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High Frequency Behavior of a Space Charge Rotating in a Magnetic Field

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A theoretical discussion is given of the propagation of electromagnetic waves in a space charge which is rotating under the influence of a uniform magnetic field. Equations are derived for the relations between the amplitudes of the electric and magnetic fields, charge density, electron velocities, the applied magnetic field, the frequency, the phase velocities, and the effective dielectric constant of the space charge region. In particular, the natural frequencies are computed for such a space charge rotating about an infinitesimal conductor and enclosed by a coaxial conducting cylinder. The predicted resonant frequencies agree well with experimentally observed values for magnetron oscillators. The theory here developed is also applicable to other devices in which a rotating space charge is utilized.

1. INTRODUCTION

MANY studies have been made of the electromagnetic waves which can propagate through a stationary space charge region. This theory has been extremely useful in studying Heaviside layer characteristics as well as plasma oscillations. More recently there has been occasion to investigate the high frequency behavior of a moving space charge, as, for example, an electron beam.[1,2] The problem has become important because devices utilizing space charge waves have been built to generate and amplify high frequency signals of the order of 1000 megacycles per second.

In this paper a study is made of the high frequency properties of a space charge which is rotating under the influence of an applied uniform magnetic field and a radial electric field. Only the theory will be given here, but applications will be kept in mind in choosing boundary conditions for the determination of natural frequencies of the system. An experimental investigation of several new devices suggested by this theoretical treatment will be reported elsewhere.

A d.c. space charge condition which is theoretically possible and easily obtainable in practice will be selected. High frequency waves of infinitesimal amplitude will then be superimposed upon these d.c. conditions, and the relations between the various components of the waves will be derived. It will be shown that the waves may be considered in general to be of two types, which are characterized by the presence or absence of an axial component of the magnetic field in the wave. The effective dielectric constant of such a space charge to the propagation of electromagnetic waves is given in terms of the applied magnetic field and the frequency.

The natural frequencies of a space charge rotating about an infinitesimal axial conductor and enclosed by a perfectly conducting cylinder are computed and are seen to agree well with experimental data for magnetron oscillators. Large signals are discussed, and a solution to the large signal equations is given for a special case which is believed to be of importance.

2. D.C. SOLUTIONS

It will be shown in this section that the application of an electric field normal to an axis of cylindrical symmetry, together with a uniform magnetic field parallel to that axis, makes possible the existence of a space charge of almost uniform density, which rotates with uniform angular velocity around the axis. Since the system is an accelerated one, an exact solution would involve the equations of general relativity. A very good approximation will be given by the special theory, however, if it is assumed that at any time the electron motion is approximately linear.[3]

The magnetic field experienced by the electrons will be composed of two parts, the uniform applied field, H_0, and the magnetic field, H_1, generated by the motion of the charges. H_1 will vary with radius. The behavior of the space charge will be completely described (in cylindrical

[1] W. C. Hahn, Gen. Elec. Rev. **42**, 258 (1939).
[2] S. Ramo, Phys. Rev. **56**, 276 (1939).
[3] Cf. Page, *Introduction to Theoretical Physics*, Second Edition (Van Nostrand, 1935), p. 526.

coordinates) by the field equations:

$$(\text{curl } H)_\theta = -\frac{\partial H_1}{\partial r} = \frac{r\dot{\theta}}{c}\rho_0, \tag{1}$$

$$\text{div } E = \frac{1}{r}\frac{\partial}{\partial r}(rE_0) = \rho_0 \tag{2}$$

and the force equations:

$$\frac{m_0}{\beta}(\ddot{r} - r\dot{\theta}^2) = eE_0 + \frac{er\dot{\theta}}{c}(H_0 + H_1), \tag{3}$$

$$\frac{1}{r}\frac{d}{dt}\left(\frac{m_0}{\beta}r^2\dot{\theta}\right) = -e\dot{r}(H_0 + H_1), \tag{4}$$

where E_0 is the radial electric field, ρ_0 is the charge density, e is the charge on the electron, m_0 is the rest mass of the electron and $\beta^2 = 1 - v^2/c^2$. (Rational units will be used throughout this paper.) These equations are satisfied by the solution:[4] (if we neglect terms of the order of $r^4\Omega_0^4/c^4$ compared with unity)

$$\dot{r} = 0; \quad \dot{\theta} = \frac{-eH_0}{2m_0c} = \Omega_0, \text{ say}; \quad \rho_0 = \frac{2m_0\Omega_0^2}{e\beta^2};$$
$$E_0 = \frac{eH_0^2 r}{4m_0c^2\beta}; \quad \beta^2 = 1 - \frac{r^2\Omega_0^2}{c^2}. \tag{5}$$

It is worthy of note that the action of the field due to the charges, and the relativistic increase in mass with velocity, have equal and opposite effects on the angular velocity, whose final value is independent of radius.

A more general solution exists in which $\dot{r} \neq 0$. In this paper, however, we shall concern ourselves only with a space charge whose behavior is described by Eqs. (5). Devices may obviously be constructed to maintain such a space charge. A common example may be the magnetron operated at or above cut-off. The authors have had the privilege of reading an unpublished discussion by L. Brillouin, which lends support to this viewpoint.

3. A.C. Solutions

Let a space charge exist having infinite extent in the z direction and subject to the conditions given in Eqs. (5). It will now be our purpose to investigate the possible oscillations in this space charge and the associated fields.

[4] This solution was obtained for the nonrelativistic case by A. W. Hull, Phys. Rev. **23**, 112 (1924).

From this point on we shall give a nonrelativistic treatment, since relativistic corrections add considerably to the complexity of the equations. Hence, in substituting from the d.c. solutions, we shall set β = unity.

The a.c. velocities, charge density, and fields must satisfy the field equations:

$$\text{curl } H = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{\rho v}{c}, \tag{6}$$

$$\text{curl } E = -\frac{1}{c}\frac{\partial H}{\partial t}, \tag{7}$$

$$\text{div } E = \rho \tag{8}$$

and the force equations:

$$\frac{d}{dt}(mv) = eE + \frac{e(v \times H)}{c}. \tag{9}$$

In this section we shall deal with a small signal only, assuming that the alternating parts of fields, velocities, and charge density are small compared with the steady parts where these exist. It is thus possible to neglect products of two or more a.c. terms, and the equations become linear. The a.c. part of the expression for field, velocity, or charge density will be assumed to have the form

$$X(r)e^{i(\omega t + n\theta + \gamma z)},$$

where ω represents 2π times the frequency of oscillation, n is the order of the wave, i.e., the number of complete waves around the circumference of a circle whose axis is the axis of symmetry; and γ is the propagation constant in the z direction. Such a wave travels with phase velocities $-\omega/n$ in the azimuthal direction and $-\omega/\gamma$ in the z direction.

Equations (9) for the a.c. velocities reduce to the form:

$$\rho_0 v_r = -\frac{2i\Omega_0^2}{\alpha}\left(E_r + \frac{r\Omega_0}{c}H_z\right), \tag{10}$$

$$\rho_0 v_\theta = -\frac{2i\Omega_0^2}{\alpha}E_\theta, \tag{11}$$

$$\rho_0 v_z = -\frac{2i\Omega_0^2}{\alpha}\left(E_z - \frac{r\Omega_0}{c}H_r\right), \tag{12}$$

where

$$\alpha = \frac{d}{dt} = \frac{\partial}{\partial t} + \dot{\theta}\frac{\partial}{\partial \theta} = \omega + n\Omega_0.$$

Thus α represents n times the difference between the azimuthal phase velocity of the wave, and the angular velocity of the electrons.

Substituting the values of a.c. charge density and velocity given by Eqs. (8), (10), (11), and (12) in Eqs. (6) and (7), we are left with the following six equations.

$$crE_r(\omega\alpha-2\Omega_0^2)+\alpha c^2\gamma rH_\theta-H_z(n\alpha c^2+[2r^2\Omega_0^3])=0, \tag{13}$$

$$\alpha\Omega_0\frac{\partial}{\partial r}(rE_r)+iE_\theta(\alpha^2-2\Omega_0^2)+i\gamma\alpha r\Omega_0E_z-i\gamma\alpha cH_r+\alpha c\frac{\partial H_z}{\partial r}=0, \tag{14}$$

$$icrE_z(\omega\alpha-2\Omega_0^2)+iH_r(n\alpha c^2+[2r^2\Omega_0^3])-\alpha c^2\frac{\partial}{\partial r}(rH_\theta)=0, \tag{15}$$

$$i\gamma E_\theta-\frac{in}{r}E_z-\frac{i\omega}{c}H_r=0, \tag{16}$$

$$\gamma E_r+i\frac{\partial E_z}{\partial r}+\frac{\omega}{c}H_\theta=0, \tag{17}$$

$$\frac{n}{r}E_r+\frac{i}{r}\frac{\partial}{\partial r}(rE_\theta)-\frac{\omega}{c}H_z=0, \tag{18}$$

where $E_re^{i(\omega t+n\theta+\gamma z)}$, etc., are the a.c. field components.

These equations can easily be reduced to two second-order partial differential equations in two of the variables. For instance, we may choose E_θ and H_θ. Eqs. (13) and (18) give E_r and H_z, and Eqs. (15) and (16) give H_r and E_z in terms of these variables. Substituting in Eqs. (14) and (17), we obtain two second-order partial differential equations, both involving E_θ and H_θ. Mathematically, four solutions should exist for these two equations. We shall find, however, that the solutions are Bessel functions of either the first or second kind, or their derivatives. Physically, this means that there are only two solutions, which differ from each other in form. Each of these two solutions will include either the term $AJ+BN$ or $A(\partial J/\partial r)+B(\partial N/\partial r)$, where J and N represent the Bessel functions of the first and second kind, and A and B are arbitrary multipliers. One of the solutions is easily seen from the form of the final second-order differential equations to be:

$$\begin{aligned}
E_r&=-\frac{\alpha\gamma c^2}{\omega}\left\{A_1\frac{\partial}{\partial r}[J_n(Dr)]+B_1\frac{\partial}{\partial r}[N_n(Dr)]\right\},\\
iE_\theta&=\frac{\alpha\gamma nc^2}{\omega r}[A_1J_n(Dr)+B_1N_n(Dr)],\\
iE_z&=-\left(\omega\alpha-2\Omega_0^2-\frac{\alpha c^2\gamma^2}{\omega}\right)[A_1J_n(Dr)+B_1N_n(Dr)],\\
iH_r&=\frac{nc}{r\omega}(\omega\alpha-2\Omega_0^2)[A_1J_n(Dr)+B_1N_n(Dr)],\\
H_\theta&=\frac{c}{\omega}(\omega\alpha-2\Omega_0^2)\left\{A_1\frac{\partial}{\partial r}[J_n(Dr)]+B_1\frac{\partial}{\partial r}[N_n(Dr)]\right\},\\
H_z&=0,
\end{aligned} \tag{19}$$

where $D^2=\omega^2/c^2-2\Omega_0^2/c^2-\gamma^2$; A_1 and B_1 are arbitrary constants; J_n and N_n represent the Bessel functions of the first and second kind and of the nth order. The a.c. charge density and velocity components follow from Eqs. (8)–(12). This solution will be referred to as solution I and evidently corresponds to the "E wave" of the wave guide[5] since $H_z=0$.

[5] Cf. J. R. Carson, S. P. Mead and S. A. Schelkunoff, Bell Sys. Tech. J. **15**, 310 (1936).

Comparison of Carson, Mead and Schelkunoff's[5] solution for the "E type" wave in a medium of dielectric constant, ϵ, with our solution I reveals the fact that for $n=0$ the rotating space charge behaves as though it were a dielectric of dielectric constant

$$\epsilon=1-2\Omega_0^2/\omega^2=1-e\rho_0/m$$

from Eqs. (5). This is the same expression as is obtained for the effective dielectric constant of a plasma in which the electron density is ρ_0.[6] Evidently, as far as the propagation of symmetrical ($n=0$) "E type" waves is concerned, it is immaterial whether the uniform concentration of electrons is maintained by the agency of a positive ion space charge, or by crossed electric and magnetic fields.

The other solution is less easily obtained since its form depends on whether or not $\alpha^2=2\Omega_0^2$.

Suppose first that $\alpha^2\neq 2\Omega_0^2$ so that the term $2r^2\Omega_0^4/c^2$ is small compared with $\alpha^2-2\Omega_0^2$. The simplest procedure for obtaining solution II is to assume that those terms of solution I which involved $J_n(Dr)$ now involve $(\partial/\partial r)[J_n(Dr)]$ and vice versa, and then to evaluate the coefficients. This gives for solution II:

$$\begin{aligned}
E_r&=\frac{(n\alpha c^2+2r^2\Omega_0^3)(\omega\alpha-2\Omega_0^2)-2r^2\Omega_0^3c^2\gamma^2}{rc(\alpha^2-2\Omega_0^2+2r^2\Omega_0^4/c^2)}[A_2J_n(Dr)+B_2N_n(Dr)],\\
iE_\theta&=-\frac{\alpha c(\omega\alpha-2\Omega_0^2)}{\alpha^2-2\Omega_0^2}\left\{A_2\frac{\partial}{\partial r}[J_n(Dr)]+B_2\frac{\partial}{\partial r}[N_n(Dr)]\right\},\\
iE_z&=\frac{2\gamma\Omega_0^3cr}{\alpha^2-2\Omega_0^2}\left\{A_2\frac{\partial}{\partial r}[J_n(Dr)]+B_2\frac{\partial}{\partial r}[N_n(Dr)]\right\},\\
iH_r&=-\gamma c^2\left\{A_2\frac{\partial}{\partial r}[J_n(Dr)]+B_2\frac{\partial}{\partial r}[N_n(Dr)]\right\},\\
H_\theta&=-\frac{\gamma nc^2}{r}[A_2J_n(Dr)+B_2N_n(Dr)],\\
H_z&=\frac{(\omega\alpha-2\Omega_0^2)^2-c^2\gamma^2(\alpha^2-2\Omega_0^2)}{\alpha^2-2\Omega_0^2+2r^2\Omega_0^4/c^2}\Big[A_2J_n(Dr)+B_2N_n(Dr)\Big],
\end{aligned}\tag{20}$$

where D^2 is defined as in Eqs. (19) and A_2 and B_2 are arbitrary constants.

This is the analog of the "H wave" in the wave guide although in the presence of the rotating space charge E_z does not disappear. Under most conditions it is small, however, compared with the other field components. Had we neglected the terms in square brackets in Eqs. (13) and (15), a procedure which is justifiable under some conditions, we should have obtained a form of solution II in which E_z is actually zero.

If $\alpha^2=2\Omega_0^2$ the problem becomes complicated and the general form of solution II has not yet been deduced. Solution II is easily obtained for a two-dimensional wave ($\gamma=0$) however. It is as follows:

$$\begin{aligned}
&E_r=A_3(nc^2+r^2\alpha\Omega_0)/n\Omega_0cr^3,\\
&iE_\theta=A_3nc/2r^3\Omega_0[1-(2r^2\Omega_0^2/c^2)\ln r]+B_3/r,\\
&H_z=A_3/r^2;\quad E_z=H_r=H_\theta=0,
\end{aligned}\tag{21}$$

where A_3 and B_3 are arbitrary constants. The case of $\alpha=0$ is of particular interest since it represents a wave which travels with the same angular velocity as do the electrons. When $\alpha=0$, the Eqs. (13)–(18) reduce to four equations:

$$E_\theta=0,\tag{22}$$

$$E_r=-r\Omega_0H_z/c,\tag{23}$$

$$E_z=r\Omega_0H_r/c,\tag{24}$$

$$\operatorname{div} H=0.\tag{25}$$

The problem is indeterminate since three equations, (23), (24) and (25) describe five

[6] L. Tonks, Phys. Rev. **37**, 1458 (1931).

variables, E_r, E_z, H_r, H_θ, and H_z. The a.c. components of electron velocity are also indeterminate, having the form 0/0. Although solutions I and II are still correct they are no longer unique. As will be shown in the next section, the condition $\alpha=0$ is one under which oscillation is generally possible.

4. Application of Boundary Conditions

It will be assumed that the rotating space charge is maintained between two perfectly conducting coaxial cylinders. A region of free space may, if it is desired, intervene between the space charge and either or both cylinders. It is then necessary to discover the conditions under which the electric fields E_θ and E_z are zero at the surface of the boundaries. If a relation between frequency and electron velocity exists which satisfies these conditions, oscillations of the type which has been postulated are possible.

The simplest case is that for which the inner boundary is infinitesimal in radius compared with the outer boundary. This corresponds in geometry to tubes of the magnetron type, except that end effects have been avoided by assuming the tube to be infinitely long. Since the N functions become infinite at the origin, their coefficients B_1 and B_2 in the two solutions must both be zero, and the solution includes only Bessel functions of the first kind. Suppose first that the space charge fills the whole region inside the outer conductor. Let the radius of the outer conductor be R. For $r=R$, E_θ and E_z are given (if $\alpha^2\neq 2\Omega_0^2$) by combining Eqs. (19) and (20):

$$iE_\theta=A_1\frac{\alpha\gamma nc^2}{\omega R}J_n(DR)-A_2\frac{\alpha c(\omega\alpha-2\Omega_0^2)}{\alpha^2-2\Omega_0^2}\frac{\partial}{\partial r}[J_n(DR)], \tag{26}$$

$$iE_z=-A_1\left(\omega\alpha-2\Omega_0^2-\frac{\alpha c^2\gamma^2}{\omega}\right)J_n(DR)+A_2\frac{2\gamma\Omega_0^3 cr}{\alpha^2-2\Omega_0^2}\frac{\partial}{\partial r}[J_n(DR)]. \tag{27}$$

The condition $E_\theta=E_z=0$ can be satisfied in several ways, three of which are not trivial. Two of these solutions are:

$$\begin{aligned} A_2&=0 \quad\text{and}\quad DR=T_n\\ A_1&=0 \quad\text{and}\quad DR=T_n', \end{aligned} \tag{28}$$

where T_n and T_n' represent roots of J_n and J_n' respectively. These conditions limit the range of oscillation to very high frequencies. The first condition requires that DR be not less than 2.4, the first root of J_0, since for $n>0$, T_n is >2.4. The only positive term in D^2 is ω^2/c^2. Hence we must have $\omega R/c>2.4$. If we assume a radius R of 1 cm, the frequency must be greater than 11,000 megacycles. The second condition requires that DR be not less than 1.8, the first root of J', since for $n\neq 1$, T_n' is >1.8. Hence $\omega R/c>1.8$ and if $R=1$ cm, the frequency must be greater than 8500 megacycles. This type of solution will not be discussed further.

The third condition which makes $E_\theta=E_z=0$ from Eqs. (26) and (27) is $\alpha=0$, A_1 and A_2 being such that the right-hand side of Eq. (27) vanishes. Since A_1 and A_2 are arbitrary, it is evident that for any value of R an oscillation can always take place whose frequency is given by

$$\omega=-n\Omega_0=neH_0/2mc. \tag{29}$$

This relation is more familiar in the form

$$\lambda H_0=21{,}400/n, \tag{30}$$

where λ = wave-length in cm and H_0 = applied magnetic field in gauss.

If $\alpha^2=2\Omega_0^2$, $\gamma=0$, E_θ and E_z are obtained by combining Eqs. (19) and (21). For $r=R$,

$$\begin{aligned} iE_\theta&=\frac{A_3nc}{2R^3\Omega_0}\left(1-\frac{2R^2\Omega_0^2}{c^2}\ln R\right)+\frac{B_3}{R}, \\ iE_z&=-A_1(\omega\alpha-2\Omega_0^2)J_n(DR). \end{aligned} \tag{31}$$

Since A_1, A_3 and B_3 are arbitrary, E_θ and E_z can be made zero for any value of R by setting $A_1=0$ and choosing values of A_3 and B_3 such that the expression for E_θ vanishes. Thus $\alpha^2=2\Omega_0^2$ is also a possible condition for oscillation. In terms of wave-length and magnetic field this expression becomes:

$$\begin{aligned} \lambda H_0=\;&51{,}700\ (n=+1)\\ &15{,}100\ (n=0)\\ &8880\ (n=-1)\\ &6280 \text{ or } 36{,}500\ (n=-2)\\ &4850 \text{ or } 13{,}500\ (n=-3),\text{ etc.} \end{aligned} \tag{32}$$

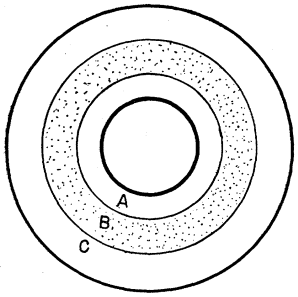

FIG. 1.

It is of interest to note that observed oscillation frequencies of the magnetron operating with space charge limited current in the so-called "electronic oscillation" mode in general satisfy relations of the form λH = constant. A particularly common value of this constant is about 15,000,[7] which would correspond to the $n=0$ wave (Eq. (32)). Other values of the constant which satisfy Eqs. (30) and (32) for low values of n have also been observed.

If, instead of filling the whole space inside the outer conductor, the space charge is confined within a cylinder of radius $R_2 < R_1$, a solution must be obtained for the region of free space between the space charge and the outer conductor of radius R_1. The field equations which must be solved are, of course, Maxwell's equations for free space:

$$\operatorname{curl} H = i\omega E/c$$
$$\operatorname{curl} E = -i\omega H/c.$$

The solutions of the equations in cylindrical coordinates are well known and can be applied directly. Since the region under consideration does not include the origin, the solution will involve Bessel functions of both the first and second kind. The free space solution must then satisfy $E_\theta = E_z = 0$ for $r = R_1$ and must match the space charge solution at $r = R_2$. A difficulty arises because R_1 is actually a function of θ and t during oscillation. A method for handling such a boundary condition will be found in a paper by W. C. Hahn.[1]

The case in which the inner conductor has a finite radius requires a still more complex treatment, since the N functions can no longer be rejected in any solution. Fig. 1 represents the most general case. Solutions for free space must be set up for the regions A and C and must satisfy the conditions $E_\theta = E_z = 0$ at both the inner and the outer conductors, and must in addition be fitted to the space charge solutions at the boundaries between A and B and between B and C.

5. Oscillations of Large Amplitude; a Special Case

The general solution for an oscillation whose amplitude can no longer be considered small compared with the d.c. quantities will be extremely complex, since it involves cross products which make contribution to the d.c. terms. A particular solution whose amplitude is not limited is obtainable, however. Eqs. (6) and (9) which are the only equations in which cross products of a.c. terms appear, can be restored to linear forms if all three a.c. components of velocity are always zero. The d.c. parts of the equations are then the same as before, and the d.c. solution can be carried over from the small signal case. A necessary condition for the disappearance of the velocities is found to be $\alpha = 0$. The situation is not the same as in the $\alpha = 0$ case described in the previous section, since the velocities are no longer indeterminate. Hence it is possible in this case to obtain a solution for the fields. It is:

$$\left.\begin{aligned}
E_r &= \frac{r\omega\gamma}{c}[A_4 J_n(Dr) + B_4 N_n(Dr)],\\
E_\theta &= 0,\\
iE_z &= -\frac{r\omega}{c}\left\{A_4\frac{\partial}{\partial r}[J_n(Dr)] + B_4\frac{\partial}{\partial r}[N_n(Dr)]\right\},\\
iH_r &= n\left\{A_4\frac{\partial}{\partial r}[J_n(Dr)] + B_4\frac{\partial}{\partial r}[N_n(Dr)]\right\},\\
H_\theta &= \frac{n^2}{r}[A_4 J_n(Dr) + B_4 N_n(Dr)],\\
H_z &= n\gamma[A_4 J_n(Dr) + B_4 N_n(Dr)],
\end{aligned}\right\} \quad (33)$$

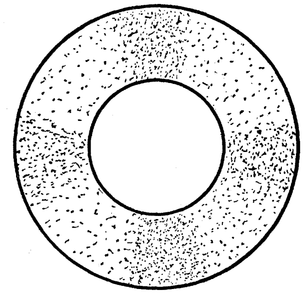

FIG. 2.

[7] E. C. S. Megaw, J. I.E.E. **72**, 330 (1933); C. W. Rice, Gen. Elec. Rev. **39**, 363 (1936).

where D^2 is defined as in Eq. (19) and A_4 and B_4 are arbitrary constants.

This solution represents a state of the rotating space charge in which "spokes" of increased density maintain a fixed position relative to a coordinate system rotating with angular velocity Ω_0. Fig. 2 is intended to represent this state of affairs. The a.c. at any point is then due only to changes in charge density and not at all to changes in electron velocity.

6. Discussion

We have assumed throughout that the boundaries are perfectly conducting and that electrons travel in circles around the axis so that no current flows from the inner to the outer conductor. Thus no d.c. power is being fed into the system, and we cannot expect to make any estimates of a.c. power output. This is also evident from the fact that currents and voltages calculated from our solutions are always $\pi/2$ out of phase. For the same reason it is impossible to estimate the resistance which a tube will present to an a.c. signal. This situation could be remedied by introducing a small d.c. radial velocity, but this increases the complexity of the a.c. equations so that they are quite unmanageable by our present technique.

It is a pleasure to acknowledge our indebtedness to Mr. W. C. Hahn for many stimulating discussions and suggestions.

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On the Problem of Degeneracy in Quantum Mechanics

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The problem of degeneracy in quantum mechanics is related to the existence of groups of contact transformations under which the Hamiltonian is invariant. The correspondence between transformations in classical and quantum theories is developed. The Fock-Bargmann treatment of the symmetry group of the hydrogenic atom comes under this theory. The symmetry group of the 2-dimensional Kepler problem is found to be the 3-dimensional rotation group; that of the n-dimensional isotropic oscillator is isomorphic to the unimodular unitary group in n dimensions. The 2-dimensional anisotropic oscillator has the same symmetry as the isotropic oscillator in classical mechanics, but the quantum-mechanical problem presents complications which leave its symmetry group in doubt.

Introduction

In the study of quantum-mechanical problems the question of the degeneracy of the energy levels plays an important role. It is often the case that this degeneracy is associated with simple symmetry properties of the Schrödinger equation, and considerable attention has been paid to the symmetry conditions associated with the rotation-reflection group and the group of permutations of identical particles.[1]

On the other hand, certain problems possess symmetry properties of more subtle types. It was shown by Fock[2] some years ago that the Schrödinger equation for the hydrogen atom actually has the symmetry of the 4-dimensional rotation group for the bound states, and the symmetry of the Lorentz group for the positive energy states. The degeneracy of the system with respect to the quantum number l is due to the invariance under these wider groups, of which the 3-dimensional rotations form a sub-group. This interpretation was verified and extended by Bargmann,[3] who showed the formal relationship of the Lenz-Pauli integrals with these groups. In spite of the satisfactory way in which these considerations clarified the hydrogen atom problem, they seem peculiar to this case, and it is not clear at once whether they can be extended to other examples.

[1] E. g., E. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (Vieweg & Sohn, 1931).
[2] V. Fock, Zeits. f. Physik **98**, 145 (1935).
[3] V. Bargmann, Zeits. f. Physik **99**, 578 (1936).