r_i \times] \\ 0 & [\hat{N}_i \times] \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -([R_0 r_i \times]^T [\hat{N}_i \times])_s \end{bmatrix} \right\} \\ &\quad \times \begin{pmatrix} \bar{v} \\ \bar{\omega} \end{pmatrix} \end{aligned} \tag{20}$$

where $\dot{\bar{q}} = (\bar{v}, \bar{\omega}) \in T_{\bar{q}_0} \bar{\mathcal{S}}_i$. Note that the above formula reduces to (15) when $r_i = 0$.

Appendix B
Proofs of Lemmas

This appendix contains proofs of some lemmas from Section V.

Lemma 5.1 The coefficients $\lambda_1, \cdots, \lambda_k$ in (7) are nonzero and unique **iff** the fingers $\mathcal{A}_1, \cdots, \mathcal{A}_k$ are all **essential**.

Proof: First we show that the uniqueness of the λ_i's is equivalent to the linear independence of the vectors $\{\hat{n}_i - \hat{n}_j\}_{1 \le i \le k, i \ne j}$, for any fixed j, $1 \le j \le k$. This is well known, see, e.g. [7]. Substituting $1 - \sum_{i=1, i \ne j}^{k} \lambda_i$ for λ_j in (7) gives

$$\sum_{i=1, i \ne j}^{k} \lambda_i (\hat{n}_i - \hat{n}_j) = -\hat{n}_j \tag{21}$$

Equation (21) can be written in matrix form as

$$\begin{aligned} &[(\hat{n}_1 - \hat{n}_j), \cdots, (\hat{n}_{j-1} - \hat{n}_j), (\hat{n}_{j+1} - \hat{n}_j), \cdots, (\hat{n}_k - \hat{n}_j)]\lambda \\ &\quad = -\hat{n}_j \end{aligned} \tag{22}$$

where $\lambda = (\lambda_1, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots, \lambda_k)^T$, and each $(\hat{n}_i - \hat{n}_j)$ is a column vector in the matrix. The vector λ is a particular solution for the inhomogeneous linear system (22). The solution is unique iff the vectors $\hat{n}_i - \hat{n}_j$ for $1 \le i \le k$, $i \ne j$, are linearly independent. Thus it suffices to show that the fingers are all essential iff the λ_i's are all nonzero and the vectors $\{\hat{n}_i - \hat{n}_j\}_{1 \le i \le k, i \ne j}$ are linearly independent, for any fixed j, $1 \le j \le k$.

First we show that if a finger $\mathcal{A}_j$ is nonessential, either $\lambda_j = 0$ or the vectors $\{\hat{n}_i - \hat{n}_j\}_{1 \le i \le k, i \ne j}$ are linearly dependent. (This is equivalent to showing that if the λ_i's are all nonzero and the vectors $\{\hat{n}_i - \hat{n}_j\}_{1 \le i \le k, i \ne j}$ are linearly independent, then the fingers are all essential.) We may assume that $j = 1$. By hypothesis, the origin is positively spanned by the vectors $\hat{n}_2, \cdots, \hat{n}_k$

$$0 = \nu_2 \hat{n}_2 + \cdots \nu_k \hat{n}_k \quad \nu_i \ge 0 \text{ and } \sum_{i=2}^{k} \nu_i = 1. \tag{23}$$

Since $1 - \sum_{i=2}^{k} \nu_i = 0$, we may add $(1 - \sum_{i=2}^{k} \nu_i)\hat{n}_1$ to the right side of (23) to obtain

$$-\hat{n}_1 = \nu_2(\hat{n}_2 - \hat{n}_1) + \cdots \nu_k(\hat{n}_k - \hat{n}_1).$$

Equating the last equation with (21), for $j = 1$, gives

$$\begin{aligned} &\nu_2(\hat{n}_2 - \hat{n}_1) + \cdots + \nu_k(\hat{n}_k - \hat{n}_1) \\ &\quad = \lambda_2(\hat{n}_2 - \hat{n}_1) + \cdots + \lambda_k(\hat{n}_k - \hat{n}_1). \end{aligned} \tag{24}$$

If $\lambda_1 = 0$ we are done. Otherwise it must be that $\nu_i \ne \lambda_i$ for some i, $2 \le i \le k$. But in this case (24) implies that the vectors $\hat{n}_2 - \hat{n}_1, \cdots, \hat{n}_k - \hat{n}_1$ are linearly dependent.

Now we prove that if the vectors $\{\hat{n}_i - \hat{n}_j\}_{i=1,\cdots,k, i \ne j}$ are linearly dependent, there exists a nonessential finger. We may assume again that $j = 1$. Let Δ be the subset of $I\!R^{k-1}$, defined as the the convex hull of the origin, and points on the coordinate axes at unit distance from the origin. Each λ_i

in (22) satisfies $0 \leq \lambda_i \leq 1$. Thus the parameters $\lambda_2, \cdots, \lambda_k$ must lie in Δ. The linear dependence assumption implies that the particular solution λ for (22) is part of an affine subspace of solutions, spanned by $\lambda + \mu$, where $[(\hat{n}_2 - \hat{n}_1) \cdots (\hat{n}_k - \hat{n}_1)]\mu = 0$. The general solution $\lambda + \mu$, being an affine subspace, is unbounded. Δ, however, is bounded. Hence there must be a solution λ^* of (22) that lies on the boundary of Δ. But every boundary point of Δ lies on a face of Δ, which is a convex combination of $k-1$ vertices of Δ. If the face contains the origin of $\mathbb{R}^{k-1}$, the jth entry of λ^* corresponding to the vertex opposite the face is zero, and $\mathcal{A}_j$ is nonessential. If λ^* lies on the face opposite the origin, we have that $\sum_{i=2}^{k} \lambda_i^* = 1$, and $\mathcal{A}_1$ is nonessential. Hence at least one finger is nonessential. □

The last paragraph of the proof implies that if a collection of $k-1$ vectors, $\{\hat{n}_2, \cdots, \hat{n}_k\}$ say, are linearly dependent, one of the fingers must be nonessential. This is stated in the following corollary.

Corollary 5.2 If the k fingers participating in the equilibrium grasp are **essential**, then any $k-1$ finger c-obstacle normals from $\{\hat{n}_1(q_0), \cdots, \hat{n}_k(q_0)\}$ are **linearly independent**.

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