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Volumes of Solids Swept Tangentially Around Cylinders

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Abstract. In earlier work ([1]-[5]) the authors used the method of sweeping tangents to calculate area and arclength related to certain planar regions. This paper extends the method to determine volumes of solids. Specifically, take a region S in the upper half of the xy plane and allow the plane to sweep tangentially around a general cylinder with the x axis lying on the cylinder. The solid swept by S is called a *solid tangent sweep*. Its *solid tangent cluster* is the solid swept by S when the cylinder shrinks to the x axis. Theorem 1: *The volume of the solid tangent sweep does not depend on the profile of the cylinder, so it is equal to the volume of the solid tangent cluster.* The proof uses Mamikon's sweeping-tangent theorem: *The area of a tangent sweep to a plane curve is equal to the area of its tangent cluster*, together with a classical slicing principle: *Two solids have equal volumes if their horizontal cross sections taken at any height have equal areas*. Interesting families of tangentially swept solids of equal volume are constructed by varying the cylinder. For most families in this paper the solid tangent cluster is a classical solid of revolution whose volume is equal to that of each member of the family. We treat forty different examples including familiar solids such as pseudosphere, ellipsoid, paraboloid, hyperboloid, persoids, catenoid, and cardioid and strophoid of revolution, all of whose volumes are obtained with the extended method of sweeping tangents. Part II treats sweeping around more general surfaces.

1. FAMILIES OF BRACELETS OF EQUAL VOLUME

In Figure 1a, a circular cylindrical hole is drilled through the center of a sphere, leaving a solid we call a *bracelet*. Figure 1b shows bracelets obtained by drilling cylindrical holes *of a given height* through spheres of different radii. A classical calculus problem asks to show that all these bracelets have equal volume, which is that of the limiting sphere obtained when the radius of the hole shrinks to zero.

It comes as a surprise to learn that the volume of each bracelet depends only on the height of the cylindrical hole and not on its radius or the radius of the drilled sphere! This phenomenon can be explained (and generalized) without resorting to calculus by referring to Figure 2.

In Figure 2a, a typical bracelet and the limiting sphere are cut by a horizontal plane parallel to the base of the cylinder. The cross section of the bracelet is a circular annulus swept by a segment of constant length, tangent to the cutting cylinder. The corresponding cross section of the limiting sphere is a circular disk whose radius is easily shown (see Figure 3) to be the length of the tangent segment to the annulus. Thus, each circular disk is a tangent cluster of the annulus which, by Mamikon's sweeping-tangent theorem, has the same area as the annulus. (See

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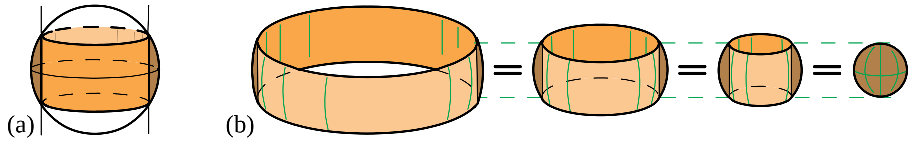


Figure 1. (a) Bracelet formed by drilling a cylindrical hole through a sphere. (b) The volume of each bracelet is the volume of a sphere whose diameter is the height of the hole.

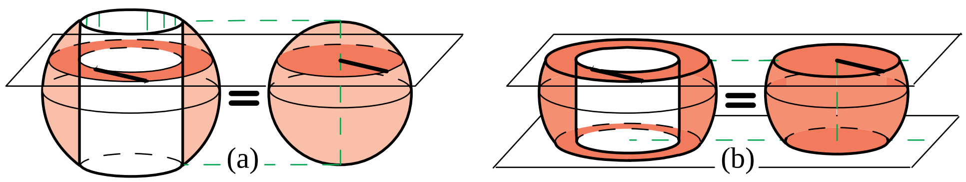


Figure 2. (a) Corresponding horizontal cross sections of bracelet and sphere have equal areas. (b) Solid slices cut by two parallel planes have equal volumes.

[2; Ch. 1], or [3].) Consequently, if the bracelet and sphere are sliced by two parallel planes as in Figure 2b, the slices have equal volumes because of the following *slicing principle*, also known as Cavalieri's principle:

Slicing principle. *Two solids have equal volumes if their horizontal cross sections taken at any height have equal areas.*

Thus, the equal volume property holds not only for all bracelets in Figure 1b, which are symmetric about the equatorial plane, but also *for any family of horizontal slices of given thickness.*

Generating bracelets by sweeping a plane region tangentially around a cylinder. Another way to generate the bracelets in Figure 1 is depicted in Figure 3a. A vertical section of the sphere cut by a plane tangent to the cylindrical hole is a circular disk whose diameter is the height of the hole. When half this disk, shown with horizontal chords, is rotated tangentially around the cylinder it sweeps out a bracelet as in Figure 3a. The tangent segment to the annulus in Figure 2a is a chord of such a semicircle, so the circular disk in Figure 2a is the planar tangent cluster of the corresponding annulus, hence *the annulus and disk have equal areas*. By the slicing principle, the bracelet and sphere in Figure 3a have equal volumes, as do arbitrary corresponding slices in 3b. We refer to each swept solid as a *solid tangent sweep* and to the corresponding portion of the limiting sphere as its *solid tangent cluster*.

Ellipsoidal bracelets. Figure 4 shows ellipsoidal bracelets swept by a given semielliptical disk rotating tangentially around circular cylinders of equal height but of different radii. The same bracelets can also be produced by drilling circular holes of given height through similar ellipsoids of revolution. The reasoning used above for spherical bracelets shows that each ellipsoiodal bracelet has the same volume as the limiting case, an ellipsoid of revolution. Moreover, *horizontal slices of these bracelets of given thickness also have equal volume*.

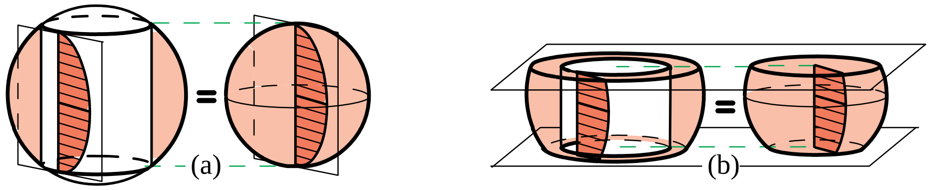


Figure 3. (a) Vertical section of sphere cut by a plane tangent to the cylindrical hole is a circular disk whose diameter is the height of the hole. (b) Arbitrary horizontal slice of (a).

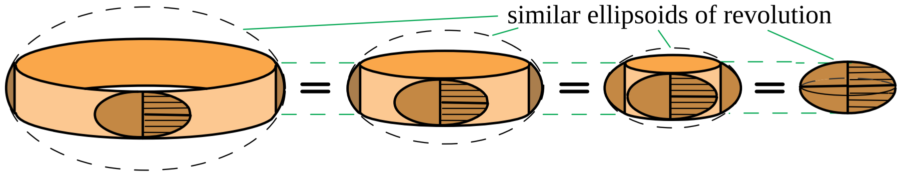


Figure 4. Ellipsoidal bracelets of equal height have the same volume as the limiting ellipsoid.

Paraboloidal bracelets. In Figure 5a a paraboloid of revolution is cut by a vertical plane, and half the parabolic cross section of height H is rotated tangentially around a circular cylinder of altitude H to sweep out a paraboloidal bracelet as indicated. The volume of this bracelet is equal to that of its solid tangent cluster, a paraboloid of revolution of altitude H. Figure 5b shows a family of paraboloidal bracelets, all of height H, cut from a given paraboloid of revolution by parallel equidistant planes. The bracelets have different radii, but each has the volume of the leftmost paraboloid of revolution of altitude H because it is easily shown that all the sweeping parabolic segments are congruent.

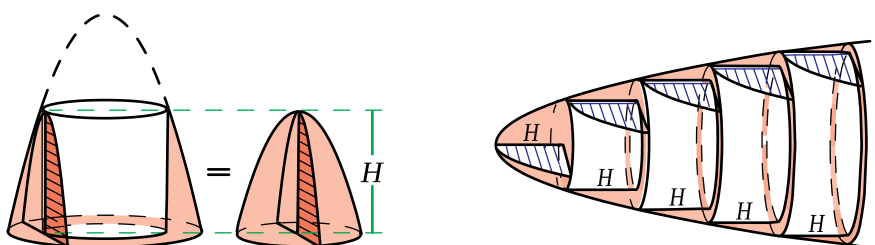
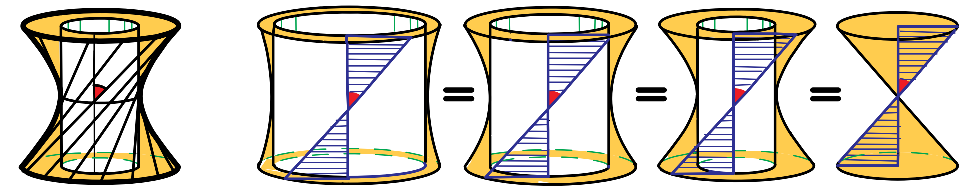


Figure 5. (a) Paraboloidal bracelet has the volume of the solid tangent cluster. (b) Family of paraboloidal bracelets of equal height and equal volume.

Hyperboloidal bracelets. Figure 6 shows a new family of bracelets, formed by drilling a cylindrical hole of given height through the center of a solid *hyperboloid of one sheet* (twisted cylinder). The generator of each hyperboloid makes the same angle with the vertical generator of the cylinder. The cylinder is tangent to the hyperboloid at its smallest circular cross section. The bracelets in Figure 6 have equal



(a) twisted cylinder (b) similar hyperboloids, swept by double triangle

Figure 6. (a) Bracelet formed by drilling a solid hyperboloid of one sheet. (b) The volume of each bracelet equals the volume of the limiting cone of the same altitude.

volume, that of the limiting cone. The same bracelets can be obtained by tangential sweeping. In Figure 6b, a vertical section of the hyperboloid tangent to that cylinder is a symmetric double triangle, shown shaded. When this double triangle is rotated tangentially around the cylinder, the solid tangent sweep is a bracelet as in Figure 6b, and the limiting cone is its solid tangent cluster.

Figure 7 shows other hyperboloidal bracelets produced by tangential sweeping, but the type of bracelet depends on the relation between the radius r of the cylindrical hole and the length b of semitransverse axis of the hyperbola. In Figure 7a, $r > b$, and the outer surface of the bracelet is a hyperboloid of one sheet somewhat

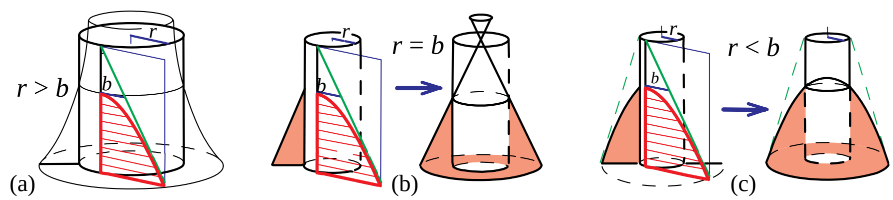


Figure 7. (a) Hyperboloidal bracelet with $r > b$. (b) Hyperboloidal bracelet with $r = b$ is a cone. (c) Hyperboloidal bracelet with $r < b$.

like those in Figure 6, except that the drilling cylinder is not tangent to the hyperboloid as in Figure 6, but intersects it. In Figure 7b, $r = b$, and the outer surface is that of a cone (a degenerate hyperboloid). In Figure 7c, $r < b$ and the outer surface is a *hyperboloid of two sheets* (only one sheet is shown). All hyperbolas in Figure 7 have the same asymptotes.

Figure 8 shows families of hyperboloidal bracelets of equal volume. Those in (a) are of one sheet; those in (b) are of two sheets (with only one sheet shown).

General oval bracelets. Figure 9a shows a bracelet swept by a semicircular disk moving tangentially around a general oval cylinder. Figure 9b shows a typical horizontal cross section of the bracelet, an oval ring swept by tangent segments of constant length. Such a ring is traced for example by a moving bicycle [1]. As in the foregoing examples, the volume of each bracelet is the volume of the limiting sphere obtained when the oval cylinder shrinks to a point. In Figure 9c, a double triangle moves tangentially around the oval cylinder to sweep out a bracelet whose outer surface is a ruled surface resembling the hyperboloid of one sheet in Figure

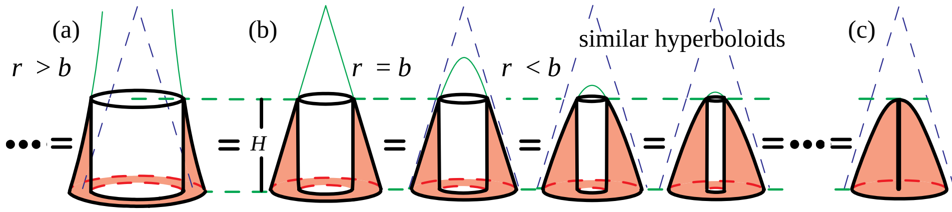


Figure 8. Hyperboloidal bracelets of one sheet in (a) and of two sheets in (b), all having equal height and equal volume, that of the limiting case in (c).

6b. A typical cross section is an oval ring, as in Figure 9b. The volume of the bracelet is that of the limiting cone as in Figure 6b.

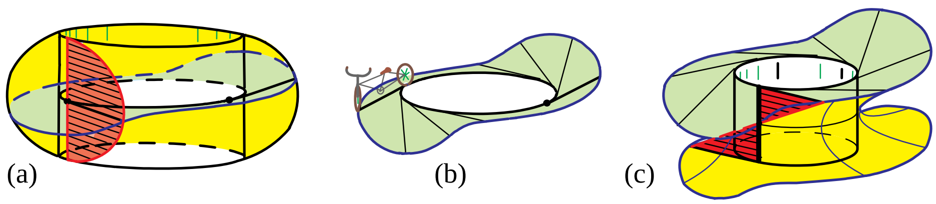


Figure 9. (a) Bracelet formed by semicircular disk swept tangentially around an oval cylinder. The volume of the solid tangent sweep is the same as that of its solid tangent cluster, a sphere. (b) Typical horizontal cross section of the bracelet in (a). (c) Bracelet formed by right triangle swept tangentially around an oval cylinder. A typical horizontal cross section is like that in (b).

2. TANGENTIAL SWEEPING AROUND A GENERAL CYLINDER

The tangentially swept solids treated above can be generalized as shown in Figure 10a. Start with a plane region S between two graphs in the same half-plane. To be specific, let S consist of all points (x, y) satisfying the inequalities

$$f(x) \leq y \leq g(x), \quad a \leq x \leq b$$

where f and g are nonnegative piecewise monotonic functions related by the inequality $0 \leq f(x) \leq g(x)$ for all x in an interval $[a, b]$. In Figure 10a, the x axis is oriented vertically, and S is in the upper half-plane having the x axis as one edge. If we rotate S around the x axis we obtain a solid of revolution swept by region S as indicated in the right portion of Figure 10a.

More generally, place the x axis along the generator of a general cylinder (not necessarily circular or closed) and, keeping the upper half-plane tangent to the cylinder, move it along the cylinder as suggested in Figure 10a. Then S generates a tangentially swept solid we call a *solid tangent sweep*. The corresponding *solid tangent cluster* is that obtained by rotating S around the x axis. When the smaller function f defining S is identically zero, the swept solid is called a *bracelet*. Clearly, by Figure 10b, any swept solid can be produced by removing one bracelet from another. We now have:

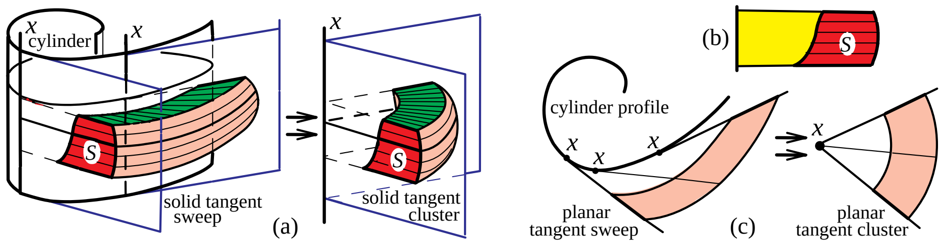


Figure 10. (a) The volume of the solid tangent sweep is the same as that of its solid tangent cluster. (b) Region S lies between two ordinate sets. (c) Top view of a typical cross section.

Theorem 1. *The volume of the solid tangent sweep does not depend on the profile of the cylinder, so it is equal to the volume of the solid tangent cluster, a portion of a solid of revolution.*

Figure 10 provides a geometric proof. A typical cross section cut by a plane perpendicular to the x axis is shown in Figure 10c. The area of the shaded band outside the cylinder is the difference of areas of two tangent sweeps of the profile of the cylinder. The area of the portion of the adjacent circular annulus swept about the x axis is the difference in areas of the corresponding tangent clusters. Therefore, by Mamikon's theorem, the shaded band and annulus in Figure 10c have equal areas. Apply the slicing principle to the solids in Figure 10a to obtain Theorem 1. □

In Section 1 we treated families of bracelets with a common property: all members of the family have the same height and the same volume, because when a given family is cut by a horizontal plane, all planar sections have equal areas.

Consequently, by simply slicing any such family by two horizontal planes at a

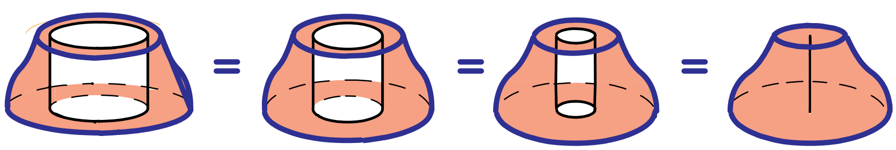


Figure 11. A family of bracelets obtained by parallel slicing of another family of bracelets. The slices also have equal height and equal volume.

given distance apart we obtain infinitely many new families with the same property because corresponding horizontal slices have equal volume. In particular, parallel slicing of families that have a horizontal plane of symmetry leads to many new families of solids with equal height and equal volume that have no horizontal plane of symmetry, as depicted in Figure 11. This greatly increases the range of applicability of our results.

3. APPLICATIONS TO TOROIDAL SOLIDS

Persoids of revolution. A torus is the surface of revolution generated by rotating a circle about an axis in its plane. The curve of intersection of a torus and a plane parallel to the axis of rotation is called a *curve of Perseus*, examples of which are shown in Figure 12. Classical examples include ovals of Cassini and leminscates of Booth and Bernoulli. Each such curve of Perseus has an axis of symmetry parallel to the axis of rotation. When the *persoidal region*, bounded by a curve of Perseus, is rotated about this axis of symmetry it generates a solid that we call a *persoid of revolution*.

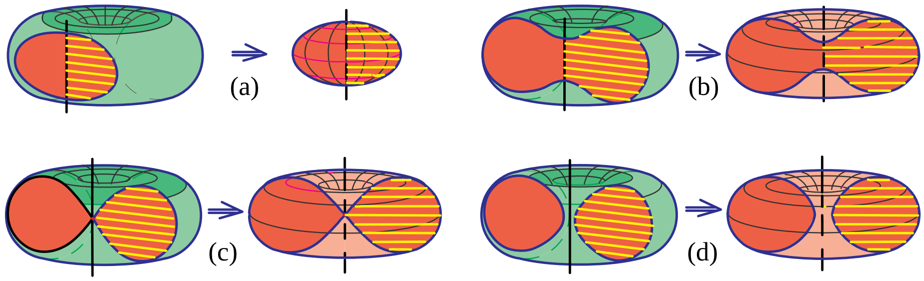


Figure 12. Each persoidal region (left) generates a solid persoid of revolution (right).

How can we calculate the volume of a persoid of revolution? We use the example in Figure 12a to illustrate a method that applies to all persoids of revolution.

When half the persoidal region in Figure 12a is swept tangentially around a circular cylinder it generates a solid tangent sweep which, by Theorem 1, has the same volume as its solid tangent cluster, in this case the persoid of revolution. To calculate this volume, we observe that the same solid can be swept by a circular segment *normal* to the cylinder as indicated in Figure 13a and in Figure 14a.

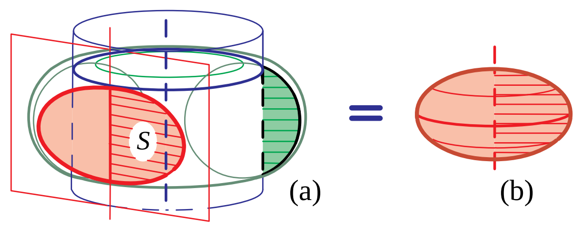


Figure 13. A tangentially swept solid with the same volume as the persoid of revolution. The same solid is swept by a circular segment normal to the cylinder.

Figure 14b shows a typical horizontal cross section of the solid, a circular annulus swept by tangential segments and by normal segments. By Pappus' theorem on solids of revolution, the volume of the solid is equal to Ad, where A is the area of the circular segment and d is the distance through which the centroid of the segment moves in sweeping out the solid. Both A and d can be determined by elementary geometry, thus giving an elementary calculation of the volume of the solid, hence also of the volume of the persoid of revolution. Moreover, according to Theorem 1, *all solids tangentially swept by a given persoidal region around a* *cylinder of any shape have the same volume as the persoid of revolution.* Only one of these solids is a solid torus.

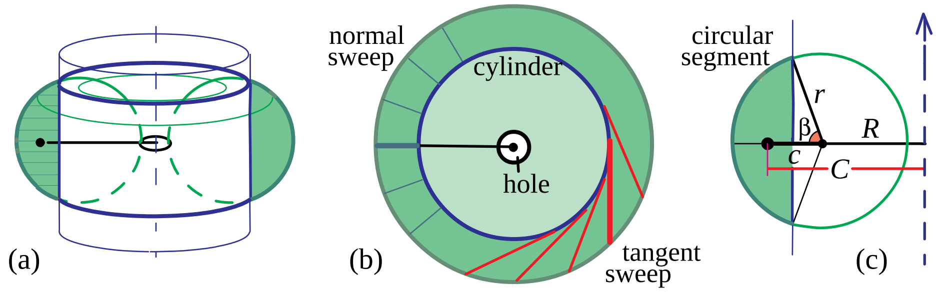


Figure 14. Calculating the volume of a swept solid using Pappus' theorem.

Volumes of classical persoids generated by ovals of Cassini and the Bernoulli lemniscate can be calculated by finding equations of the Perseus curves and using integral calculus. The foregoing discussion provides an elementary derivation that does not require equations or integration. In particular, the curve of Perseus in Figure 12c, known as a Booth lemniscate (with a cusp), generates a persoid of revolution whose volume is equal to that of the entire solid torus, $2\pi^2 r^2 R$. Here r is the radius of the circle that generates the torus as its center moves around a circle of radius R. Cassinian ovals can be defined as sections cut by a plane at a distance r from the axis of the torus. Their shapes are represented by the examples in Figure 12. When $R > 2r$, the oval consists of two symmetric disconnected pieces as in Figure 12d, and again the persoid of revolution has volume equal to that of the torus. When $R = 2r$, the Cassinian oval and the Booth lemniscate in Figure 12c become a Bernoulli lemniscate, and the persoid of revolution has volume $4\pi^2 r^3$. We summarize as follows:

Proposition. *When $R \geq 2r$ the persoid of revolution has volume $2\pi^2 r^2 R$, which is that of the solid torus.*

When $R < 2r$, as in Figures 12a and b, the persoidal region consists of one piece, and the volume V of the persoid of revolution is given by Pappus' theorem as

$$V = 2\pi CA, \tag{1}$$

where A is the area of the circular segment shaded in Figure 14c, and C is the centroidal distance of the segment from the axis of rotation. We show now that this volume is given by the explicit formula

$$V = \frac{4}{3}\pi(r\sin\beta)^3 + \pi R r^2(2\beta - \sin 2\beta). \tag{2}$$

Here r is the radius of the circle that generates the torus as its center moves around a circle of radius R, and β is half the angle that subtends the circular segment of radius r. In our geometric proof we assume that $0 \leq \beta \leq \pi/2$, but formula (2) is valid for all β. The area A of the segment is

$$A = r^2(\beta - \sin\beta\cos\beta). \tag{3}$$

Figure 14c shows that $C = c + R$, where c is the centroidal distance of the segment from the center of the circle of radius r. Hence $CA = cA + RA$. But $2\pi cA = \frac{4}{3}\pi(r\sin\beta)^3$, the volume of a spherical bracelet of height $r\sin\beta$, so (1) and (3) give (2).

For a Cassinian oval as depicted in Figures 12b and c, we have $R + r\cos\beta = r$, which gives $\cos\beta = 1 - R/r$. This determines the value of β to used in (2).

Hierarchy of toroidal solids. We can construct a hierarchy of toroidal solids as follows. Start with a plane oval region and rotate it around an axis at a positive distance from the oval to generate a toroidal solid, which we call the *initial toroid*. Cut this toroid through its hole by planes parallel to the axis at varying distances from it. Each cut produces two new oval sections with an axis of symmetry between them. Rotation of one them about the axis of symmetry generates a new toroidal solid, and the family of such toroidal solids obtained by all possible cuts we call toroids of the 1st generation. By analogy to the persoid of revolution treated in Figure 12d, each solid in this generation has the same volume as the initial toroidal solid. This extends the result for initial circular toroids described in the foregoing Proposition.

Now we repeat the process, taking as initial toroid any member of the 1st generation. For each such member we can produce a new family of toroids of the 2nd generation. Each member of the 2nd generation can also be taken as initial toroid to produce a 3rd generation, and so on. Remarkably, *all toroids so produced have the same volume as the initial toroid* we started with. It seems unbelievable that so many families exist sharing the same volume property as the classical family of drilled bracelets in Figure 1.

The next section describes another principle that aids in calculating volumes of solid clusters (hence of solid sweeps) without using calculus.

4. VOLUME OF SOLID CLUSTERS VIA CONICAL SHELLS

Conical shell principle. Figure 15a shows a triangle with its base on a horizontal axis. The area centroid of the triangle is at a distance one-third its altitude from the base, which we denote by c. When the same triangle is translated so that the upper vertex is on the axis, its centroid is at distance $2c$ from the axis.

By rotating each triangular configuration about the horizontal axis we form two solids of revolution, called *conical shells*, shown in Figure 15b. By a theorem of

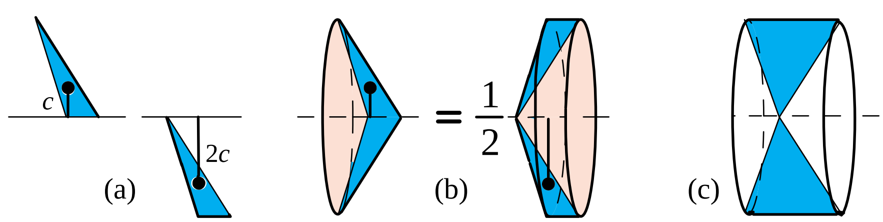


Figure 15. A conical shell principle for volumes of solids of revolution.

Pappus, the volume of each shell is the area of the triangle times the length of the path of the centroid of the triangle. Apply this to the solids in Figure 15b to obtain:

Conical shell principle. *The solid on the right of Figure 15b has twice the volume of that on the left.*

This principle implies that the punctured cylinder in Figure 15c has volume 2/3 that of the cylinder. It also leads to a basic theorem (Theorem 2 below) concerning tangent sweeps and tangent clusters that we turn to next.

Figure 16a shows the graph of a monotonic function we use as a tangency curve. Tangent segments (not necessarily of the same length) from this curve to the horizontal axis generate the tangent sweep of this curve. Figure 16a also shows the tangent cluster obtained by translating all the tangent segments so the points of tangency are brought to a common point P on the horizontal axis. Consider the region between any two tangent segments in the tangent sweep, and the corresponding portion of the tangent cluster, both shown shaded in Figure 16a. We know from Mamikon's sweeping-tangent theorem that these two shaded regions have equal areas.

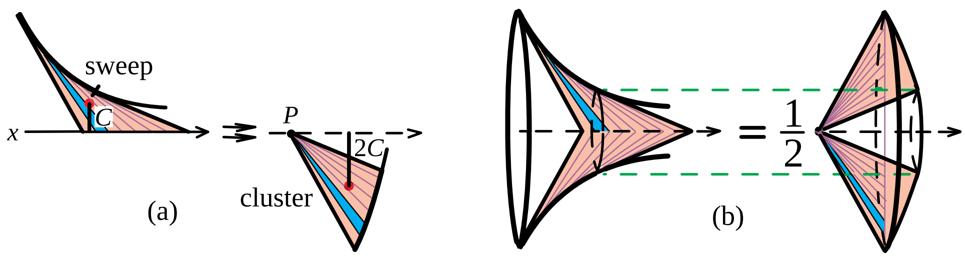


Figure 16. Geometric meaning of Theorem 2.

Now we obtain a simple relation connecting their area centroids and also the volumes of the two solids they generate by rotation about the axis. Decompose each region into tiny triangles akin to those shown in Figure 15a. We deduce that if C is the centroidal distance of the tangent sweep from the horizontal axis, then the centroidal distance of the tangent cluster from the same axis is $2C$, as indicated in Figure 16a. This proves part (a) of Theorem 2. Part (a), together with Pappus' theorem, gives part (b) of Theorem 2.

Theorem 2. (a) *If C is the centroidal distance of the tangent sweep from a horizontal axis, then the centroidal distance of the tangent cluster from the same axis is $2C$.*

(b) *The volume of the solid obtained by rotating the tangent sweep about the horizontal axis is one-half the volume of the solid obtained by rotating the corresponding tangent cluster about the same axis.*

Now we apply Theorem 2 to several examples of solids of revolution.

Tractrix and pseudosphere. When the tangent sweep of the entire tractrix shown in Figure 17a is rotated about the x axis it generates a solid of revolution which is half a pseudosphere. If the cusp of the tractrix is at height H above its asymptote, the

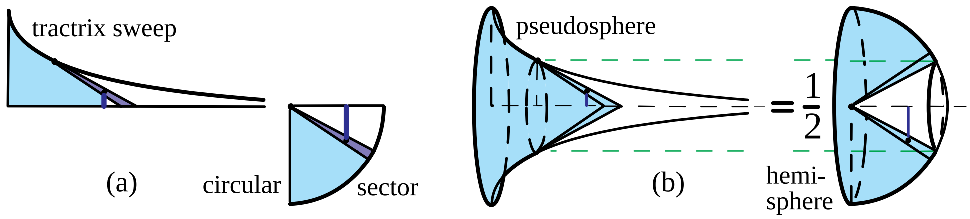


Figure 17. Determining the volume of a portion of a pseudosphere without calculus.

volume of half the pseudosphere is $\frac{2}{3}\pi H^3$, half the volume of a sphere of radius H, a result known from integral calculus. We shall obtain the same result and more (without calculus) as a direct application of Theorem 2b. Because all tangent segments to the tractrix cut off by the x axis have constant length, the tangent cluster shown in Figure 17a is a circular sector, and each small triangle contributing to the tangent sweep has a corresponding translated triangle in the tangent cluster. Therefore, Theorem 2b tells us that *the volume of any portion of the half pseudosphere is half that of the corresponding portion of the hemisphere,* as indicated in Figure 17b.

Exponential. Next we rotate the tangent sweep of an exponential function, shaded in Figure 18a, around the x axis to form a solid of revolution shown in Figure 18b. To determine its volume, refer to Figure 18a which shows the corresponding tangent cluster, a right triangle whose base is the constant length of the subtangent

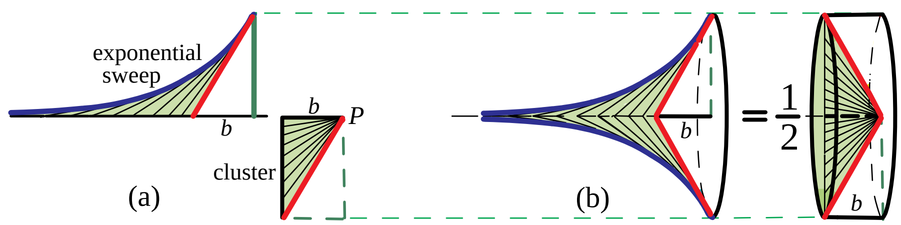


Figure 18. The volume of the solid generated by rotating an exponential ordinate set is half that of a cylinder whose altitude is the length of the constant subtangent.

to the tangency curve indicated as b in Figure 18a. (See [2; p. 16] or [3]). When this tangent sweep is rotated about the x axis it generates a solid of revolution whose volume, according to Theorem 2, is half that of the solid cluster of revolution. Consequently, *the volume of the solid obtained by rotating the ordinate set of the exponential (which includes the unshaded right triangle) is equal to half the volume of the circular cylinder whose altitude is the length b of the constant subtangent.*

Generalized pursuit curve. Figure 19a shows a tangency curve with tangent segments cut off by a horizontal axis. At each point, a tangent segment of length t cuts off a subtangent of length b. For a tractrix, t is constant, and for an exponential, b

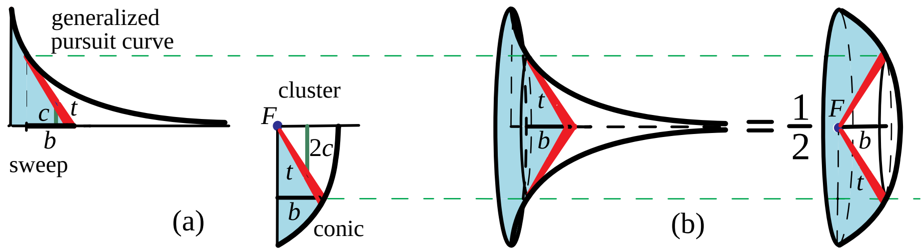


Figure 19. (a) Tangent cluster of a generalized pursuit curve is bounded by a portion of a conic section. (b) Solid obtained by rotating the generalized pursuit curve has half the volume of the solid of revolution of the conic.

is constant. If a convex combination of t and b is constant, say $\mu t + \nu b = C$ for some choice of nonegative μ and ν, with $\mu + \nu = 1$, the tangency curve is called a generalized pursuit curve. We know (see [2; p. 348], or [3]) that the tangent cluster of a generalized pursuit curve is bounded by a conic section with eccentricity ν/μ and a focus at the common point F to which each tangent segment is translated, as shown in Figure 19a. For example, when $\mu = \nu$ the pursuit curve is the classical dog-fox pursuit curve. A fox runs along the horizontal line with constant speed and is chased by a dog running at the same speed. In this case, the tangent cluster is bounded by part of a parabola.

When the general pursuit curve is rotated about the horizontal axis, its tangent sweep generates a solid of revolution as depicted in Figure 19b. By Theorem 2, *the volume of this solid is half that of the solid generated by rotating the tangent cluster.*

Paraboloidal segment. Figure 20a shows the parabola $y = x^2$ with the tangent sweep consisting of tangent segments cut off by the y axis. A corresponding tangent cluster is shaded in Figure 20b, whose curved boundary is easily shown to be the vertically dilated parabola $y = 2x^2$. Now we form two solids by rotating the tangent sweep and tangent cluster about the y axis. According to Theorem 2,

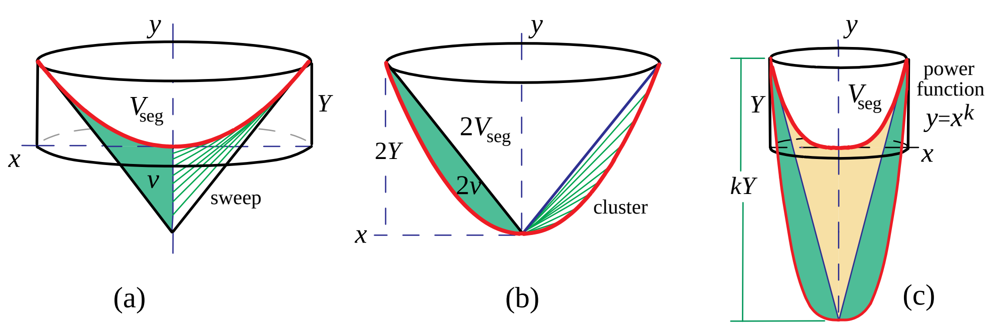


Figure 20. Volume of a paraboloidal segment.

the volume v of the solid obtained by rotating the sweep is one-half the volume

V obtained by rotating the cluster. This enables us to determine the volume V_{seg} of the paraboloidal segment obtained by rotating the parabola $y = x^2$ about the y axis. The volume of the paraboloidal segment in Figure 20b is $2V_{\text{seg}}$. Both Figures 20a and 20b show the same cone of volume V_{cone}. From Figure 20a we see that $V_{\text{seg}} = V_{\text{cone}} - v$, and from Figure 20b we find $2V_{\text{seg}} - V_{\text{cone}} = 2v$. Eliminating v we find $4V_{\text{seg}} = 3V_{\text{cone}}$. But $3V_{\text{cone}}$ is twice the volume of the circumscribing cylinder shown in Figure 20a. Consequently, we find Archimedes' result: *The volume V_{seg} of a paraboloidal segment is one-half that of its circumscribing cylinder.* In other words, the surface of revolution obtained by rotating the parabola around the y axis divides its circumscribing cylinder into two pieces of equal volume. Theorem 2 also yields a corresponding result for the power function $y = x^k$ in Figure 20c. The surface of revolution about the y axis divides the circumscribing cylinder into two solids whose volumes are in the ratio $k : 2$.

5. MODIFIED TREATMENT FOR VOLUMES OF SOLID CLUSTERS

The next theorem modifies the conical shell principle for treating volumes of solids obtained by rotating the ordinate set of a monotonic function about the x axis.

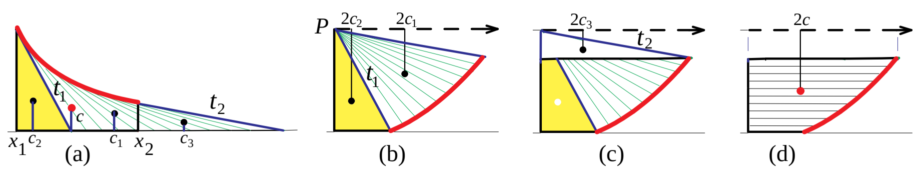


Figure 21. An abscissa set in (d) formed from the ordinate set in (a). They have equal areas and centroidal distances in the ratio 2:1.

Figure 21a shows the graph of a monotonic function and part of its tangent sweep between the graph and the x axis determined by two tangential segments t_1 and t_2 as shown. We are interested in the ordinate set above the interval $[x_1, x_2]$. This ordinate set can be formed from the tangent sweep by adding the right triangle with hypotenuse t_1 and subtracting the right triangle with hypotenuse t_2. The tangency points of t_1 and t_2 are brought to the same point P on the tangent cluster. From the corresponding tangent cluster we form its *abscissa set* shown in Figure 21d in two steps: add right triangle with hypotenuse t_1 as in Figure 21b, and subtract right triangle with hypotenuse t_2 as in Figure 21c. The resulting abscissa set in Figure 21d has the same area as the ordinate set in Figure 21a above $[x_1, x_2]$. The 2:1 relation of centroidal distances in Figure 15a yields the same relation for the components in Figures 21 a, b, and c. Now rotate the ordinate set about the x axis, and rotate the abscissa set about the polar axis p (the axis through P parallel to the x axis) to produce the two solids in Figure 22a. Argue as in Theorem 2 to get:

Theorem 3. (a) *The area of the ordinate set of any monotonic graph is equal to the area of the abscissa set of the corresponding tangent cluster.*

(b) *If C is the centroidal distance of the ordinate set from the horizontal axis, then the centroidal distance of the abscissa set of the corresponding tangent cluster from the polar axis is $2C$.*

(c) *The volume of the solid obtained by rotating the ordinate set about the horizontal axis is one-half the volume of the solid obtained by rotating the abscissa set of the corresponding tangent cluster about the polar axis.*

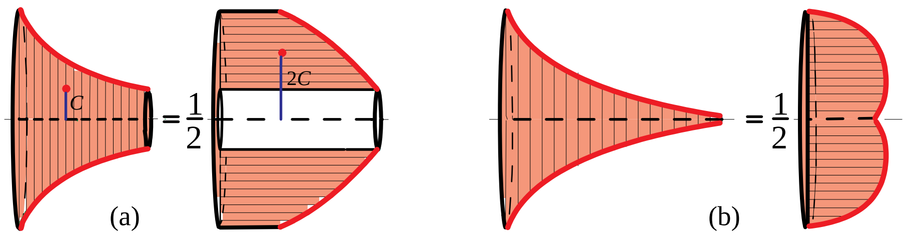


Figure 22. (a) Theorem 3a, b and c. (b) Special case where graph touches the x axis.

The geometric meaning of Theorem 3 is shown in Figure 22a. Figure 22b illustrates the special case where the graph touches the x axis.

Cut pseudosphere. When Theorem 3 is applied to a cut portion of a pseudosphere and its mirror image obtained from Figure 17b, it reveals that *the volume of that portion of a pseudosphere is half the volume of a spherical bracelet,* as indicated in Figure 23.

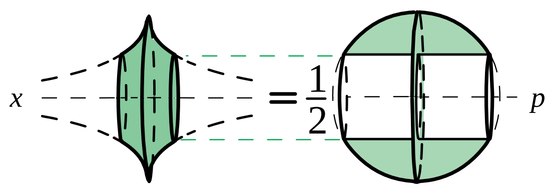


Figure 23. Cut pseudosphere has half the volume of a spherical bracelet.

Paraboloidal solid funnel. The shaded region in Figure 24a is a parabolic segment between the curve $y = x^2$ and the interval $[0, X]$. Figure 24b shows a tangent sweep of the parabola and a corresponding tangent cluster, whose curved boundary is part of the parabola $y = (2x)^2$. This figure was used in [2; p. 476] and in [3] to calculate the area of the parabolic segment in Figure 24a by Mamikon's sweeping tangent method. Now we use it to determine the volume v of the paraboloidal funnel in Figure 24c which is obtained by rotating the ordinate set in Figure 24a about the x axis. The upper shaded region in Figure 24b is the abscissa set of the cluster. By Theorem 3, v is one-half the volume V of the solid obtained by rotating the upper shaded region about the x axis. This implies that v is one-fourth the volume of the solid obtained by rotating the unshaded region in Figure 24a around the x axis. Hence the curved surface of the funnel divides its circumscribing cylinder into two pieces whose volumes are in the ratio $4 : 1$. Therefore *the volume*

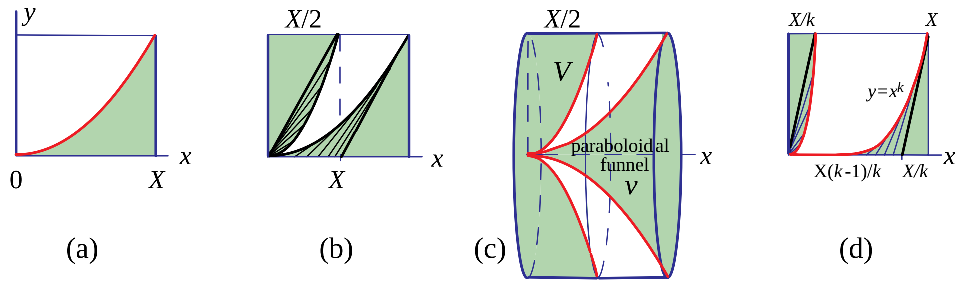


Figure 24. Volume of a paraboloidal solid funnel.

of the paraboloidal funnel is 1/5 *that of its circumscribing cylinder*. In the same manner, Theorem 3 shows that if we rotate the curve $y = x^k$ in Figure 24d about the x axis, the surface of revolution divides the circumscribing cylinder into two pieces whose volumes are in the ratio $2k : 1$

Rotated cycloidal cap. Figure 25 shows one arch of a cycloid generated by a point on the boundary of a rolling circular disk, together with a circumscribing rectangle. The disk rolls along the base of this rectangle, and a tangent sweep is the "cap" formed by drawing tangent segments from the cycloid to the upper edge of the

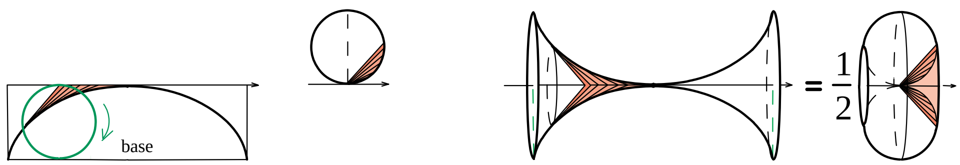


Figure 25. Solid of revolution swept by cycloidal cap.

rectangle as indicated. It is known that the area of the cap is equal to that of the disk because the disk is the tangent cluster of this tangent sweep (see [2; p. 35], or [4]). By Theorem 3, *the horn-shaped solid obtained by rotating the cycloidal cap about the upper edge has volume equal to half that of the toroidal-type solid obtained by rotating the disk about the same edge.* If the disk has radius a this volume is $\pi^2 a^3$.

A family generalizing the cycloid and tractrix. Figure 26 shows a cycloid (flipped over) and a tractrix, with tangent clusters to each obtained in similar fashion. For the cycloid the tangent cluster segments emanate from a common point P at one

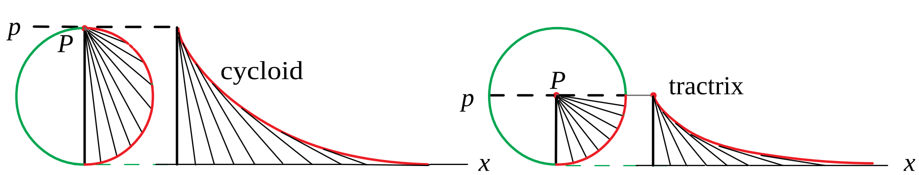


Figure 26. Common method for generating cycloid and tractrix.

end of the vertical diameter of a circle; for the tractrix they emanate from the center P of a circle.

Figure 27 shows how to produce a family of curves generalizing the cycloid and tractrix by allowing the tangent segments of the cluster to emanate from a common point P anywhere on the diameter. We consider the symmetric solids of revolution

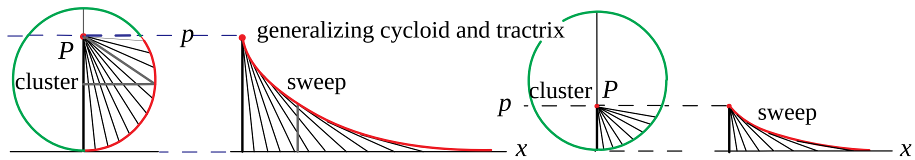


Figure 27. Family that includes cycloid and tractrix.

swept by rotating about the x axis the ordinate sets of these curves together with their mirror images through the y axis. Figure 28 shows how Theorem 3 determines the volume of a symmetrically cut portion of such solids. *Each volume is half that of a toroidal bracelet*, the corresponding rotated abscissa set of the cluster, whose volume can be easily found by Pappus' rule as was done earlier for persoids of revolution.

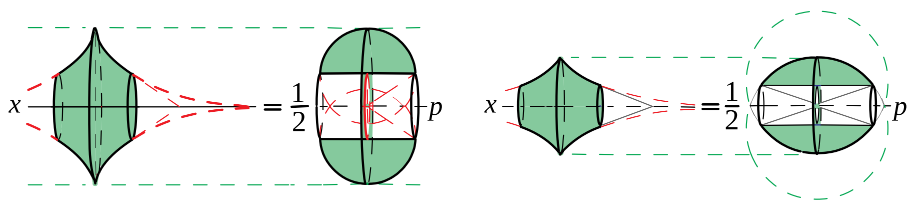


Figure 28. Volume relations for solids obtained by rotating the ordinate sets in Figure 27.

6. VOLUMES SWEPT BY COMPLEMENTARY REGIONS

According to Pappus, the solid of revolution obtained by rotating a plane region of area A around an axis has volume $V = 2\pi cA$, where c is the centroidal distance of the region from the axis of rotation. Therefore, for a region of given area A, determining V is equivalent to determining centroidal distance c. We exploit this fact to derive a surprising and useful comparison lemma for volumes swept by two complementary regions whose union is a rectangle.

Figure 29a shows a rectangle divided into two complementary regions of areas A_1 and A_2. In Figure 29b, the region of area A_1 has been rotated about the lower edge l_1 of the rectangle to generate a solid of revolution of volume V_1. In Figure 29c, the complementary region of area A_2 has been rotated about the upper edge l_2 of the rectangle to generate another solid of revolution of volume V_2. Both solids are circumscribed by a cylinder of volume $V = \pi RH$ obtained by rotating the rectangle of area R and height H around either horizontal edge. Let $a_1 = A_1/R$

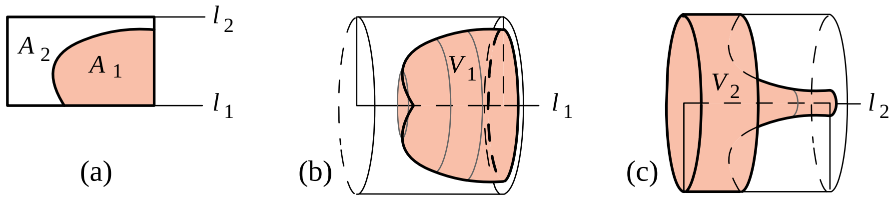


Figure 29. Solids obtained by rotating complementary parts of a rectangle around its lower and upper edges.

and $a_2 = A_2/R$ denote the fractional areas relative to the rectangle, so that $a_1 + a_2 = 1$. Similarly, let $v_1 = V_1/V$ and $v_2 = V_2/V$ denote the fractional volumes relative to the cylinder. (Relative areas and relative volumes are dimensionless.) Then we have the following surprising relation, which we state as a lemma:

Comparison Lemma for Complementary Regions. *The difference of relative volumes is equal to the corresponding difference of relative areas:*

$$v_2 - v_1 = a_2 - a_1. \tag{4}$$

To prove (4), let c_1 denote the distance of the area centroid of A_1 from the lower axis l_1, and let c_2 denote the centroidal distance of area A_2 from the upper axis l_2. Then $c = H/2$ is the centroidal distance of the area R of the rectangle from either axis. By equating area moments about the lower axis l_1 we find $c_1a_1 + (2c - c_2)a_2 = c$, which can be rewritten as follows:

$$c_1a_1 - c_2a_2 = c(1 - 2a_2) = c(a_1 - a_2). \tag{5}$$

From Pappus' theorem we have

$$v_2 - v_1 = \frac{2\pi}{V}(c_2A_2 - c_1A_1) = \frac{2}{H}(c_2a_2 - c_1a_1) = \frac{1}{c}(c_2a_2 - c_1a_1),$$

which, together with (5), gives (4).

Examples: Cycloidal and paraboloidal solids. To illustrate how this can be used in practice, refer to Figure 30. Figure 30a shows the solid swept by rotating one arch of a cycloid around its base. If the rolling disk generating the cycloid has radius a, then the volume V_{cap} of the solid of revolution swept by the cycloidal cap in Figure 25 is $V_{\text{cap}} = \pi^2a^3$. The arch and cap are complementary regions with relative areas 3/4 and 1/4, whose difference is 1/2. The cylinder has volume $8\pi^2a^3$ so the volume of the cap relative to that of the cylinder is $v_{\text{cap}} = 1/8$. By (4) in the comparison lemma, the volume of the arch relative to that of the cylinder is $v_{\text{arch}} = v_{\text{cap}} + 1/2 = 5/8$. Therefore $V_{\text{arch}} = 5\pi^2a^3$.

Now we use the lemma again to determine the relative volume v_2 of the solid in Figure 30b obtained by rotating the complement of the parabolic segment in Figure 24a about the upper edge of the circumscribing rectangle. In Figure 24c we found that the volume v_1 of a paraboloidal funnel is 1/5 that of the circumscribing cylinder, so by (4) we have $v_2 = v_1 + a_2 - a_1 = 1/5 + 2/3 - 1/3 = 8/15$. In Figure 30b, *the volume of the shaded solid is* 8/15 *that of its circumscribing cylinder*.

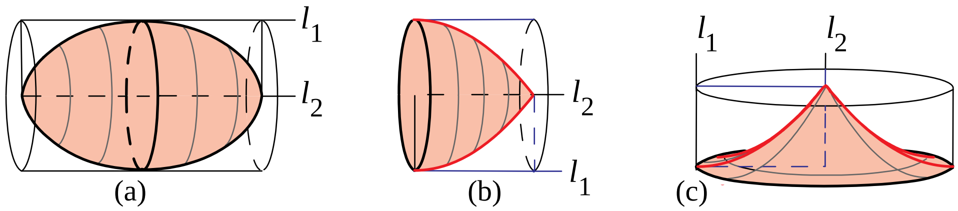


Figure 30. (a) Cycloidal arch rotated about its base has volume 5/8 that of its circumscribing cylinder. (b) Solid obtained by rotating a parabolic region around the upper edge has volume 8/15 that of the circumscribing cylinder. (c) Paraboloidal funnel has volume 1/6 that of its circumscribing cylinder.

Finally, we use the lemma once more to determine the relative volume v_2 of the paraboloidal funnel in Figure 30c obtained by rotating the parabolic segment in Figure 24a around axis l_2. In Figure 20 we found that the relative volume v_1 of the complementary paraboloidal segment rotated around l_1 is $1/2$, so by (4) we have $v_2 = 1/2 + a_2 - a_1 = 1/2 + 1/3 - 2/3 = 1/6$. In other words, in Figure 30c *the volume of the paraboloidal funnel is* 1/6 *that of the circumscribing cylinder*.

The lemma has a surprising consequence. For the special case in which $a_1 = a_2$ we find $v_1 = v_2$. In other words:

If the rectangle is divided into two regions of equal area, then the two solids obtained by rotating one region about the upper edge and the other about the lower edge have equal volumes!

Figure 31 shows three interesting examples. In Figure 31a, a cycloid generated by a rolling disk of radius 1 divides the shaded rectangle of altitude 3/2 into two regions of equal area. Hence the solid obtained by rotating the portion of the rectangle above the arch around the upper edge of this rectangle has the same volume as the solid obtained by rotating the cycloidal arch around the lower edge, which was treated in Figure 30a.

In Figure 31b a parabolic segment of height 1 is inside a rectangle of altitude 4/3. The parabola divides the shaded rectangle into two regions of equal area, so the solid obtained by rotating one region around the upper edge has the same volume as the solid obtained by rotating the complementary region around the lower edge. Figure 31c is similar, with the regions rotated about the right and left edge of the rectangle.

Examples. Generalized strophoidal solids.

(a) Solid of revolution generated by tangent sweep to unit circle. The shaded region in Figure 32a is the tangent sweep to a unit circle, where each tangent segment is cut off by a horizontal line p through the center of the circle. The corresponding tangent cluster is shown in Figure 32b. Now we rotate each of these regions about the horizontal axis to produce two solids of revolution. The volume V_a of the solid in Figure 32a (inside the cone and outside the sphere) is easy to find. It is equal to that of the cone minus the volume of the inscribed spherical segment. The volume

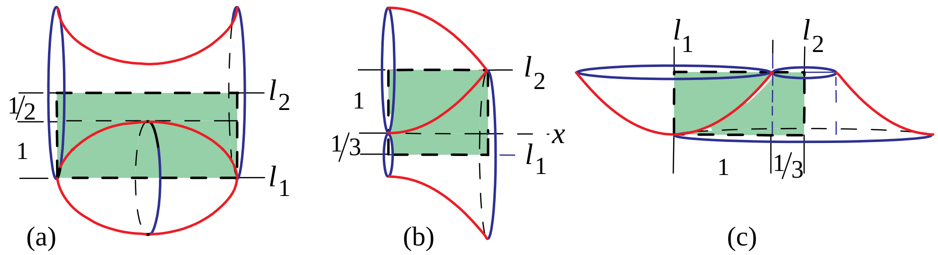


Figure 31. Three examples of solids of equal volume obtained by rotating complementary regions of equal area around opposite edges of a rectangle.

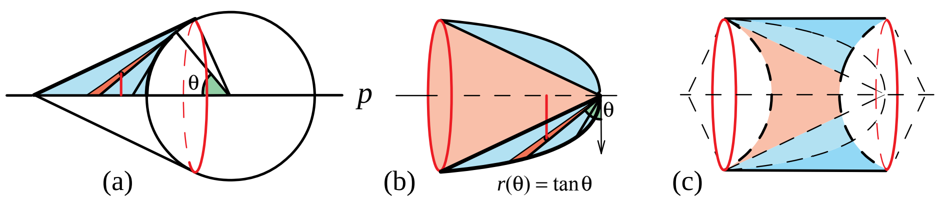


Figure 32. Solids swept by (a) tangent sweep to unit circle, and by (b) its tangent cluster. (c) Calculating the volume of the full solid in (b).

V_b of the solid in Figure 32b (outside the cone) is twice V_a, according to Theorem 2, so V_b is twice the volume of the cone minus twice that of the spherical segment in Figure 32a. The cones in Figures 32a and 32b are congruent. Therefore, if we adjoin the inside cone to the solid of volume V_b in Figure 32b, we obtain a solid whose volume is three times that of the cone minus twice that of the spherical segment in Figure 32a. But three times the volume of cone is the volume of its circumscribed cylinder, shown in Figure 32c. Consequently, *the volume of the full solid is that of the circumscribing cylinder minus twice that of the spherical segment.*

The full solid in Figure 32b can be generated another way. It is part of the solid of revolution obtained by rotating the plane curve with polar equation $r = \tan\theta$ around its horizontal axis of symmetry. The volume of that solid can also be calculated by using integral calculus, but the foregoing calculation is simpler and more revealing.

(b) Solid of revolution generated by strophoid. Figure 33a shows a tangent sweep like that in Figure 32a, except that the tangent segments to the unit circle are cut off by a horizontal line p tangent to the circle at point P. The corresponding tangent cluster, with the points of tangency brought to the common point P, is shown in the lower part of Figure 33a. This cluster is bounded by a curve which, as we will show later, is a classical right strophoid. The strophoid consists of two parts, a loop and a leftover portion with a horizontal asymptote. The region bounded by the loop is the tangent cluster of the portion of the tangent sweep circumscribed by the rectangle in Figure 33b. Therefore the loop area is $2 - \pi/2$. Tangent sweeping

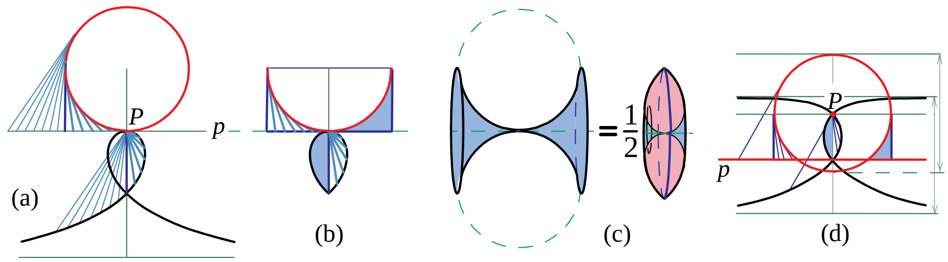


Figure 33. (a) Tangent sweep to a circle and its tangent cluster. (b) Portion of sweep in (a) corresponding to the loop. (c) Volume relation for solids obtained by rotating regions in (b). (d) Generalized strophoid.

can also be used to show that the area of the region between the strophoid and its asymptote is $2 + \pi/2$.

Now we determine the volume of the solid obtained by rotating the loop about the horizontal axis p. According to Theorem 2, *its volume is twice that of a toroidal cavity* (the solid obtained by rotating the corresponding tangent sweep around p). To find that volume, in turn, we apply the Comparison Lemma. Rotation of the complementary region about the upper edge of the rectangle in Figure 33b gives a sphere of volume $4\pi/3$, which means that the relative volume v_1 is $2/3$ that of its circumscribing cylinder. The relative areas of the complementary regions are $a_1 = \pi/4$ and $a_2 = 1 - \pi/4$, so $a_2 - a_1 = 1 - \pi/2$ and (4) gives $v_2 = 5/3 - \pi/2$ as the relative volume of the rotated tangent sweep. Therefore the absolute volume of the solid on the left of Figure 33c is $2\pi(5/3 - \pi/2) = \pi(10/3 - \pi)$. The volume of the solid obtained by rotating the loop is twice that.

The volume of the solid generated by rotating, about the asymptote, the region in Figure 33a between the strophoid and its asymptote can also be determined, but we omit the details.

An infinite family of generalized strophoids can be constructed by parallel motion of the line p which cuts off the tangent segments to the circle, as indicated in Figure 33d. Tangent sweeping can be used to determine corresponding areas and volumes of revolution, but we shall not present the details.

Different descriptions of the classical strophoid. The classical strophoid has been described in three different ways by Roberval, Barrow, and Newton. Figure 34a shows Newton's description as the locus of corner A of a carpenter's square, as the end point B of edge BA slides along a horizonal line while the perpendicular edge touches a fixed peg P at distance AB above B. Figure 34b shows that our description of the strophoid in Figure 33a is equivalent to that of Newton. And Figure 34c leads to a known polar description of the right strophoid.

7. APPLICATIONS TO HYPERBOLOIDS

Hyperboloid of two sheets. Figure 35a shows the lower half of right circular cone with a cylindrical hole drilled through its axis. A tangent plane to the cylinder intersects the cone along one branch of a hyperbola, forming a hyperbolic cross

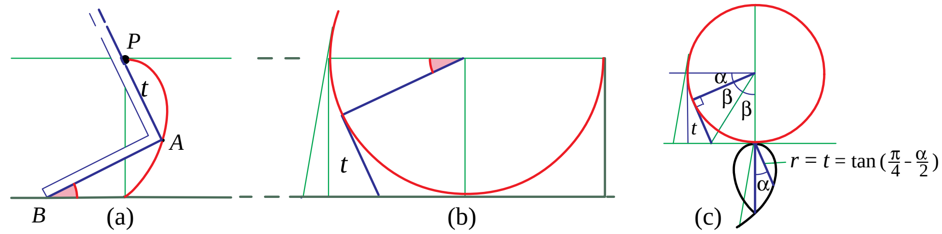


Figure 34. (a) Newton's description of strophoid. (b) Tangent sweep to a unit circle used to describe strophoid. (c) Polar equation describing strophoid.

section that generates a bracelet by tangential sweeping. *The volume of the hyperboloidal bracelet shown is equal to the volume of the solid of revolution generated by the hyperbolic cross section.*

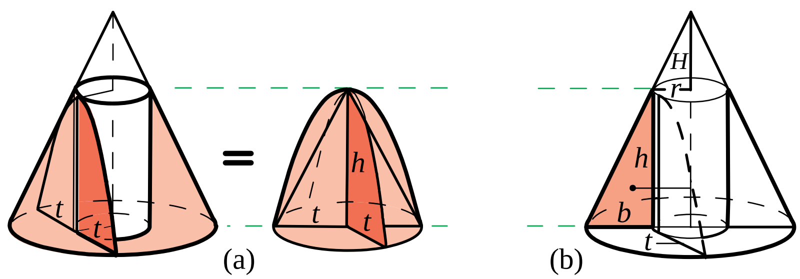


Figure 35. (a) Bracelet cut from a cone by an axial hole has the same volume as that of a hyperboloid of revolution. (b) Diagram for proving Archimedes' volume relation in (6).

Archimedes showed in [6; *On Conoids and Spheroids*, Prop. 25] that this volume (call it V) bears a simple relation to the volume V_{cone} of the inscribed right circular cone in Figure 35a with the same base and axis. This cone has altitude h and base t, the base radius of the hyperboloid of revolution. Archimedes showed that

$$\frac{V}{V_{\text{cone}}} = \frac{3H + h}{2H + h}, \tag{6}$$

where H, indicated in Figure 35b, is the length of the semimajor axis of the hyperbola. A simple proof of (6) can be given from our observation that the bracelet can be swept by rotating tangentially around the cylindrical hole the shaded triangle of base b and altitude h in Figure 35b. The area of the triangle is $bh/2$, and the area centroid of the triangle is at distance $r + b/3$ from the axis of rotation, where r is the radius of the cylindrical hole.

By Pappus, volume V is the product of the area of the triangle and the distance its centroid moves in one revolution, giving us

$$V = 2\pi(r + \frac{b}{3})\frac{bh}{2} = \frac{\pi}{3}(b + 3r)bh. \tag{7}$$

The cone with the same base and axis has altitude h and base t, where t is the length of the tangent to the hole in Figure 35b. By similar triangles, $b/t = t/(2r+b)$, so $t^2 = (b+2r)b$. Hence

$$V_{\text{cone}} = \frac{\pi}{3}t^2 h = \frac{\pi}{3}(b+2r)bh. \tag{8}$$

Now divide (7) by (8) and use the similarity relation $r/b = H/h$ to obtain (6).

Equilateral hyperbola rotated about an asymptote. Figure 36a shows an equilateral hyperbola and its orthogonal asymptotes. The portion of any tangent to the hyperbola between the asymptotes is bisected at the point of tangency. As the point of tangency moves to the right, the lower half of the tangent segment forms a tangent sweep with the hyperbola as tangency curve. A corresponding tangent cluster is formed by translating each tangent segment so the point of tangency is at the origin. The free end of the translated segment traces the mirror image of the original hyperbola, as suggested in Figure 36b. By Theorem 2, the volume of the solid obtained by rotating the tangent sweep about the horizontal axis is one-half the volume of the solid obtained by rotating the corresponding tangent cluster about the same axis. As the point of tangency moves from some initial position to ∞, the swept solid is a hyperboloid of revolution of volume V_{hyp}, say, punctured by a right circular cone of volume V_{cone} generated by rotating the initial tangent segment. On the other hand, the cluster solid is the same hyperboloid together with a cone congruent to the puncturing cone. Consequently, $V_{\text{hyp}} + V_{\text{cone}} = 2(V_{\text{hyp}} - V_{\text{cone}})$, hence $V_{\text{hyp}} = 3V_{\text{cone}}$, which, in turn, is the volume of the cylinder attached to the hyperboloid, as shown in Figure 36c.

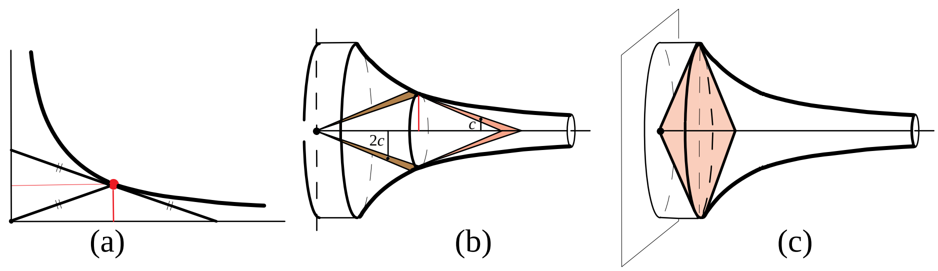


Figure 36. Hyperboloid of revolution and attached cylinder of equal volume.

Figure 37 shows an interesting interpretation of the foregoing result. The hyperboloid of revolution can be regarded as a "monument" of infinite extent supported by a cylindrical pedestal whose base rests on a plane through the other asymptote. We have just shown that *the volume of such a monument is equal to the volume of its pedestal.* It seems appropriate to refer to this as a "monumental result." It can, of course, also be easily verified by integration.

General hyperbola rotated about one asymptote. An even deeper monumental result will be obtained for a general hyperbola rotated about one of its asymptotes. The volume of the solid hyperboloid is again equal to the volume of its pedestal, but now the more general pedestal consists of two parts, a cylindrical part together with

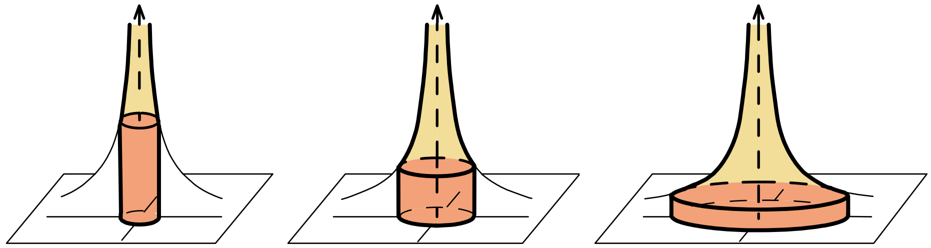


Figure 37. Each hyperbolic monument has the same volume as its cylindrical pedestal.

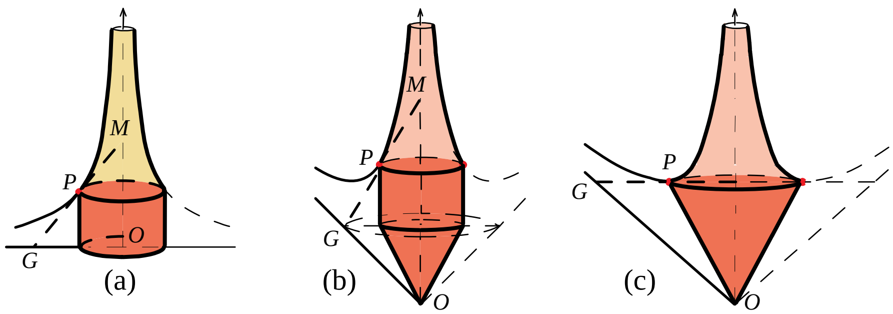


Figure 38. Hyperboloid of revolution and attached pedestal of equal volume.

an attached conical part whose shape depends on the angle between the asymptotes, as illustrated in Figure 38b. The conical part disappears when the asymptotes are orthogonal as in Figure 38a, and the cylindrical part disappears when the monument touches the ground, as in Figure 38c.

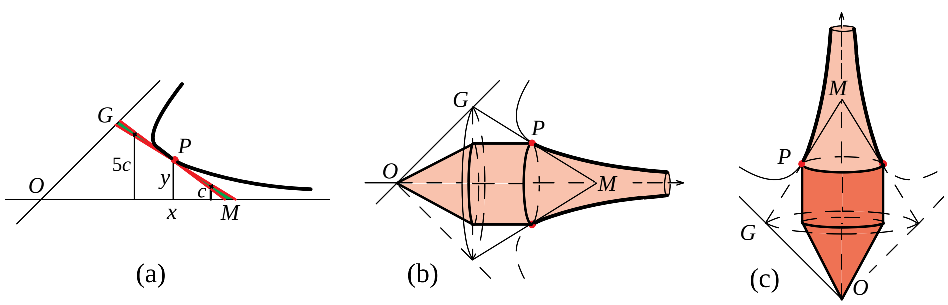


Figure 39. (a) Centroidal distance to upper tangent sweep is 5 times that to the lower tangent sweep. (b) and (c) Hyperboloid of revolution and attached pedestal of equal volume.

Figure 39a shows one branch of the hyperbola oriented so that the asymptote of rotation is along the x axis, together with a tangent segment at a point $P = (x, y)$ cut off by the two asymptotes at points G and M in Figure 39b. The asymptotes intersect at O. For any hyperbola, the point of tangency P bisects segment GM.

We wish to determine the volume of the solid of revolution obtained by rotating about the x axis the ordinate set of this hyperbola above the interval $[x, \infty)$.

The ordinate set consists of two parts, a lower tangent sweep generated by moving PM from x to ∞, plus the triangle between the initial tangent PM and its subtangent. Figure 39a shows a small triangle contributing to the lower tangent sweep; its centroid is at height $c = y/3$ above the x axis. The corresponding triangle cut off by the other asymptote, which is part of the another (upper) tangent sweep, has its centroid at height $y + 2y/3 = 5c$ above the x axis. The ratio 5 to 1 of these centroidal distances for the hyperbola has the following profound consequence which we state as a lemma:

Lemma. *The solid obtained by rotating the upper tangent sweep of the hyperbola about the x axis has volume 5 times that of the solid obtained by rotating the lower tangent sweep about the same axis.*

The lemma follows from Pappus' theorem. The volume of the conical shell generated by rotating each small triangle in the lower tangent sweep is $2\pi c$ times the area of the triangle. The corresponding triangle in the upper tangent sweep has the same area, so the corresponding conical shell has volume 5 times as great.

The lemma now follows from the fact that each solid of rotation is the union of such conical shells.

Now we show that *the volume of the hyperboloid of revolution is equal to the volume of the composite pedestal, cone plus cylinder.* First, we express each of these volumes in terms of the volume V_{hyp} of the hyperboloid of revolution and various related cones. The volume generated by the lower tangent sweep is $V_{\text{hyp}} - V_{\text{cone}}$, where V_{cone} is the volume of the cone of slant height PM swept by the right triangle below the initial tangent segment. The volume generated by the upper tangent sweep is equal to that generated by the lower tangent sweep plus the volume V_{double} of the double cone generated by rotating triangle OGM in Figure 39b. By the lemma we have

$$(V_{\text{hyp}} - V_{\text{cone}}) + V_{\text{double}} = 5(V_{\text{hyp}} - V_{\text{cone}}),$$

which gives us

$$4V_{\text{hyp}} = V_{\text{double}} + 4V_{\text{cone}}. \tag{9}$$

From Figure 39b it is easy to see that $V_{\text{double}} = 8V_{\text{cone}} + V_O$, where V_O is the volume of the cone with slant height OG. But $V_O = 4V_{\text{base}}$, where V_{base} is the volume of the base cone with vertex O and radius y. Hence (9) implies

$$V_{\text{hyp}} = V_O + 3V_{\text{cone}} = V_O + V_{\text{cyl}}, \tag{10}$$

where V_{cyl} is the volume of the cylinder joining the bases of the base cone and the cone with slant height PM. This completes the proof that the volume of the hyperboloid of revolution is equal to the volume of the composite pedestal, cone plus cylinder.

8. FURTHER EXAMPLES OF TANGENTIALLY SWEPT SOLIDS

Cardioid. In the next example we rotate one lune of a cardioid about the axis of the cardioid to generate a solid of revolution. Here the cardioid is a pedal curve as described in [2; p. 24]. (Point P is the pedal point and F denotes the foot of the perpendicular from P to an arbitrary tangent line to the large circle. The cardioid is the locus of all such points F constructed for all tangent lines.) One lune of the cardioid is swept by tangents to the large circle as indicated in Figures 40a and b. The left half of the small disk is the tangent cluster of that part of the lune swept

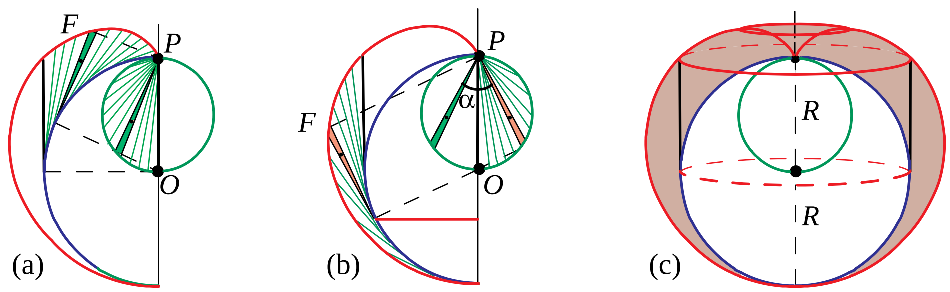


Figure 40. (a) and (b): Tangential sweeping of one lune of a cardioid. (c) Solid of revolution.

by tangents from the horizontal position at P to the vertical position in Figure 40a. The right half of the small disk is the tangent cluster of the remaining part of the lune as in Figure 40b. Hence, the area of the lune is equal to the area of the small disk.

When we rotate the cardioid about its axis of symmetry it generates an apple-like solid depicted in Figure 40c. Classical integration in polar form shows that its volume is twice the volume of the large central sphere. In other words, *the punctured apple (the shaded portion between the sphere and the apple) has the same volume as the sphere.* We shall give a geometric proof of this result.

In Figure 41 a thin shaded triangle of altitude t of the tangent sweep of the upper part of the lune makes an angle α with the axis of rotation. The corresponding triangle of the same altitude for the lower part of the lune that makes the same angle α is also shown. The two triangles have equal area (which we denote by ΔA) and the sum of their centroidal distances from the axis of rotation is $(R\cos\alpha - c) + (R\cos\alpha + c) = 2R\cos\alpha$, where R is the radius of the large central circle. When rotated together around the axis they sweep a solid of volume $4\pi(R\cos\alpha)\Delta A$, according to a theorem of Pappus.

The two thin triangles can be combined to form a rectangle shown in Figure 41a as a thin horizontal slice of the large semicircular disk. The area of the rectangle is $2\Delta A$ and its centroidal distance from the axis is $\frac{1}{2}R\cos\alpha$. Two symmetric copies of this rectangle are shown. When the rectangles are rotated around the axis they generate two symmetric slices of the sphere which together, by Pappus, have the same volume as the solid swept by the two thin triangles. As α varies from 0 to $\pi/2$, the rotated triangles sweep the punctured apple, and the corresponding rectangles

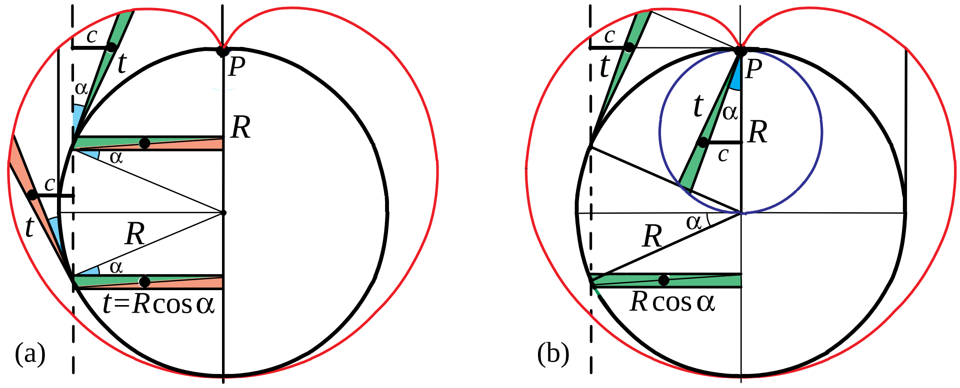


Figure 41. Diagram showing that punctured apple has same volume as the sphere.

sweep the large interior sphere. This shows that *the punctured apple has the same volume as the sphere.*

We can gain further insight by regarding the punctured apple as a piece of pottery with two parts, an upper one (the cap) shown in Figure 42a, and a lower one shown in Figure 42b. We will show that *the volume* V_{upper} *of the cap is equal to that of the large hemisphere minus that of the small sphere* obtained by rotating the tangent cluster disk in Figure 42 whose area is that of the lune. Consequently, the *volume* V_{lower} *of the lower part is that of the large hemisphere plus the volume of the small sphere.*

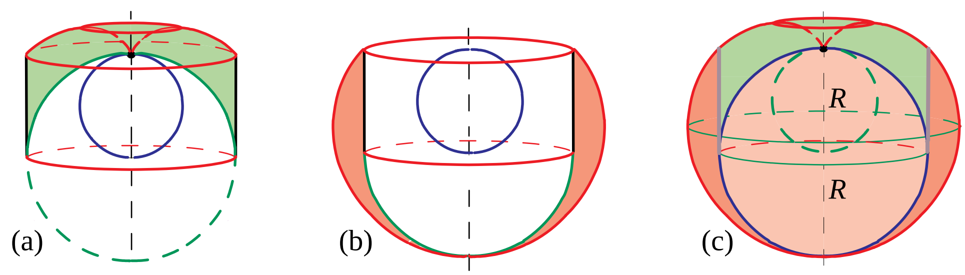


Figure 42. Upper part (a) and lower part (b) of punctured apple. Volume of upper part is that of the large hemisphere in (c) minus that of small sphere. Volume of lower part is that of large hemisphere plus that of small sphere.

Volume of the upper part: Figure 41b shows a thin triangle in the tangent sweep of the upper part of the lune and its counterpart in the tangent cluster, which makes an angle α with the axis of rotation. The triangles have equal area (which we call ΔA), and the sum of their centroidal distances from the axis of rotation is $R\cos\alpha$. The triangles of this part of the tangent sweep generate the cap and those of the tangent cluster generate the small interior sphere. The two thin triangles can be combined to form the familiar rectangle shown in Figure 41b as a thin horizontal slice of the large semicircular disk. The area of the rectangle is $2\Delta A$ and its centroidal distance from the axis is $\frac{1}{2}R\cos\alpha$. When all these rectangles are rotated around the axis they sweep a hemispherical solid whose volume is equal that of the volume swept by all the above triangles. This shows that V_{upper} is the volume of the large hemisphere minus the volume of the small sphere.

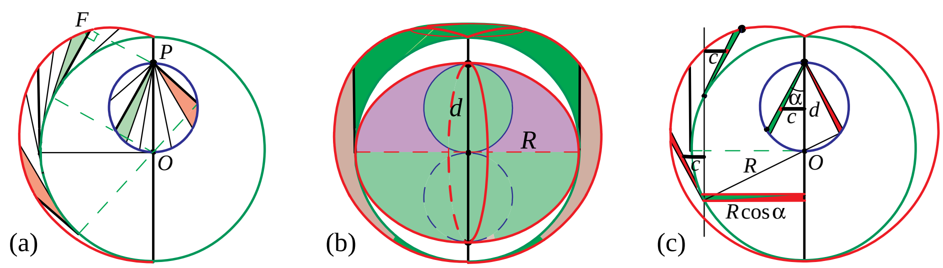


Figure 43. Analysis for cardioid modified for the Limaçon of Pascal.

Limaçon. Not surprisingly, a similar argument works when the cardioid is replaced by any Limaçon of Pascal, an example of which is shown in Figure 43. In this case, *the volume of the punctured apple is equal to that of an ellipsoid of revolution* obtained by rotating an ellipse of semiaxes R and d around the major axis, as indicated in Figure 43b. We also note that volume V_{upper} of the upper part is equal to that of the large semiellipsoid minus that of the small sphere of diameter d in Figure 43b. The volume V_{lower} of the lower part is that of the same semiellipsoid plus that of the small sphere of diameter d. For the proof observe that the thin triangles now have area smaller than the area ΔA for the cardioid by a factor $(d/R)^2$, where d is the diameter of the small circle in Figure 43c. The rest of the argument is like that for the cardioid.

Catenoid. Figure 44a shows a portion of a catenary, the graph of a hyperbolic cosine, $y = \cosh x$, for $0 \le x \le X$. When the ordinate set of this graph is rotated about the x axis the solid of revolution is a catenoid whose volume, expressed as an integral, is $V_{\text{ch}} = \pi \int_0^X \cosh^2 x\,dx$. The corresponding volume of the solid obtained by rotating the ordinate set (Figure 44b) of a hyperbolic sine, $y = \sinh x$, over the same interval is $V_{\text{sh}} = \pi \int_0^X \sinh^2 x\,dx$. But $\cosh^2 x - \sinh^2 x = 1$, so the difference of the volumes is

$$V_{\text{ch}} - V_{\text{sh}} = \pi X. \tag{11}$$

The result in (11) can be obtained without integration by using sweeping tangents to show geometrically that the difference of volumes $V_{\text{ch}} - V_{\text{sh}}$ is the volume of the cylinder of altitude X and radius 1 shown in Figure 44c.

The method of sweeping tangents also reveals the nonobvious result that the sum of the volumes is the same as the volume of another solid of revolution, shown in Figure 45a. This solid is generated by rotating the rectangle in Figure 45b with vertex X about the x axis. That rectangle appears in [2; p. 346] and in [5; p. 413] where its area is shown by tangent sweeping to be equal to that of the ordinate set

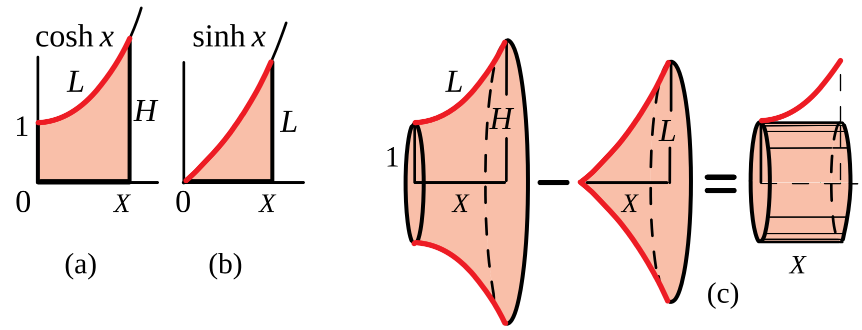


Figure 44. The difference of volumes $V_{\text{ch}} - V_{\text{sh}}$ is the volume of a cylinder.

of the catenary. The rectangle has base 1, altitude L and diagonal of length H.

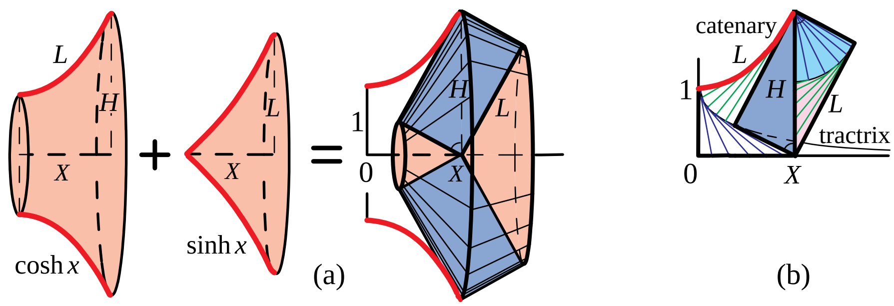


Figure 45. The sum of volumes $V_{\text{ch}}+V_{\text{sh}}$ is the volume of a solid obtained by rotating the rectangular region in (b).

Here $L = \sinh X$ is the arclength of the catenary, and $H = \cosh X$. The rectangle reveals that $H^2 = L^2 + 1$. An easy calculation shows that the solid has volume

$$V_{\text{ch}} + V_{\text{sh}} = \pi LH. \tag{12}$$

From (11) and (12) we obtain V_{ch} and V_{sh} separately without integration:

$$V_{\text{ch}} = \frac{\pi}{2}(LH + X), \quad V_{\text{sh}} = \frac{\pi}{2}(LH - X). \tag{13}$$

9. VERTICAL SECTIONS OF SOLID SWEEP AND CLUSTER

We know that a solid tangent sweep and its solid tangent cluster have equal volumes because corresponding horizontal cross sections of these solids have equal areas. We turn next to surprising properties relating their *vertical* cross sections.

Area balance of axial sections. Figure 46 shows vertical cross sections of bracelets in Figures 1, 4, 5a, 6 and 8 taken through the axis of revolution, indicated by the arrow. The section of the solid tangent cluster is shown on the right of the axis, and a section of a typical solid tangent sweep is shown on the left.

From Pappus' rule for volumes, we obtain the following balance-revolution principle (introduced in [2; p. 410].) *The areas of two plane regions are in equilibrium*

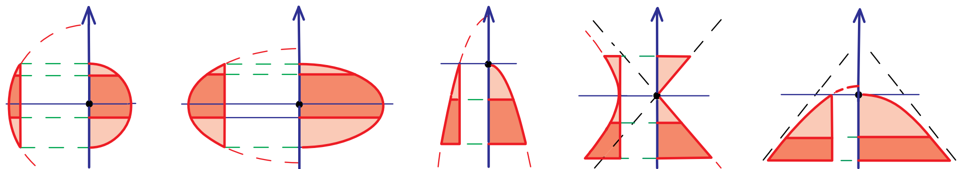


Figure 46. Area equilibrium of axial sections of bracelets with respect to the axis of rotation.

with respect to a balancing axis if, and only if, the solids of revolution generated by rotating them about the balancing axis have equal volumes. Applying this to the solids in Figure 1, we find that the semicircular disk in Figure 46a is in area balance with the circular segment on the left of the axis. The same holds true for the semielliptical disk in Figure 46b, the semiparabolic segment in Figure 46c, and the hyperbolic segments in Figures 46d and e. Because any slice of a family of bracelets has the equal height-equal volume property, each area equilibrium in Figure 46 holds slice-by-slice and, in the limiting case, chord-by-chord.

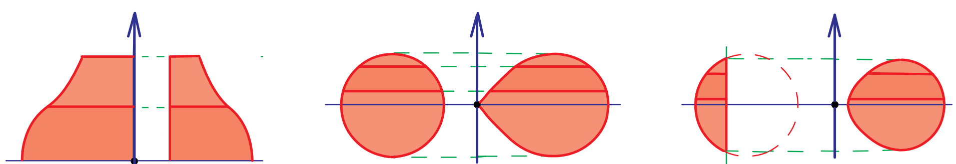


Figure 47. Area equilibrium of axial sections of more general sweeps and clusters.

Figure 47a shows the same principle applied to vertical cross sections of a more general tangentially swept solid and its solid cluster. Figures 47b and c are special cases obtained by vertical cross sections in Figure 13. We were pleasantly surprised to learn that the circular disk and lemniscate in Figure 47b are in chord-by-chord equilibrium. Tangential sweeping reveals unexpected area balancing relations without knowing the areas themselves, their centroids, or cartesian equations representing the boundary curves.

Congruent sections. Figure 48 reveals a new fact concerning vertical sections of a solid sweep around a circular cylinder of radius a and its solid cluster, for a general sweeping region S.

Each vertical section of the solid cluster at distance d from its rotation axis is geometrically congruent to the vertical section of the solid sweep at distance D from its rotation axis, where $D = (d^2 + a^2)^{1/2}$.

To prove this it suffices to show that their corresponding chords PQ and $P'Q'$ in a typical two-dimensional horizontal section of the two vertical sections are congruent.

Figure 48 shows a typical horizontal section of (a) a solid sweep, and (b) it solid cluster. In (a) the inner circle is the profile of the tangency cylinder, and AT is the horizontal section of the tangent plane to the cylinder. Point A is the outer edge of the tangent segment of the sweeping region S. Its inner edge B, where the vertical

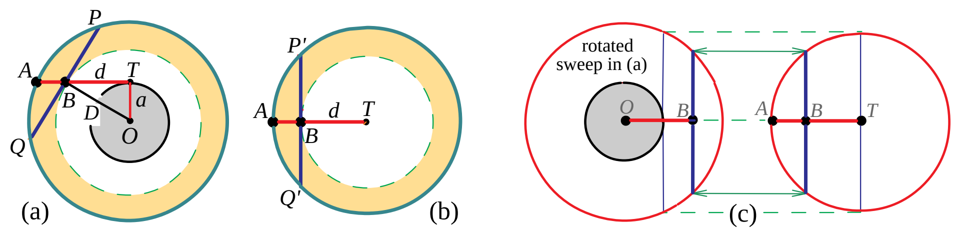


Figure 48. Chord PQ in (a) and $P'Q'$ in (b) are congruent because they sweep out annuli of equal area. (c) Matching congruent vertical sections of the sweep and cluster.

section intersects AT, and can be anywhere on AT. In (b), the circle through A has as radius the translated segment AT, with the position of B at distance d from T. In (a) and (b), the points B of all horizontal sections lie on a vertical line, which is an axis of symmetry of the corresponding vertical section. The outer edge A and corresponding inner edge can vary from layer to layer.

To prove congruency of chords PQ and $P'Q'$, we note that the annulus swept by AT in (a) has the area of the circle of radius AT in (b). Also, the annulus swept by BT in (a) has the area of the circle of radius BT in (b). Hence their area differences (those of the lighter shaded annuli) are also equal. Therefore the tangent segments BP in (a) and BP' in (b) are congruent (otherwise the areas they sweep would not be equal). Because B is the midpoint of PQ in Figure 48a, and of $P'Q'$ in Figure 48b, chords PQ and $P'Q'$ are congruent. Figure 48c shows how to match directly any two congruent vertical sections of the sweep and cluster.

Example: Bernoulli lemniscate. A Bernoulli lemniscate (see Section 3) is the boundary of a vertical cross section internally tangent to a solid torus generated by a circular disk S of radius r rotated around an axis at distance $2r$ from the center of S. Using the axis as the edge of a half-plane as in Figure 10a, rotate the same disk S tangentially around any circular cylinder to produce a solid tangent sweep whose solid cluster is the solid torus. When the solid sweep is cut by a vertical plane internally tangent to the sweep, its cross section is a region congruent to that bounded by the Bernoulli lemniscate. Because the radius of the tangency cylinder is arbitrary, this process produces infinitely many congruent Bernoulli lemniscates, all generated by the same disk S.

10. CONCLUDING REMARKS

We began this paper with the classical calculus result that all bracelets obtained by drilling cylindrical holes of given height through solid spheres of different radii have equal volume. We derived this result without calculus, and then showed that the same bracelets can be produced differently by a method of tangential sweeping of plane regions around general cylinders. Tangential sweeping, in turn, leads to infinitely many new families of solids that share the equal height-equal volume property and also gives a new way of calculating volumes of many solids of revolution, some familiar and some unfamiliar, without the use of calculus. A knowledge of the volume of a solid of revolution, in turn, also gives the centroidal distance from the axis of the planar sweeping region if its area is known, which is the case in most of our examples. Some results of this paper also appear in [7].

Another view of swept solids and their clusters. Figure 49 shows another way to see visually why a solid tangent sweep has the same volume as its solid cluster. In Figure 49a we take a solid of revolution and slice it into wedges by vertical planes passing through its axis. The vertical faces are shown there as rectangles, but they could have a more general shape like that in Figure 10a, as suggested by the shading. Now slide the wedges radially away from the axis in such a way that common faces continually touch each other. The new configuration is a prismatic

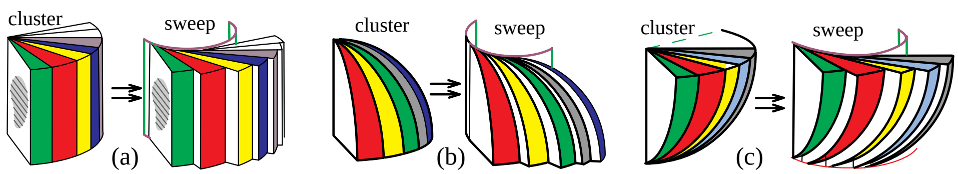


Figure 49. (a) Solid of revolution sliced into wedges that are spread tangentially around a cylinder. Sliced upper hemisphere in (b), lower hemisphere in (c).

solid of the same volume, surrounding a prismatic cavity. As the number of wedges increases indefinitely, the cavity becomes more like a cylinder along which the prismatic solid is swept tangentially. The original solid of revolution is its tangent cluster. Figure 49 also reveals that *the volume centroids of any solid tangent sweep and its cluster lie in the same horizontal plane.*

Extensions to n-space. Many results in this paper can be readily extended to higher dimensions. For example, to extend the results for the family of spherical bracelets in Figure 1, we puncture an n-sphere by a coaxial n-cylinder to produce an n-dimensional bracelet. As in Figure 1, those bracelets of equal height also have equal volume, that of the n-sphere with diameter equal to the height of the cylindrical hole. We can also regard the general n-dimensional tangent sweep as being swept tangentially by an $(n-1)$-dimensional hemisphere as in Figure 3.

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