"RIEMANNIAN" DIFFERENTIAL GEOMETRY IN ABSTRACT SPACES[1]

BY A. D. MICHAL

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated July 26, 1935

1. *Introduction*.—In this note brief indications are given of a set of postulates for "Riemannian" differential geometry in abstract spaces. The detailed treatment of this geometry has been included in a comprehensive memoir that I intend to publish elsewhere. An introduction to more general abstract differential geometries with one or several linear connections has been given by me in another comprehensive memoir.[2]

It has become clear in recent years that a differential geometry is concerned with *several* spaces together with their interconnections by means of interspace functional transformations. Moreover, the differentiability of at least one of these functional transformations enters into the theory in an essential manner. It is evident from these remarks that some of the spaces must be sufficiently restricted so as to be capable of supporting a theory of differentials.[3]

Let R be the class of real numbers, which we take as a known system.[4] The universe of discourse of our abstract Riemannian geometry consists of two classes E_1 and E_2 of undefined elements, two relations of equality,[5] two operations of addition, two operations of multiplication by elements of R, two norms, one interspace inner product $[\,.\,,\,.\,]$, two metric functions, $g(.,.)$, $\bar{g}(.,.)$, two "parallelism" functions $g(.,.,.)$, $f(.,.,.)$ and two "curvature" functions $g(.,.,.,.)$, $f(.,.,.,.)$. The theory of this universe of discourse subject to the *sixty postulates* (classified into eight groups) with $[\delta x, g(x, \delta x)]^{1/2}$ as element of arc length constitutes our abstract Riemannian differential geometry. In the first comprehensive memoir referred to above we have stated the sixty postulates in such a manner as to make each postulate intelligible independently of the remaining ones. In order to avoid lengthy statements in the present note we have not done this for the thirty postulates that are explicitly given in section 2.

The classes E_1 and E_2 have been shown to have all the properties of a complete normed vector space.[6] The commutativity and associativity of each of the two additions, the usual three properties of each of the two equalities, the existence of the two zero elements and several continuity, differentiability and substitution properties are not postulated but proved as theorems.

Besides the classical finite dimensional examples and the author's infinitely dimensional instances,[7] several new instances and specializations of the general abstract theory have been found.

2. *The Postulates.*—

Group I: Postulates for E_1
Group II: Postulates for E_2
Group III: Postulates for interspace inner product $[\xi,\eta]$

(1) If $\xi\epsilon E_1$, $\eta\epsilon E_2$ then $[\xi, \eta]\epsilon R$; (2) $[\xi_1 + \xi_2, \eta] = [\xi_1, \eta] + [\xi_2, \eta]$; (3) $[\xi, \eta_1 + \eta_2] = [\xi, \eta_1] + [\xi, \eta_2]$; (4) $|[\xi, \eta]| \leq M\|\xi\| \|\eta\|$; (5) $[\xi, \eta] = 0$ for all ξ implies $\eta + \eta_1 = \eta_1$ for all η_1; (6) $[\xi, \eta] = 0$ for all η implies $\xi + \xi_1 = \xi_1$ for all ξ_1.

Group IV: Postulates for metric $g(x, \xi)$ and its inverse $\bar{g}(x, \eta)$

(1) If $x\epsilon E_1((x_0)_a)$, $\xi\epsilon E_1$, then $g(x, \xi)\epsilon E_2$; (2) $\|g(x, \xi)\| \leq M_1 \|\xi\|$; (3) if $x_1 = x_2$, then $g(x_1, \xi) = g(x_2, \xi)$; (4) if $x\epsilon E_1((x_0)_a)$, $\eta\epsilon E_2$, then $\bar{g}(x, \eta)\epsilon E_1$; (5) if $x_1 = x_2$, then $\bar{g}(x_1, \eta) = \bar{g}(x_2, \eta)$; (6) $g(x, \bar{g}(x, \eta)) = \eta$; (7) $\bar{g}(x, g(x, \xi)) = \xi$; (8) $[\xi_1, g(x, \xi_2)] = [\xi_2, g(x, \xi_1)]$; (9) $[\xi, g(x, \xi)] \geq 0$; (10) if $[\xi, g(x, \xi)] = 0$, then $\xi + \xi_1 = \xi_1$ for all ξ_1.

Group V: First group of postulates for parallelism

(1) If $x\epsilon E_1((x_0)_a)$, ξ, $y\epsilon E_1$, then $g(x, \xi, y)\epsilon E_2$; (2) $g(x, \xi, y_1 + y_2) = g(x, \xi, y_1) + g(x, \xi, y_2)$; (3) given an $\epsilon > 0$ there exists a $\delta > 0$ such that $\|y\| < \delta$ implies $\|g(x, \xi, y)| < \epsilon$; (4) given an $\epsilon > 0$ there exists a $\delta > 0$ such that $\|y\| < \delta$ implies $\|(g(x + y, \xi) + (-1)g(x, \xi)) + (-1)g(x, \xi, y)\| \leq \epsilon \|y\|$; (5) if $x_1 = x_2$, then $g(x_1, \xi, y) = g(x_2, \xi, y)$; (6) if $\xi_1 = \xi_2$, then $g(x, \xi_1, y) = g(x, \xi_2, y)$; (7) if $y_1 = y_2$, then $g(x, \xi, y_1) = g(x, \xi, y_2)$.

Group VI: Second group of postulates for parallelism

(1) If $x\epsilon E_1((x_0)_a)$, ξ_1, $\xi_2\epsilon E_1$, then $f(x, \xi_1, \xi_2)\epsilon E_2$; (2) $[y, f(x, \xi_1, \xi_2)] = [\xi_1, g(x, \xi_2, y)]$; (3) given an $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\xi_1\| < \delta$ implies $\|f(x, \xi_1, \xi_2)\| < \epsilon$.

Group VII: First group of postulates for curvature

(1) If $x\epsilon E_1((x_0)_a)$, ξ, y_1, $y_2\epsilon E_1$, then $g(x, \xi, y_1, y_2)\epsilon E_2$; ...; (4) given an $\epsilon > 0$ there exists a $\delta > 0$ such that $\|y_2\| < \delta$ implies $\|(g(x + y_2, \xi, y_1) + (-1)g(x, \xi, y_1)) + (-1)g(x, \xi, y_1, y_2)\| \leq \epsilon \|y_2\|$; ....

Group VIII: Second group of postulates for curvature

(1) If $x\epsilon E_1((x_0)_a)$, ξ_1, ξ_2, $y\epsilon E_1$, then $f(x, \xi_1, \xi_2, y)\epsilon E_2$; ...; (4) given an $\epsilon > 0$ there exists a $\delta > 0$ such that $\|y\| < \delta$ implies $\|(f(x + y, \xi_1, \xi_2) + (-1)f(x, \xi_1, \xi_2)) + (-1)f(x, \xi_1, \xi_2, y)\| \leq \epsilon \|y\|$; ....

3. *Parallelism and Curvature.*—Each function considered in this section has been shown to be continuous jointly in all its arguments and, except for the dependence on the variable x, linear in each argument. The differentials[8] $g(x, \xi; \delta x)$, $g(x, \xi; \delta_1 x; \delta_2 x)$, $f(x, \xi_1, \xi_2; \delta x)$ exist equal to $g(x, \xi, \delta x)$, $g(x, \xi, \delta_1 x, \delta_2 x)$, $f(x, \xi_1, \xi_2, \delta x)$, respectively.

The following two functions $\Gamma_1(x, \xi_1, \xi_2)$ and $\Gamma_2(x, \eta, \xi)$ play a rôle analogous to that of the Christoffel symbols of the second kind in n-dimensional classical Riemannian geometry. They are defined by

$$\begin{aligned}\Gamma_1(x, \xi_1, \xi_2) &\equiv {}^1/_2\, \bar{g}(x, g(x, \xi_1; \xi_2) + g(x, \xi_2; \xi_1) - f(x, \xi_1, \xi_2)) \\ \Gamma_2(x, \eta, \xi) &\equiv {}^1/_2 \left\{ g(x, \bar{g}(x, \eta); \xi) + f(x, \bar{g}(x, \eta), \xi) - g(x, \xi; \bar{g}(x, \eta)) \right\}\end{aligned} \quad (1)$$

and have been shown to satisfy the identity

$$[\Gamma_1(x, \xi_1, \xi_2), \eta] = [\xi_1, \Gamma_2(x, \eta, \xi_2)].$$

The "covariant" differential $g(x, \xi_1/\xi_2)$ defined by

$$g(x, \xi_1/\xi_2) \equiv g(x, \xi_1; \xi_2) - g(x, \Gamma_1(x, \xi_1, \xi_2)) - \Gamma_2(x, g(x, \xi_1), \xi_2)$$

vanishes. Moreover, the equation $g(x, \xi_1/\xi_2) = 0$ considered as a functional equation with the functions Γ_1 and Γ_2 as unknowns has the unique solution (1). With the aid of the functions Γ_1 and Γ_2 it is possible to develop a theory of parallel displacements and geodesics.

Let

$$\gamma_1(x, \xi_1, \xi_2) \equiv g(x, \Gamma_1(x, \xi_1, \xi_2)), \; \gamma_2(x, \xi_1, \xi_2) \equiv \Gamma_2(x, g(x, \xi_1), \xi_2)$$

then the function $R_1(x, \xi_1, \xi_2, \xi_3)$ defined as follows plays a rôle similar to the Riemann Christoffel curvature tensor in classical Riemannian geometry

$$\begin{aligned}R_1(x, \xi_1, \xi_2, \xi_3) \equiv {}^1/_2 \{ & g(x, \xi_2; \xi_1; \xi_3) + f(x, \xi_1, \xi_3; \xi_2) - f(x, \xi_1, \xi_2; \xi_3) \\ & - g(x, \xi_3; \xi_1; \xi_2) \} + \gamma_2(x, \bar{g}(x, \gamma_1(x, \xi_1, \xi_3)), \xi_2) \\ & - \gamma_2(x, \bar{g}(x, \gamma_1(x, \xi_1, \xi_2)), \xi_3).\end{aligned}$$

The following identities have been shown to hold for the curvature form R_1:

$R_1(x, \xi_1, \xi_2, \xi_3) = - R_1(x, \xi_1, \xi_3, \xi_2)$, $[\xi, R_1(x, \xi_1, \xi_2, \xi_3)] = -[\xi_1, R_1(x, \xi, \xi_2, \xi_3)]$, $[\xi, R_1(x, \xi_1, \xi_2, \xi_3)] = [\xi_2, R_1(x, \xi_3, \xi, \xi_1)]$, $R_1(x, \xi_1, \xi_2, \xi_3) + R_1(x, \xi_3, \xi_1, \xi_2) + R_1(x, \xi_2, \xi_3, \xi_1) = 0$.

The addition of a few postulates has enabled us to study abstract Riemannian geometry with torsion while an evident modification of the postulates has led to a geometry with abstract Hermitian differential metric. As an interesting by-product we mention the fact that a slight modification of the first six groups of postulates alone suffices for an abstract dynamics with ${}^1/_2\, [dx/dt, g(x, dx/dt)]$ as kinetic energy.

[1] Presented at the 1933 Pasadena meeting of the American Math. Soc. The present drastically reduced set of postulates was presented at the 1935 Stanford meeting of the Amer. Math. Soc. Cf. *Bull. Am. Math. Soc.*, **39**, 879 (Nov., 1933), and **41**, 195 (March, 1935).

[2] Presented at the 1933 Pasadena meeting of the Am. Math. Soc. Cf. *Bull. Am. Math. Soc.*, **39**, 880 (Nov., 1933).

[3] Fréchet, M., *Ann. Sc. Ec. Norm. Sup.*, *t.* **62**, 293–323 (1925).

[4] A more general theory will result if in the place of R we employ an abstract normed ring.

[5] We use the symbols $=$, $+$, $\|\ldots\|$, ... in more than one sense. No confusion need arise as the context makes clear the meaning of each such symbol. It is worth while to mention here that the relation of equality $=$ for E_1 as well as for E_2 is not an independent primitive idea; for, an equivalent set of postulates can be given in which the equality $=$ for E_1 as well as for E_2 is *defined* by means of the properties of the corresponding norm function $\|\ldots\|$.

[6] Banach, S., *Théorie des Opérations Linéaires*, chapter IV, Warsaw, 1932; see also Fréchet, M., *Espaces Abstraits*, Paris, 1928.

[7] Michal, A. D., "Affinely Connected Function Space Manifolds," *Am. Jour. Math.*, **50**, 473–517, especially 509–517 (1928); "Differential Geometries of Function Space," these PROCEEDINGS, **16**, 88–94 (Jan., 1930). See also Peterson, T. S., "The Analogue of Weyl's Conformal Curvature Tensor in a Michal Functional Geometry," *Annali di Mat.*, **13**, 55–62 (1934).

[8] The first partial differential with respect to x of a function $F(x, y_1, \ldots, y_n)$ is denoted by $F(x, y_1, \ldots, y_n;\ \delta x)$. All differentials are taken in the Fréchet sense.

ASYMPTOTIC REPRESENTATIONS OF CONFLUENT HYPERGEOMETRIC FUNCTIONS

BY E. FISHER

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Communicated July 18, 1935

In a number of problems in quantum mechanics there is need for simple asymptotic approximations to the confluent hypergeometric function $M_{k,m}(z)$. In the following note a combination of the recurrence and differential properties of these functions has led to such expressions for the cases $|z| \gg |k|, 1$; $|m| \gg |k|, 1$ and $|z| \gg |m|, 1$. The notation and fundamental formulas used are from Whittaker and Watson, *Modern Analysis*.

Case I. $|z| \gg |k|, 1$.

If we let $F_{k,m}(z) = z^{-k}e^{z/2}W_{k,m}(z)$, the function F satisfies the two equations,

$$\frac{m^2 - (k - {}^1/_2)^2}{z^2} F_{k-1,m} + \left(1 - \frac{2k}{z}\right) F_{k,m} = F_{k+1,m}. \qquad (1)$$

$$\frac{d}{dz} F_{k,m} = \frac{(k - {}^1/_2)^2 - m^2}{z^2} F_{k-1,m}. \qquad (2)$$

From equation (1), neglecting $0(z^{-1})$, we obtain

$$0 = \frac{m^2}{z^2} F_{k,m} + F_{k+1,m} - F_{k+2} = \left(\delta + {}^1/_2 + \frac{1}{2} D\right)\left(\delta + {}^1/_2 - \frac{1}{2} D\right) F_{k,m}$$