LEMMA 1. *If $n \equiv j \bmod 4$, so that $n - 2^{a+2}t^2 \equiv j \bmod 4$, and if*

$$N[n = S_j + 2^{a+2}t^2] = A\Sigma\psi(n - 2^{a+2}t^2), \qquad (6)$$

where A is independent of n and ψ is simple as first defined, then (6) *implies*

$$N[n = S_j] = A\psi(n), \qquad (7)$$

and (7) *implies* (6).

That (7) implies (6) is obvious. To see that (6) implies (7), equate the generating functions of the functions on the left and right of (6), to get an identity in the parameter q of the generating function. Divide both sides of this identity by

$$\Sigma q^{2^{a+2}t^2}(t = 0, \pm 1, \pm 2, \ldots),$$

and thus get the identity which implies (7). Clearly the lemma holds also for ψ not simple.

LEMMA 2. *If n is an arbitrary integer* >0, *and A is independent of n, each of the following implies the other,*

$$N[n = S_j + t^2] = A\Sigma\psi(n - t^2), N[n = S_j] = A\psi(n).$$

[1] E. T. Bell, *Bull. Am. Math. Soc.*, **35**, 695 (1929).
[2] E. T. Bell, *J. London Math. Soc.*, **4**, 279 (1929).
[3] E. T. Bell, *J. für die r.u.a. Math.*, to appear shortly.
[4] E. T. Bell, *Amer. J. Math.*, **42**, 168 (1920).

THE DIFFERENTIAL GEOMETRY OF A CONTINUOUS INFINITUDE OF CONTRAVARIANT FUNCTIONAL VECTORS[1]

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1. *Introduction.*—A general theory of function space affinely connected manifolds has been developed by the author in several publications.[2] In this paper I propose to give a number of new results pertaining to the differential geometry and invariant theory of a continuous infinitude of contravariant functional vectors. An application is made of these results to the differential geometry of functional group vectors of infinite groups of functional transformations. It is my intention to publish the complete results and proofs elsewhere.

2. *Infinitude of Contravariant Functional Vectors.*—Let $\eta_i[y]$ be a covariant functional vector, then the pair of functionals $\xi^\alpha[y]$, $\xi^\alpha_i[y]$ will

be said to constitute an infinitude of contravariant functional vectors if

$$L_i[\eta] = \xi^{(i)}\eta_i + \xi_i^\alpha \eta_\alpha$$

is an absolute functional form in $\eta_i[y]$. Throughout the paper we shall assume that $\xi^i[y] \neq 0$ and that there does not exist a functional $\lambda^i[y]$, other than $\lambda^i \equiv 0$, that satisfies the integral equation (linear independence)

$$\lambda^i[y]\xi^i[y] + \lambda^\alpha[y]\,\xi_\alpha^i[y] = 0.$$

In other words we shall assume that the Fredholm determinant

$$D[\xi_\mu^\lambda/\xi^{(\mu)}] \neq 0.$$

THEOREM I. *A necessary and sufficient condition that the pair of functionals $\xi^\alpha[y]$, $\xi_i^\alpha[y]$ form an infinitude of contravariant functional vectors is that they transform in accordance with the law*

$$\left.\begin{aligned} \bar{\xi}^i[\bar{y}] &= \xi^i[y]\,{}_{i,}\bar{y}^{(i)} \\ \bar{\xi}_i^\alpha[\bar{y}] &= \xi_i^\alpha[y]\,{}_{\alpha,}\bar{y}^{(\alpha)} + \xi_i^\lambda\,\bar{y}_{,\lambda}^\alpha + \xi^{(i)}\,\bar{y}_{,i}^\alpha \end{aligned}\right\} \quad (1)$$

under the functional transformations

$$\bar{y}^i = f^i[y].$$

It is to be observed that the inversion of the law of transformation (1) involves the solution of a system of mixed linear *integral equations.*

THEOREM II. *The Fredholm determinant*

$$D[\xi_\sigma^\rho/\xi^{(\sigma)}]$$

is a relative scalar functional invariant of weight minus one of the infinitude of contravariant functional vectors $\xi_\sigma^\rho[y]$, $\xi^\rho[y]$.

Put

$$\eta^i[y] = 1/\xi^i[y],\ \eta_\alpha^i[y] = -\frac{D_\alpha^i[\xi_\mu^\lambda/\xi^{(\mu)}]}{\xi^i D[\xi_\tau^\sigma/\xi^{(\tau)}]}\,.$$

3. *Asymmetric Functional Affine Connection.*

THEOREM III. *The functionals $\xi^i[y]$; $\xi_\alpha^i[y]$ determine the asymmetric functional affine connection*

$$\left.\begin{aligned} L_{\alpha\beta}^i[y] &= -\,\xi_{,\beta}^i\eta_\alpha^i - \xi_{\alpha,\beta}^i\,\eta^{(\alpha)} - \xi_{\sigma,\beta}^i\,\eta_\alpha^\sigma - {}_{\beta,}\xi_\beta^i\,\eta_\alpha^{(\beta)} \\ M_\alpha^i[y] &= -\,{}_{\alpha,}\xi_\alpha^i\,\eta^{(\alpha)} \\ N_\alpha^i[y] &= -\,{}_{i,}\xi^{(i)}\,\eta_\alpha^i - {}_{i,}\xi_\alpha^{(i)}\,\eta^{(\alpha)} - {}_{i,}\xi_\sigma^{(i)}\,\eta_\alpha^\sigma \\ O_\alpha^i[y] &= -\,\xi_{,\alpha}^i\,\eta^i \\ P^i[y] &= -\,{}_{i,}\xi^i\,\eta^i. \end{aligned}\right\} \quad (2)$$

Let

$$\left.\begin{aligned} \omega^i &= P^i\,\delta y^i + O_\alpha^i\,\delta y^\alpha \\ \omega_\sigma^i &= N_\sigma^i\,\delta y^i + M_\sigma^i\,\delta y^{(\sigma)} + L_{\sigma\alpha}^i\,\delta y^\alpha, \end{aligned}\right\} \quad (3)$$

where the functionals L, M, N, O, P constitute an arbitrary asymmetric connection, not necessarily the one defined by (2). In terms of the Pfaffian functional forms ω^i and ω^i_σ, the covariant functional differential of ξ^i, ξ^i_α is given by

$$\left. \begin{aligned} D\xi^k[y] &= \delta\xi^k[y] + \xi^k\,\omega^k \\ D\xi^i_k[y] &= \delta\xi^i_k[y] + \xi^i_k\,\omega^i + \xi^\sigma_k\,\omega^i_\sigma + \xi^{(k)}\,\omega^i_k \end{aligned} \right\} \tag{4}$$

THEOREM IV. *The covariant functional derivative of the infinitude of functional vectors* ξ^i, ξ^i_α; *based on an arbitrary affine connection is given by the sequence of functionals* ${}_{k;}\xi^{(k)}$, $\xi^k_{;l}$, ${}^{i;}\xi^i_k$, ${}_{k;}\xi^i_k$, $\xi^i_{k;l}$ *where*

$$\left. \begin{aligned} {}_{k;}\xi^{(k)} &= {}_{k,}\xi^{(k)} + \xi^k\,P^k \\ \xi^k_{;l} &= \xi^k_{,l} + \xi^k\,O^k_l \\ {}^{i;}\xi^i_k &= {}^{i,}\xi^i_k + \xi^i_k\,P^i + \xi^\sigma_k\,N^i_\sigma + \xi^{(k)}\,N^i_k \\ {}_{k;}\xi^i_k &= {}_{k,}\xi^i_k + \xi^{(k)}\,M^i_k \\ \xi^i_{k;\,l} &= \xi^i_{k,\,l} + \xi^i_k\,O^i_l + \xi^{(l)}_k\,M^i_l + \xi^\sigma_k\,L^i_{\sigma l} + \xi^{(k)}L^i_{kl}. \end{aligned} \right\} \tag{5}$$

THEOREM V. *The covariant functional derivative of* ξ^i, ξ^i_α *based on the functional affine connection* (2) *vanishes*:

$${}_{k;}\xi^{(k)} = 0,\ \xi^k_{;l}\ 0, = {}^{i;}\xi^i_k = 0,\ {}_{k;}\xi^i_k = ,\ \xi^i_{k;l} = 0.$$

As a consequence of this theorem, the result is obtained that a sequence of five functionals (the functional curvature tensor of the asymmetric connection) vanishes.

4. *Functional Group Vectors.*—From the asymmetric functional affine connection (2), a symmetric one is defined by the four functionals

$$\left. \begin{aligned} \mathfrak{L}^i_{\alpha\beta} &= {}^1/_2\,(L^i_{\alpha\beta} + L^i_{\beta\alpha}) \\ \mathfrak{M}^i_\alpha &= M^i_\alpha \\ \mathfrak{N}^i_\alpha &= {}^1/_2\,(N^i_\alpha + O^i_\alpha) \\ \mathfrak{P}^i &= P^i \end{aligned} \right\} \tag{6}$$

Let the five functionals of the curvature tensor based on (6) be denoted by $B^i_{\alpha\beta\gamma}$, $C^i_{\alpha\beta}$, $D^i_{\alpha\beta}$, E^i_α, F^i_α.

If we now *specialize our functional vectors* and consider $\xi^i[y]$, $\xi^i_\alpha[y]$ to be the functional group vectors of the first parameter group of a functional group with one arbitrary function,[3] the following theorem holds good.

THEOREM VI. *The nineteen functionals that make up the covariant functional derivative of the curvature tensor* $B^i_{\alpha\beta\gamma}$, $C^i_{\alpha\beta}$, $D^i_{\alpha\beta}$, E^i_α, F^i_α *all vanish.*

[1] Presented to the Amer. Math. Society, Des Moines meeting, December, 1929.

[2] A. D. Michal, *Amer. J. Math.*, **50,** 473–517 (1928) (presented to the Amer. Math. Soc. at the New York meeting, October, 1927—cf. *Bull. Amer. Math. Soc.*, **34,** 8–9 (1928)); these PROCEEDINGS, **16,** 88–94 (1930) (presented to the Amer. Math. Soc. at the New York meeting, March, 1929—cf. *Bull. Amer. Math. Soc.*, **35,** 438–439 (1929)).

[3] G. C. Moisil, *Comptes Rendus*, **188,** 691–692 (1929).