PROJECTIVE FUNCTIONAL TENSORS AND OTHER ALLIED FUNCTIONALS

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Communicated December 27, 1929

1. *Introduction.*—Some time ago I discovered a sequence of functionals[1] that is the correspondent in function space[2] of Weyl's projective curvature tensor in n dimensions.[3] Since then, I have succeeded in finding a sequence of non-tensor functionals[4] with the property that each functional of the sequence remains unaltered under an arbitrary projective functional transformation[1] of the functional affine connection. It is the object of this note to outline briefly the results obtained in the above studies. A detailed presentation with proofs as well as an account of numerous results and theorems that flow out of the ideas and methods of the present note is reserved for a series of papers to be published elsewhere.

2. *Projective Functional Transformations of Functional Affine Connections.*—Let $\Gamma_{\alpha\beta}^{i}[y]$, $M_{\alpha}^{i}[y]$, $N_{\alpha}^{i}[y]$, $P^{i}[y]$ be the functionals of an arbitrary symmetric functional affine connection.[2] The most general projective functional transformation of such a functional affine connection is given by

$$\left.\begin{aligned} '\Gamma_{\alpha\beta}^{i}[y] &= \Gamma_{\alpha\beta}^{i}[y],\ 'M_{\alpha}^{i}[y] = M_{\alpha}^{i}[y] \\ 'N_{\alpha}^{i}[y] &= N_{\alpha}^{i}[y] + \varphi_{\alpha}[y],\ 'P^{i}[y] = P^{i}[y], \end{aligned}\right\} \qquad (1)$$

where $\varphi_{\alpha}[y]$ is an arbitrary covariant functional vector. More generally, the projective functional transformation of an asymmetric functional affine connection is determined by

$$\left.\begin{aligned} '\Gamma_{\alpha\beta}^{i}[y] &= \Gamma_{\alpha\beta}^{i}[y],\ 'M_{\alpha}^{i}[y] = M_{\alpha}^{i}[y],\ 'S_{\alpha}^{i}[y] = S_{\alpha}^{i}[y] + \varphi_{\alpha}[y], \\ '\Omega_{\alpha\beta}^{i} &= \Omega_{\alpha\beta}^{i},\ 'T_{\alpha\beta}^{i}[y] = T^{i}[y] + \varphi_{\alpha}[y],\ 'P^{i}[y] = P^{i}[y], \end{aligned}\right\} \qquad (2)$$

where $\varphi_{\alpha}[y]$ is again an arbitrary covariant functional vector and

$$\left.\begin{aligned} \Gamma_{\alpha\beta}^{i} &= {}^{1}/_{2}(L_{\alpha\beta}^{i} + L_{\beta\alpha}^{i}),\ S_{\alpha}^{i} = {}^{1}/_{2}(N_{\alpha}^{i} + O_{\alpha}^{i}) \\ \Omega_{\alpha\beta}^{i} &= {}^{1}/_{2}(L_{\alpha\beta}^{i} - L_{\beta\alpha}^{i}),\ T_{\alpha}^{i} = {}^{1}/_{2}(N_{\alpha}^{i} - O_{\alpha}^{i}). \end{aligned}\right\} \qquad (3)$$

The functionals $\Omega_{\alpha\beta}^{i}[y]$ and $T_{\alpha}^{i}[y]$ are the two distinct functionals of the *torsion functional tensor*. Incidentally in this connection we state the theorem that

$$\Omega_{jk}^{i}\, \xi^{j}\, \eta^{k} u_{i} + T_{k}^{i} u_{i}\, (\xi^{i}\eta^{k} - \xi^{k}\, \eta^{i}) \qquad (4)$$

is an absolute cubic functional form in the contravariant functional vectors $\xi^{i}[y]$, $\eta^{i}[y]$ and in the covariant functional vector $u_{i}[y]$.

3. *The Analogue of Weyl's Tensor in Function Space.*—Consider now a symmetric functional affine connection.[2] Let

$$B^i_{\alpha\beta\gamma}[y],\ C^i_{\alpha\beta}[y],\ D^i_{\alpha\beta}[y],\ E^i_{\alpha}[y],\ F^i_{\alpha}[y]$$

stand for the five distinct functionals of the *functional curvature tensor*.[2] Define now a sequence of five other functionals

$$\mathfrak{B}^i_{\alpha\beta\gamma}[y],\ \mathfrak{C}^i_{\alpha\beta}[y],\ \mathfrak{D}^i_{\alpha\beta}[y],\ \mathfrak{E}^i_{\alpha}[y],\ \mathfrak{F}^i_{\alpha}[y]$$

by

$$\left.\begin{aligned}
\mathfrak{B}^i_{\alpha\beta\gamma}[y] &= B^i_{\alpha\beta\gamma}[y]\\
\mathfrak{C}^i_{\alpha\beta}[y] &= C^i_{\alpha\beta}[y]\\
\mathfrak{D}^i_{\alpha\beta}[y] &= D^i_{\alpha\beta}[y] - \frac{1}{b-a}\int_a^b D^\sigma_{\alpha\beta}[y]d\sigma\\
E^i_{\alpha}[y] &= E^i_{\alpha}[y] - \frac{1}{b-a}\int E^\sigma_{\alpha}[y]d\sigma\\
F^i_{\alpha}[y] &= F^i_{\alpha}[y].
\end{aligned}\right\} \quad (5)$$

This new sequence of five functionals can be considered as the five distinct functional coefficients of a quartic functional form[2] $Q[\xi, \eta, \zeta, u]$ in the contravariant functional vectors $\xi^i[y]$, $\eta^i[y]$, $\zeta^i[y]$ and covariant functional vector $u_i[y]$, with symmetry conditions

$$\left.\begin{aligned}
&Q[\xi, \eta, \zeta, u] = -Q[\xi, \zeta, \eta, u]\\
&Q[\xi, \eta, \zeta, u] + Q[\zeta, \xi, \eta, u] + Q[\eta, \zeta, \xi, u] = 0.
\end{aligned}\right\} \quad (6)$$

Consequently the functionals $\mathfrak{B}^i_{\alpha\beta\gamma}$, $\mathfrak{C}^i_{\alpha\beta}$, $\mathfrak{D}^i_{\alpha\beta}$, $\mathfrak{E}^i_{\alpha}$, $\mathfrak{F}^i_{\alpha}$ are the functionals of a functional tensor of rank four, covariant of rank three and contravariant of rank one, with symmetry conditions that are easily obtained from (6). But, what is more important for the purpose of this paper is that this same sequence of functionals possesses the remarkable property that each functional of the sequence is an invariant functional of the four functionals $\Gamma^i_{\alpha\beta}$, M^i_α, N^i_α, P^i and their first functional derivatives under the infinite group of projective functional transformations (1). The set of functionals $\mathfrak{B}^i_{\alpha\beta\gamma}, \ldots, \mathfrak{F}^i_\alpha$ will be said to constitute the *projective curvature tensor in function space.*

4. *Projective Functional Connections.*—From an asymmetric[2] functional connection $L^i_{\alpha\beta}$, M^i_α, N^i_α, O^i_α, P^i it is possible to form the sequence of functionals $\lambda^i_{\alpha\beta}$, μ^i_α, ν^i_α, o^i_α, π^i defined by

$$\left.\begin{aligned}
\lambda^i_{\alpha\beta} &= L^i_{\alpha\beta},\ \mu^i_\alpha = M^i_\alpha,\ \nu^i_\alpha = N^i_\alpha - \frac{1}{b-a}\int_a^b N^\sigma_\alpha d\sigma,\\
o^i_\alpha &= O^i_\alpha,\ \pi^i = P^i.
\end{aligned}\right\} \quad (7)$$

Each functional of this sequence is invariant under the infinite group determined by (2).

In particular, if we specialize our work and consider a symmetric functional connection $\Gamma^i_{\alpha\beta}$, M^i_α, N^i_α, P^i, each member of the sequence of functionals $\gamma^i_{\alpha\beta}$, μ^i_α, ν^i_α, π^i defined by

$$\gamma^i_{\alpha\beta} = \Gamma^i_{\alpha\beta},\ \mu^i_\alpha = M^i_\alpha,\ \nu^i_\alpha = N^i_\alpha - \frac{1}{b-a}\int_a^b N^\sigma_\alpha d\sigma,\ \pi^i = P^i \qquad (8)$$

will have the property of remaining unaltered under the group (1). Because of this property, the four functionals defined by (8) will be said to constitute a *projective functional connection*[5] (symmetric—as distinguished from the asymmetric case (7)). I find that the law of transformation of the projective functional connection is given by

$$\left.\begin{array}{l}
{}_{i,}y^{(i)}\,\bar{\gamma}^i_{jk} + y^i_{,\alpha}\,\bar{\gamma}^\alpha_{jk} = y^i_{,jk} - y^i_{,j}\,{}^{j}y^j_{,k}\,(1/{}_{j,}y^{(j)}) - y^i_{,k}\,{}^{k,}y^{(k)}_{,j}\,(1/{}_{k,}y^{(k)}) \\
+ \gamma^i_{jk}\,{}_{j,}y^{(j)}\,{}_{k,}y^{(k)} + \gamma^i_{j\sigma}\,{}_{j,}y^{(j)}\,y^\sigma_{,k} + \gamma^i_{\sigma k}\,y^\sigma_{,j}\,{}_{k,}y^{(k)} + \gamma^i_{\sigma\tau}y^\sigma_{,j}\,y^\tau_{,k} \\
+ \mu^i_j\,{}_{j,}y^{(j)}\,y^j_{,k} + \mu^i_k\,y^{(k)}_{,j}\,{}_{k,}y^k + \mu^i_\sigma\,y^\sigma_{,j}\,y^\sigma_{,k} + (\nu^i_k - \nu^{(j)}_k)\,y^i_{,j}\,{}_{k,}y^{(k)} \\
+ (\nu^i_\sigma - \nu^{(j)}_\sigma)\,y^i_{,j}\,y^\sigma_{,k} + (\nu^i_j - \nu^{(k)}_j)\,{}_{j,}y^{(j)}\,y^i_{,k} + (\nu^i_\sigma - \nu^{(k)}_\sigma)\,y^\sigma_{,j}\,y^i_{,k} \\
+ \pi^i\,y^i_{,j}\,y^i_{,k} - \pi^{(j)}\,y^i_{,j}\,y^j_{,k} - \pi^{(k)}\,y^i_{,k}\,y^k_{,j} \\
{}_{i,}y^{(i)}\bar{\mu}^i_j + y^i_{,\alpha}\,\bar{\mu}^\alpha_j = {}_{j,}y^i_{,j} - y^i_{,j}\,{}_{jj,}y^{(j)}\,(1/{}_{j,}y^j) - y^i_{,j}\,\pi^{(j)}\,{}_{j,}y^j \\
+ \mu^i_j\,({}_{j,}y^{(j)})^2 \\
{}_{i,}y^{(i)}\,\bar{\nu}^i_j = {}^{i,}y^i_{,j} - \dfrac{{}_{i,}y^{(i)}}{b-a}\displaystyle\int_a^b (1/{}_{\sigma,}y^\sigma)\,{}^{\sigma,}y^\sigma_{j}\,,\ d\sigma \\
+ \nu^i_j\,{}_{i,}y^{(i)}\,{}_{j,}y^{(j)} + \nu^i_\sigma\,{}_{i,}y^{(i)}\,y^\sigma_{,j} \\
+ \pi^i\,{}_{i,}y^{(i)}\,y^i_{,j} - \dfrac{{}_{i,}y^{(i)}}{b-a}\displaystyle\int_a^b \pi^\sigma\,y_{,j}^\sigma\,d\sigma \\
{}_{i,}y^i\,\bar{\pi}^i = {}_{ii,}y^{(i)} + \pi^i\,({}_{i,}y^{(i)})^2
\end{array}\right\} \qquad (9)$$

5. *Concluding Remarks.*—If in the projective transformation of a given symmetric functional connection $\Gamma^i_{\alpha\beta}$, M^i_α, N^i_α, P^i, we choose the functional vector $\varphi_\alpha[y]$ as

$$\varphi_\alpha[y] = -\frac{1}{b-a}\int_a^b N^\sigma_\alpha\, d\sigma, \qquad (10)$$

we obtain a unique *normal functional connection*,[5] $'\Gamma$, $'M$, $'N$, $'P$, for an arbitrarily given functional coördinate system y^i, with the property

$$\int_a^b {}'N^\sigma_\alpha\, d\sigma = 0. \qquad (11)$$

For a *projective functional parameter*[5] p along a functional geodesic

$$p = k\left\{\int e^{-\frac{2}{b-a}\int\left[\int_a^b N^\sigma_\alpha \frac{\partial y^\alpha(u)}{\partial u}\, d\sigma\right] du}\right\} ds, \qquad (12)$$

we have the theorem that a projective functional parameter p determined by y^i is an affine functional parameter for the normal functional connection determined by y^i.

Finally, a large number of theorems have been proved by me in connection with a theory of projective normal coördinates in function space and a theory of hybrid projective functional tensors under algebro-functional groups of the type

$$\left. \begin{aligned} x &= \bar{x} + F[y^i] \\ y^i &= f^i\,[\bar{y}^\alpha]. \end{aligned} \right\}$$

[1] Michal, A. D., *Bull. Amer. Math. Soc.*, **35** (1929), 438–439.

[2] Michal, A. D., *Amer. J. Math.*, **50** (1928), 473–517; *Bull. Amer. Math. Soc.*, **34** (1928), 8–9; these PROCEEDINGS, **16** (1930), 88–94; *Ibid.*, **16** (1930), 162–164.

[3] Weyl, H., *Gottinger Nachrichten*, **1921**, pp. 99–112.

[4] Michal, A. D., *Des Moines Meeting of Amer. Math. Soc.*, December, 1929.

[5] Although our algebra and analysis is from the very nature of our subject different from those existent in differential geometries in a finite number of dimensions, I have emphasized as much as possible the many analogies that exist by a suitable choice of notation and terminology. A point in question is the projective connection and projective parameter of Tracy Yerkes Thomas, whose brilliant work on a projective theory of n-dimensional affinely connected manifolds (*Math. Zeitschrift*, **25** (1926), 723–733) together with the notable work of O. Veblen (these PROCEEDINGS, **14** (1928), pp. 154–166) and many others on the same subject has opened up beautiful vistas for the mathematician. For a masterly account of some of these n-dimensional researches as well as for references to the important work of Weyl, Cartan, and Schouten, the reader is referred to L. P. Eisenhart's colloquium lectures on "Non-Riemannian Geometry," Amer. Math. Soc., New York, 1927.

NON-ABELIAN GROUPS ADMITTING MORE THAN HALF INVERSE CORRESPONDENCIES

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Communicated January 13, 1930

Let G represent any non-abelian group of order g which admits an automorphism under which more than half the operators correspond to their inverses. All such automorphisms must be of order 2 since in the square thereof more than half of the operators of G correspond to themselves and hence all of these operators have then this property. Let H represent one of the largest subgroups of G which satisfy the condition that each of their operators corresponds to its inverse under the given automorphism of G. It is known that H is abelian and that if s represents any operator of G which corresponds to its inverse under this automorphism but is not found in H then the number of operators which correspond to their inverses in the co-set of G with respect to H which involves s is exactly equal to the number of operators in H which are commutative with s. Moreover, every co-set of G with respect to H must involve at least one operator which corresponds to its inverse in this automorphism.

The special case when H is of index 2 is comparatively simple. In this case the number of operators which correspond to their inverses