QUADRATIC FUNCTIONAL FORMS IN A COMPOSITE RANGE[1]

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1. *Transformations of Third Kind in a Composite Range.*—Let $\bar{y}^1$, $\bar{y}^2$, ..., $\bar{y}^n$ be n independent variables and $\bar{y}^\alpha$ a real continuous function of a real variable α, defined for $a \leq \alpha \leq b$. Consider the linear functional transformation from $\bar{y}^\alpha$, $\bar{y}^1$, ..., $\bar{y}^n$ to new variables y^α, y^1, ..., y^n

$$\left.\begin{aligned} y^\alpha &= K^\alpha \bar{y}^\alpha + K^\alpha_\beta \bar{y}^\beta + K^\alpha_j \bar{y}^j \qquad (K^\alpha \neq 0) \\ y^i &= \qquad\qquad K^i_\beta \bar{y}^\beta + K^i_j \bar{y}^j \end{aligned}\right\} \tag{1}$$

In (1) we assume that K^α, K^α_1, ..., K^α_n, K^1_α, ..., K^n_α are continuous functions of α, defined for $a \leq \alpha \leq b$, and that K^α_β is a continuous function of α and β, defined for $a \leq \alpha, \beta \leq b$. Here and throughout this paper we shall understand that any Greek letter used as an index can range over the closed continuous interval (a, b), while any Latin letter can take on any integral value 1, 2, ..., n. We shall use the convention of denoting Riemann integration on (a, b) by the repetition of a Greek subscript and superscript in a term except when an index is inclosed in a parenthesis. A similar convention is to hold for summation from 1 to n on the Latin indices.

The totality of such transformations (1) whose bordered Fredholm determinants do not vanish form a group G with inverses. By the bordered Fredholm determinant D is meant the following functional

$$|K^i_j| + \sum_{m=1}^{\infty} \frac{1}{m!} \Delta^{\alpha_1 \ldots \alpha_m}_{\alpha_1 \ldots \alpha_m}, \tag{2}$$

where

$$\Delta^{\alpha_1 \ldots \alpha_m}_{\beta_1 \ldots \beta_m} = \begin{vmatrix} \frac{K^{\alpha_1}_{\beta_1}}{K^{(\beta_1)}} & \cdots & \frac{K^{\alpha_1}_{\beta_m}}{K^{(\beta_m)}} & K^{\alpha_1}_1 & \cdots & K^{\alpha_1}_n \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{K^{\alpha_m}_{\beta_1}}{K^{(\beta_1)}} & \cdots & \frac{K^{\alpha_m}_{\beta_m}}{K^{(\beta_m)}} & K^{\alpha_m}_1 & \cdots & K^{\alpha_m}_n \\ \frac{K^1_{\beta_1}}{K^{(\beta_1)}} & \cdots & \frac{K^1_{\beta_m}}{K^{(\beta_m)}} & K^1_1 & \cdots & K^1_n \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{K^n_{\beta_1}}{K^{(\beta_1)}} & \cdots & \frac{K^n_{\beta_m}}{K^{(\beta_m)}} & K^n_1 & \cdots & K^n_n \end{vmatrix}.$$

We shall denote the bordered Fredholm determinant corresponding to (1) by

$$D\begin{bmatrix} \dfrac{K^\alpha_\beta}{K^{(\beta)}} & K^\alpha_j \\ \dfrac{K^i_\beta}{K^{(\beta)}} & K^i_j \end{bmatrix}$$

2. *The Form and a Functional Invariant.*—Consider a functional form with continuous coefficients

$$\left. \begin{array}{c} g_{\alpha\beta}\, y^\alpha y^\beta + g_\alpha (y^\alpha)^2 + 2g_{\alpha i}\, y^\alpha y^i + g_{ij}\, y^i y^j; \\ g_\alpha \neq 0, \quad g_{\alpha\beta} = g_{\beta\alpha}, \quad g_{ij} = g_{ji} \end{array} \right\} \qquad (3)$$

in the composite range $y^x, y^1, \ldots, y^n$. We assume that we are dealing with an absolute form under the group G.

As a consequence of the law of transformation of the coefficients g, it follows that the continuity of the coefficients, the non-vanishing of g_α, and the symmetry relations of $g_{\alpha\beta}$ and g_{ij} are invariant properties under G.

Since the bordered Fredholm functional of the product of two transformations of G is the product of the bordered Fredholm functionals of the transformations, it follows after some reductions that

$$D\begin{bmatrix} \dfrac{\bar{g}_{\alpha\beta}}{\bar{g}_\beta} & \bar{g}_{\alpha j} \\ \dfrac{\bar{g}_{i\beta}}{\bar{g}_\beta} & \bar{g}_{ij} \end{bmatrix} = \left(D\begin{bmatrix} \dfrac{K^\alpha_\beta}{K^{(\beta)}} & K^\alpha_j \\ \dfrac{K^i_\beta}{K^{(\beta)}} & K^i_j \end{bmatrix} \right)^2 D\begin{bmatrix} \dfrac{g_{\alpha\beta}}{g_\beta} & g_{\alpha j} \\ \dfrac{g_{i\beta}}{g_\beta} & g_{ij} \end{bmatrix}$$

under the group G. Hence we are led to the theorem.

THEOREM I. *The bordered Fredholm determinant of the coefficients of the form* (3) *is a relative functional invariant of weight two under the group* G.

$$D\begin{bmatrix} \dfrac{g_{\alpha\beta}}{\sqrt{g_\alpha g_\beta}} & \dfrac{g_{\alpha j}}{\sqrt{g_\alpha}} \\ \dfrac{g_{i\beta}}{\sqrt{g_\beta}} & g_{ij} \end{bmatrix}$$

is another form of the same invariant that exhibits explicitly the symmetric character of the original matrix.

3. *Forms Quadratic in a Function and Its Derivative.*—Let w^α be a function of α with a continuous derivative y^α. The form with continuous coefficients

$$\left.\begin{aligned} &A_{\alpha\beta}w^\alpha w^\beta + 2B_{\alpha\beta}w^\alpha y^\beta + C_{\alpha\beta}y^\alpha y^\beta + A_\alpha (w^\alpha)^2 + 2B_\alpha w^\alpha y^\alpha \\ &+ C_\alpha(y^\alpha)^2; \quad A_{\alpha\beta} = A_{\beta\alpha}, C_{\alpha\beta} = C_{\beta\alpha} \end{aligned}\right\} \tag{5}$$

does not possess a unique expansion, so that its vanishing for all admissible w^α does not entail the identical vanishing of all the coefficients. Denoting w^a by Y and defining

$$E^\alpha = 1, \; E^\alpha_\beta = \begin{cases} 0 \text{ for } \alpha < \beta \leq b \\ 1 \text{ for } a \leq \beta \leq \alpha \end{cases}$$

we may write

$$w^\alpha = E^\alpha \, Y + E^\alpha_\beta y^\beta. \tag{6}$$

An application of the functional transformation (6) to the form (5) is instrumental[1] in yielding the following theorem.

THEOREM II. *Any form* (5) *with continuous coefficients can be thrown over into a form of type* (3) *with* $n = 1$, $y^1 = Y$ *and with continuous coefficients given as follows:*

$$\left.\begin{aligned} g_{\alpha\beta} &= A_{\gamma\delta}E^\gamma_\alpha E^\delta_\beta + B_{\gamma\beta}E^\gamma_\alpha + B_{\gamma\alpha}E^\gamma_\beta + C_{\alpha\beta} \\ &\qquad + A_\gamma E^\gamma_\alpha E^\gamma_\beta + B_\alpha E^{(\alpha)}_\beta + B_\beta E^{(\beta)}_\alpha \\ g_\alpha &= C_\alpha \\ g_{\alpha 1} &= A_{\gamma\delta}E^\gamma E^\delta_\alpha + B_{\gamma\alpha}E^\gamma + A_\gamma E^\gamma_\alpha + B_\alpha \\ g_{11} &= A_{\gamma\delta}E^\gamma E^\delta + A_\gamma E^\gamma. \end{aligned}\right\} \tag{7}$$

COROLLARY. *A necessary and sufficient condition that form* (5) *vanish for all admissible functions* w^α *is that* $g_{\alpha\beta}$, g_α, $g_{\alpha 1}$, g_{11} *defined by* (7) *all vanish identically.*

Finally the following theorem has been proved by us:

THEOREM III. *If* $'K^\alpha_\gamma = \frac{\partial}{\partial\alpha} K^\alpha_\gamma$ *exists, is bounded and is integrable, then the following relation holds*

$$D\begin{bmatrix} 'K^\alpha_\gamma E^\gamma_\beta & 'K^\alpha_\gamma E^\gamma \\ K^\alpha_\gamma E^\gamma_\beta & 1 + K^\alpha_\gamma E^\gamma \end{bmatrix} = D[K^\alpha_\beta],$$

where $D[K^\alpha_\beta]$ *is the well-known Fredholm determinant of* K^α_β.

[1] Presented to the American Mathematical Society, June 20, 1930. A complete account will be published in the *Trans. Am. Math. Soc.* Paragraphs 1 and 2 of the present note involve as special cases well-known algebraic theories as well as some recent results on functional forms. For the latter, see A. D. Michal, *Am. J. Math.*, **50** (1928), in particular pp. 476–480; A. D. Michal and T. S. Peterson, a forthcoming paper in the *Ann. Math.*