order 2 besides the identity. When this automorphism is of order 2 then the four two-thirds automorphisms of G must generate the dihedral group of order 12, and the three conjugate two-thirds automorphisms of G must always generate the symmetric group of order 6. It may also be noted that *whenever at least one of the four two-thirds automorphisms of G is an inner automorphism then all of them are inner automorphisms and they generate a group of order* 6. The characteristic two-thirds automorphism must be the identity automorphism in this case.

A necessary and sufficient condition that there is at least one group of a given order g which admits two-thirds automorphisms is that g is divisible by 6, and the only case when there is only one such group is when g is of the form $6m$, where m is the product of distinct odd prime numbers. In view of the elementary properties of the category of groups which are characterized by the fact that each of them admits at least one two-thirds automorphism it is easy to determine the number of these groups of any order which is a multiple of 6. It is only necessary to observe that these groups are completely characterized by the abelian subgroups of index 3 involved therein and their cross-cut. That is, *the number of the distinct groups of order* $6m$ *which admit separately a two-thirds automorphism is equal to the sum of the numbers of the sets of subgroups of index* 2 *contained in the abelian groups of order* $2m$ *such that each set is composed of all these subgroups which are conjugate under the groups of isomorphisms of the corresponding groups of order* $2m$. For instance, when $m = 4$ there are four such groups since each of two of the abelian groups of order 8 involves only one set of conjugate subgroups of order 4 while the third involves two such sets.

[1] G. A. Miller, these PROCEEDINGS, **15**, 369, 1929.

DIFFERENTIAL GEOMETRIES OF FUNCTION SPACE

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In a previous paper[1] the author initiated the study of a species of functional differential geometries. These geometries are the function space analogues of the n-dimensional theories of affinely connected manifolds. An attempt to develop a projective theory in function space was instrumental in showing that the functional geometries which were developed in the cited paper[1] were a truncated form of a much more general situation. It is the object of the present note to set forth in outline some of the results obtained in these more general function space differential geometries. A detailed treatment of the subject with proofs will appear elsewhere.

Let $\bar{y}^x$ stand for an arbitrary continuous function of x defined over the interval $a \leq x \leq b$. We shall be concerned with $1 - 1$ continuous functional transformations

$$y^x = y^x[\bar{y}^s] \tag{1}$$

that transform a function $\bar{y}^x$ into a continuous function y^x and that have differentials of the form

$$\delta y^x = {}_{x,}y^x\, \delta\bar{y}^{(x)} + y^x_{,t}\, \delta\, \bar{y}^t \qquad ({}_{x,}y^{(x)} \neq 0) \tag{2}$$

with non-vanishing Fredholm determinants

$$D[y^x_{,t}/{}_{t,}y^{(t)}] \neq 0. \tag{3}$$

The notation used in the above formulæ is essentially in accord with the one adopted by me in my former paper.[1] *We shall adhere to the convention of letting the repetition of a continuous index in a term, once as a subscript and once as a superscript, stand for integration with respect to that index over the fundamental interval* (a, b). *A parenthesis about a continuous index will denote an exception to this integration convention.*

A functional $\xi_\alpha[y^x]$ which under (1) transforms in accordance with the law

$$\bar{\xi}_\alpha[\bar{y}] = \xi_\alpha[y]_{\alpha,}y^{(\alpha)} + \xi_\sigma\, y^\sigma_{,\alpha} \tag{4}$$

will be called a covariant vector while a functional $\xi^\alpha[y^x]$ which under (1) transforms in accordance with the law

$$\bar{\xi}^\alpha[\bar{y}] = \xi^\alpha[y]_{\alpha,}\bar{y}^{(\alpha)} + \xi^\sigma\, \bar{y}^\alpha_{,\sigma} \tag{5}$$

a contravariant vector. A pair of functionals $t_{\alpha\beta}[y]$ and $t_\alpha[y]$ will be said to constitute a covariant tensor of rank two if under (1) it transforms in accordance with

$$\left.\begin{aligned} \bar{t}_{\alpha\beta}[\bar{y}] &= t_{\alpha\beta}[y]_{\alpha,}y^{(\alpha)}\,{}_{\beta,}y^{(\beta)} + t_{\sigma\beta}\, y^\sigma_{,\alpha}\,{}_{\beta,}y^{(\beta)} + t_{\alpha\sigma}\, y^\sigma_{,\beta}\,{}_{\alpha,}y^{(\alpha)} \\ &\quad + t_{\sigma\tau}\, y^\sigma_{,\alpha}\, y^\tau_{,\beta} + t_\alpha[y]\, y^\alpha_{,\beta}\,{}_{\alpha,}y^{(\alpha)} + t_\beta\, y^\beta_{,\alpha}\,{}_{\beta,}y^{(\beta)} + t_\sigma\, y^\sigma_{,\alpha}\, y^\sigma_{,\beta} \\ \bar{t}_\alpha[\bar{y}] &= ({}_{\alpha,}y^{(\alpha)})^2\, t_\alpha[y]. \end{aligned}\right\} \tag{6}$$

Theorem I.—A necessary and sufficient condition that $\xi_\alpha[y]\delta y^\alpha$ be an absolute linear functional differential form is that ξ_α be a covariant functional vector. A necessary and sufficient condition that

$$t_{\alpha\beta}[y]\delta y^\alpha\, \delta y^\beta + t_\alpha(\delta y^\alpha)^2$$

be an absolute quadratic functional differential form is that the pair of functionals $t_{\alpha\beta}[y]$, $t_\alpha[y]$ form a covariant functional tensor of rank two.

Theorem II.—If

$$f_{\alpha\beta\ldots\gamma\delta}\, y^{\delta} + \varphi_{\alpha\beta\ldots\gamma}\, y^{(\alpha)} + \psi_{\alpha\beta\ldots\gamma}\, y^{(\beta)} + \ldots\ldots + \omega_{\alpha\beta\ldots\gamma}\, y^{(\gamma)} = 0,$$

for all continuous functions w^{α} such that $y^{a} = y^{b} = 0$, then in the field of continuous functions

$$f_{\alpha\beta\ldots\gamma\delta} \equiv 0, \quad \varphi_{\alpha\beta\ldots\gamma} \equiv 0, \quad \psi_{\alpha\beta\ldots\gamma} \equiv 0, \quad \ldots, \quad \omega_{\alpha\beta\ldots\gamma} \equiv 0.$$

Define the functional $Q^{i}[z^{\alpha}, w^{\alpha}]$ by

$$Q^{i}[z^{\alpha}, w^{\alpha}] = L^{i}_{\alpha\beta}[y] z^{\alpha} w^{\beta} + M^{i}_{\alpha}[y] z^{\alpha} w^{\alpha} + N^{i}_{\alpha}[y] z^{i} w^{\alpha} + O^{i}_{\alpha}[y] z^{\alpha} w^{i} + P^{i} z^{i} w^{i}.$$

Theorem III.—A necessary and sufficient condition that

$$Q^{i}[z^{\alpha}, w^{\alpha}] = Q^{i}[w^{\alpha}, z^{\alpha}]$$

for all continuous functions w^{α} and z^{α} which vanish at the end points of the fundamental interval (a, b) is that

$$L^{i}_{\alpha\beta} = L^{i}_{\beta\alpha}, \quad N^{i}_{a} = O^{i}_{\alpha}.$$

The differentials of the functionals L, M, N, O, P will be assumed to be of the form

$$\left.\begin{aligned} \delta L^{i}_{\alpha\beta} &= {}_{\alpha,}L^{i}_{\alpha\beta}\, \delta y^{(\alpha)} + {}_{\beta,}L^{i}_{\alpha\beta}\, \delta y^{(\beta)} + {}_{i,}L^{i}_{\alpha\beta}\, \delta y^{(i)} + L^{i}_{\alpha\beta,\gamma}\, \delta y^{\gamma} \\ \delta M^{i}_{\alpha} &= {}_{\alpha,}M^{i}_{\alpha}\, \delta y^{(\alpha)} + {}_{i,}M^{i}_{\alpha}\, \delta y^{(i)} + M^{i}_{\alpha,\beta}\, \delta y^{\beta} \\ &\qquad\qquad (\text{similarly for } \delta N^{i}_{\alpha} \text{ and } \delta O^{i}_{\alpha}) \\ \delta P^{i} &= {}_{i,}P^{i}\, \delta y^{(i)} + P^{i}_{,\alpha}\, \delta y^{\alpha}. \end{aligned}\right\} \quad (7)$$

Theorem IV.—A necessary and sufficient condition that the functional (covariant functional differential)

$$\Delta\xi^{i} = \delta\xi^{i}[y] + L^{i}_{\alpha\beta}\, \xi^{\alpha}\, \delta y^{\beta} + M^{i}_{\alpha}\, \xi^{\alpha}\, \delta y^{\alpha} + N^{i}_{\alpha}\, \xi^{i}\, \delta y^{\alpha} + O^{i}_{\alpha}\, \xi^{\alpha}\, \delta y^{i} + P^{i}\, \xi^{i}\, \delta y^{i} \quad (8)$$

transform in accordance with the functional contravariant law (5) whenever $\xi^{i}[y]$ is a contravariant functional vector with a differential

$$\delta\xi^{i} = {}_{i,}\xi^{(i)}\, \delta y^{i} + \xi^{i}_{,\alpha}\, \delta y^{\alpha}$$

is that the functionals $L^{i}_{\alpha\beta}$, M^{i}_{α}, N^{i}_{α}, O^{i}_{α}, P^{i} have the following laws of transformation, under the functional transformations (1):

$$\left.\begin{aligned} \bar{L}^{i}_{jk}[\bar{y}] &= {}_{i,}\bar{y}^{i}\, \alpha^{(i)}_{jk} + \bar{y}^{i}_{,\sigma}\, \alpha^{\sigma}_{jk} + \bar{y}^{i}_{,j}\, \beta^{(j)}_{k} + \bar{y}^{i}_{,k}\, \gamma^{(k)}_{j} \\ \bar{M}^{i}_{j}[\bar{y}] &= {}_{i,}\bar{y}^{i}\, \epsilon^{(i)}_{j} + \bar{y}^{i}_{,\sigma}\, \epsilon^{\sigma}_{j} + \bar{y}^{i}_{,j}\, \zeta^{(j)} \\ \bar{N}^{i}_{j}[\bar{y}] &= {}_{i,}\bar{y}^{i}\, \beta^{(i)}_{j} \\ \bar{O}^{i}_{j}[\bar{y}] &= {}_{i,}\bar{y}^{i}\, \gamma^{(i)}_{j} \\ \bar{P}^{i}[\bar{y}] &= {}_{i,}\bar{y}^{i}\, \zeta^{(i)} \end{aligned}\right\} \quad (9)$$

The functionals α, β, γ, ϵ, ζ in (9) are defined by

$$\left.\begin{aligned}
\alpha^i_{jk} &= y^i_{,jk} + L^i_{jk}\,{}_{j,}y^{(j)}\,{}_{k,}y^{(k)} + L^i_{j\sigma}\,{}_{j,}y^{(j)}\,y^\sigma_{,k} + L^i_{\sigma k}\,y^\sigma_{,j}\,{}_{k,}y^{(k)} \\
&\quad + L^i_{\sigma\tau}\,y^\sigma_{,j}\,y^\tau_{,k} + M^i_j\,{}_{j,}y^{(j)}\,y^j_{,k} + M^i_k\,y^{(k)}_{,j}\,{}_{k,}y^k + M^i_\sigma\,y^\sigma_{,j}\,y^\sigma_{,k} \\
&\quad + N^i_k\,y^i_{,j}\,{}_{k,}y^{(k)} + N^i_\sigma\,y^i_{,j}\,y^\sigma_{,k} + O^i_j\,{}_{j,}y^{(j)}\,y^i_{,k} + O^i_\sigma\,y^\sigma_{,j}\,y^i_{,k} \\
&\quad + P^i\,y^i_{,j}\,y^i_{,k} \\
\beta^j_k &= {}_{j,}y^{(j)}_{,k} + N^j_k\,{}_{j,}y^{(j)}\,{}_{k,}y^{(k)} + N^j_\sigma\,{}_{j,}y^{(j)}\,y^\sigma_{,k} + P^j\,{}_{j,}y^{(j)}\,y^j_{,k} \\
\gamma^j_k &= {}_{j,}y^{(j)}_{,k} + O^j_k\,{}_{j,}y^{(j)}\,{}_{k,}y^{(k)} + O^j_\sigma\,{}_{j,}y^{(j)}\,y^\sigma_{,k} + P^j\,{}_{j,}y^{(j)}\,y^j_{,k} \\
\epsilon^i_j &= {}_{j,}y^i_{,j} + M^i_j\,({}_{j,}y^{(j)})^2 \\
\zeta^i &= {}_{ii,}y^{(i)} + P^{(i)}\,({}_{i,}y^i)^2
\end{aligned}\right\} \quad (10)$$

For obvious reasons, the sequence of five functionals L^i_{jk}, M^i_j, N^i_j, O^i_j, P^i will be said to determine an *affine connection in our function space.* An affine connection for which

$$L^i_{jk} = \Gamma^i_{jk} \qquad (\Gamma^i_{jk} = \Gamma^i_{kj})$$
$$N^i_j = O^i_j$$

will be called a symmetric affine connection. It is clear from (9) that the conditions

$$\Gamma^i_{jk} = \Gamma^i_{kj}, \qquad N^i_j = O^i_j \qquad (11)$$

are invariant under an arbitrary change of functional variable y^i. Thus a symmetric affine connection is determined by a sequence of four functionals Γ^i_{jk}, M^i_j, N^i_j, P^i whose law of transformation is obtained from (9) in an obvious manner. From now on we shall deal explicitly only with the symmetric affine connection. However, much that we shall have to say has corresponding parts in a theory of an asymmetric affine connection.

Let us define a pair of functionals $\xi^i_{;\alpha}$, ${}_{i;}\xi^{(i)}$ by

$$\left.\begin{aligned}
{}_{i;}\xi^{(i)} &= {}_{i,}\xi^{(i)} + \xi^\sigma N^i_\sigma + \xi^i P^i \\
\xi^i_{;\alpha} &= \xi^i_{,\alpha} + \xi^\sigma \Gamma^i_{\sigma\alpha} + \xi^{(\alpha)} M^i_\alpha + \xi^i N^i_\alpha.
\end{aligned}\right\}. \quad (12)$$

The sequence ${}_{i;}\xi^{(i)}$, $\xi^i_{;\alpha}$ so defined will be called the *covariant functional derivative* of ξ^i with respect to the affine connection Γ, M, N, P. This definition is justified by the theorem.

Theorem V.—The covariant functional derivative of a contravariant functional vector $\xi^i[y]$ constitute a mixed functional tensor of rank two covariant of rank one and contravariant of rank one, i.e., the sequence of functionals ${}_{i;}\xi^{(i)}$, $\xi^i_{;\alpha}$ transforms in accordance with the law

$$\left.\begin{aligned}
{}_{i;}\bar{\xi}^{(i)} &= {}_{i;}\xi^{(i)} \\
\bar{\xi}^i_{;\alpha} &= \xi^i_{;\alpha}\,{}_{\alpha,}y^{(\alpha)}\,{}_{i,}\bar{y}^{(i)} + \xi^i_{;\sigma}\,y^\sigma_{,\alpha}\,{}_{i,}\bar{y}^{(i)} + \xi^\sigma_{;\alpha}\,{}_{\alpha,}y^{(\alpha)}\,y^i_{,\sigma} \\
&\quad + \xi^\sigma_{;\tau}\,y^\tau_{,\alpha}\,\bar{y}^i_{,\sigma} + {}_{i;}\xi^i\,{}_{i,}\bar{y}^{(i)}\,y^i_{,\alpha} + {}_{\alpha;}\xi^\alpha\,{}_{\alpha,}y^{(\alpha)}\,\bar{y}^i_{,\alpha} + {}_{\sigma;}\xi^\sigma\,y^\sigma_{,\alpha}\,\bar{y}^i_{,\sigma}.
\end{aligned}\right\} \quad (13)$$

Clearly the *covariant functional differential* of ξ^i for the symmetric case can be written as

$$\Delta \xi^i = {}_{i;}\xi^i \, \delta y^{(i)} + \xi^i_{;\alpha} \, \delta y^\alpha. \tag{14}$$

Similar results hold good for the covariant functional derivative ${}_{i;}\xi_i$, $\xi_{i;\alpha}$ of the covariant functional vector ξ_i

$$\left.\begin{aligned} {}_{i;}\xi_i &= {}_{i,}\xi_i - \xi_\sigma M^\sigma_i - \xi_i P^{(i)} \\ \xi_{i;\alpha} &= \xi_{i,\alpha} - \xi_\alpha N^{(\alpha)}_i - \xi_i N^{(i)}_\alpha - \xi_\sigma \Gamma^\sigma_{i\alpha}. \end{aligned}\right\} \tag{15}$$

The functionals ${}_{i;}\xi_i$, $\xi_{i;\alpha}$ constitute a covariant functional tensor of rank two and hence transform in accordance with the law (6).

Theorem VI.—The conditions for the complete integrability of the variational equation[2]

$$\delta\xi^i + \Gamma^i_{\alpha\beta}\, \xi^\alpha \,\delta y^\beta + M^i_\alpha \,\xi^\alpha\, \delta y^\alpha + N^i_\alpha \,\xi^i \,\delta y^\alpha + N^i_\alpha \,\xi^\alpha\, \delta y^i + P^i \,\xi^i \,\delta y^i = 0 \tag{16}$$

are

$$B^i_{\alpha\beta\gamma} = 0, \quad C^i_{\alpha\beta} = 0, \quad D^i_{\alpha\beta} = 0, \quad E^i_\alpha = 0, \quad F^i_\alpha = 0 \tag{17}$$

where

$$\left.\begin{aligned} B^i_{\alpha\beta\gamma} &= \Gamma^i_{\alpha\beta,\gamma} - \Gamma_{\alpha\gamma,\beta} + \Gamma^\sigma_{\alpha\beta}\Gamma^i_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma}\Gamma^i_{\sigma\beta} + \Gamma^i_{\alpha\beta} N^i_\gamma - \Gamma^i_{\alpha\gamma} N^i_\beta \\ &\quad + N^{(\alpha)}_\beta \Gamma^i_{\alpha\gamma} - N^{(\alpha)}_\gamma \Gamma^i_{\alpha\beta} + N^{(\beta)}_\alpha \Gamma^i_{\beta\gamma} - N^{(\gamma)}_\alpha \Gamma^i_{\gamma\beta} + \Gamma^{(\gamma)}_{\alpha\beta} M^i_\gamma \\ &\quad - \Gamma^{(\beta)}_{\alpha\gamma} M^i_\beta \\ C^i_{\alpha\beta} &= {}_{\alpha,}\Gamma^i_{\alpha\beta} - M^i_{\alpha,\beta} - M^\sigma_\alpha \Gamma^i_{\sigma\beta} - M^i_\alpha N^i_\beta + N^{(\alpha)}_\beta M^i_\alpha - P^{(\alpha)}\Gamma^i_{\alpha\beta} \\ &\quad - M^{(\beta)}_\alpha M^i_\beta \\ D^i_{\alpha\beta} &= {}_{i,}\Gamma^{(i)}_{\alpha\beta} - N^i_{\alpha,\beta} + \Gamma^\sigma_{\alpha\beta} N^i_\sigma + \Gamma^i_{\alpha\beta} P^i - N^i_\alpha N^i_\beta + N^{(\alpha)}_\beta N^i_\alpha \\ &\quad + N^{(\beta)}_\alpha N^i_\beta \\ E^i_\alpha &= {}_{i,}M^{(i)}_\alpha - {}_{\alpha,}N^i_\alpha + M^\sigma_\alpha N^i_\sigma + M^i_\alpha P^i + P^{(\alpha)} N^i_\alpha \\ F^i_\alpha &= {}_{i,}N^{(i)}_\alpha - P^i_{,\alpha}. \end{aligned}\right\} \tag{18}$$

Let $\xi^i[y]$, $\eta^i[y]$, $\zeta^i[y]$ be arbitrary contravariant functional vectors and $u_i[y]$ an arbitrary covariant functional vector. Then the general normal quadrilinear functional form $Q[\xi, \eta, \zeta, u]$ will have fifteen functional coefficients. The conditions

$$\left.\begin{aligned} Q[\xi, \eta, \zeta, u] &= -Q[\xi, \zeta, \eta, u] \\ Q[\xi, \eta, \zeta, u] + Q[\zeta, \xi, \eta, u] &+ Q[\eta, \zeta, \xi, u] = 0 \end{aligned}\right\} \tag{19}$$

for arbitrary ξ, η, ζ, u cut down the number of independent functional coefficients to five. The functionals $B^i_{\alpha\beta\gamma}$, $C^i_{\alpha\beta}$, $D^i_{\alpha\beta}$, E^i_α, F^i_α, which constitute the *functional curvature tensor* based on a general symmetric affine connection of our function space, can be thought of as the functional coefficients of an absolute quadrilinear functional form $Q[\xi, \eta, \zeta, u]$ with symmetry conditions (19).

Definition of an absolute functional tensor.—A sequence of functionals of y^i that are the coefficients of an absolute multilinear functional form $M[\xi, \ldots, \eta, u, \ldots, v]$ in r contravariant vectors $\xi^i[y], \ldots, \eta^i[y]$ and s covariant vectors $u_i[y], \ldots, v_i[y]$ will be said to constitute an absolute functional tensor of rank $r + s$, covariant of rank r and contravariant of rank s.

The law of transformation of a functional tensor possesses the properties of linear homogeneity and transitivity.

A theory of covariant functional differentiation based on $\Gamma^i_{\alpha\beta}$, M^i_{α}, N^i_{α}, P^i is obtained by an extension of the elimination methods of my previous paper.[1] An alternative method is developed based on the covariant differential of the multilinear form M.

If we now assume that $y^x(t)$ is an analytic function of t for $t_0 \leq t \leq t_1$ and that $\Gamma^i_{\alpha\beta}$, M^i_{α}, N^i_{α}, P^i for an arbitrarily given $y^x(t)$ are analytic functions of t, it is possible to make the *integro-differential equation*

$$\left.\begin{aligned} &\frac{\partial y^j(t)}{\partial t}\left(\frac{\partial^2 y^i(t)}{\partial t^2} + \Gamma^i_{\alpha\beta}\frac{\partial y^\alpha}{\partial t}\frac{\partial y^\beta}{\partial t} + M^i_{\alpha}\left(\frac{\partial y^\alpha}{\partial t}\right)^2 + 2N^i_{\alpha}\frac{\partial y^i}{\partial t}\frac{\partial y^\alpha}{\partial t} \right.\\ &\qquad \left. + P^i\left(\frac{\partial y^i}{\partial t}\right)^2\right) \\ &= \frac{\partial y^i}{\partial t}\left(\frac{\partial^2 y^j(t)}{\partial t^2} + \Gamma^j_{\alpha\beta}\frac{\partial y^\alpha}{\partial t}\frac{\partial y^\beta}{\partial t} + M^j_{\alpha}\left(\frac{\partial y^\alpha}{\partial t}\right)^2 + 2N^j_{\alpha}\frac{\partial y^j}{\partial t}\frac{\partial y^\alpha}{\partial t} \right.\\ &\qquad \left. + P^j\left(\frac{\partial y^j}{\partial t}\right)^2\right) \end{aligned}\right\} \tag{20}$$

the basis of a "geometry of paths" in our function space. For an "affine parameter" s defined along a functional path, the equations (20) for the path take the form

$$\frac{\partial^2 y^i(s)}{\partial s^2} + \Gamma^i_{\alpha\beta}\frac{\partial y^\alpha}{\partial s}\frac{\partial y^\beta}{\partial s} + M^i_{\alpha}\left(\frac{\partial y^\alpha}{\partial s}\right)^2 + 2N^i_{\alpha}\frac{\partial y^i}{\partial s}\frac{\partial y^\alpha}{\partial s} + P^i\left(\frac{\partial y^i}{\partial s}\right)^2 = 0. \tag{21}$$

For the theory of normal functional coördinates z^i we have the following theorem.

Theorem VII.—An arbitrary functional coördinate transformation

$$\bar{y}^i = \bar{y}^i[y]$$

induces a linear homogeneous functional transformation of the third kind

$$\bar{z}_i = ({}_{i,}\bar{y}^{(i)})_q\, z^i + (\bar{y}^i_{,\alpha})_q\, z^\alpha \qquad (D[\bar{y}^i_{,\alpha}/{}_{\alpha,}\bar{y}^{(\alpha)}])_q \neq 0$$

in the normal functional coördinates z^i, $\bar{z}^i$ with the same origin $y^i = q^i$.

The whole theory of the extensions of tensors (in particular the general theory of covariant differentiation), normal tensors, replacement theorems, etc., of n dimensions have their analogues in our function space geometries. The proofs of the various general results are materially simplified by the employment of differentials of functionals rather than the functional derivatives, and the multilinear functional forms rather than the sequence of functionals that constitute the functional tensor.

A Riemannian function space geometry based on a functional quadratic differential form

$$ds^2 = g_{\alpha\beta}[y]\delta y^\alpha \delta y^\beta + g_\alpha[y](\delta y^\alpha)^2, \quad g_\alpha \neq 0, \quad g_{\alpha\beta} = g_{\beta\alpha}, \quad D[g_{\alpha\beta}/g_\alpha] \neq 0 \tag{22}$$

possesses geodesics that satisfy an integro-differential equation of the form (21).

Theorem VIII.—The covariant functional derivative $g_{\alpha\beta;\gamma}$, $_{\alpha;}g_{\alpha\beta}$, $_{\alpha;}g_\alpha$, $g_{\alpha;\beta}$ of the tensor $g_{\alpha\beta}$, g_α vanishes.

Corollary.—The functionals $g_{\alpha\beta}[y]$, $g_\alpha[y]$ are solutions of the system of variational equations

$$\begin{aligned} \delta g_{\alpha\beta} &= \lambda_{\alpha\beta\gamma}[y]\delta y^\gamma + \mu_{\alpha\beta}[y]\ \delta y^{(\alpha)} + \mu_{\beta\alpha}[y]\ \delta y^{(\beta)} \\ \delta g_\alpha &= \nu_{\alpha\beta}[y]\delta y^\beta + \rho_\alpha[y]\delta y^{(\alpha)}, \end{aligned}$$

where

$$\lambda_{\alpha\beta\gamma} = g_{\alpha\beta} N_\gamma^{(\alpha)} + g_{\alpha\beta} N_\gamma^{(\beta)} + g_{\gamma\beta} N_\alpha^{(\gamma)} + g_{\gamma\alpha} N_\beta^{(\gamma)} + g_{\sigma\beta}\Gamma^\sigma_{\alpha\gamma} + g_{\sigma\alpha}\Gamma^\sigma_{\beta\gamma} + g_\alpha \Gamma^{(\alpha)}_{\beta\gamma} + g_\beta \Gamma^{(\beta)}_{\alpha\gamma}$$

$$\mu_{\alpha\beta} = g_{\alpha\beta} P^{(\alpha)} + g_{\sigma\beta} M^\sigma_\alpha + g_\alpha N_\beta^{(\alpha)} + g_\beta M_\alpha^{(\beta)}$$

and $\Gamma^i_{\alpha\beta}$, M^i_α, N^i_α, P^i is the affine connection determined by (22).

Theorem IX.—The Fredholm determinant of $g_{\alpha\beta}/\sqrt{g_\alpha g_\beta}$ is a relative functional invariant of weight two of the quadratic functional form (22).

[1] A. D. Michal, *Am. J. Math.*, **40**, pp. 473–517, 1928. This paper was presented to the American Mathematical Society at the New York meeting, October, 1927. (Cf. *Bulletin of the Am. Math. Soc.*, **34**, pp. 8–9, 1928.)

[2] We have here the beginnings of a theory of variational equations that is so vital to the development of a group theory and invariant theory in function space.