WEAK PERTURBATIONS OF THE P–LAPLACIAN

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Abstract. We consider the p-Laplacian in $\mathbb{R}^d$ perturbed by a weakly coupled potential. We calculate the asymptotic expansions of the lowest eigenvalue of such an operator in the weak coupling limit separately for $p > d$ and $p = d$ and discuss the connection with Sobolev interpolation inequalities.

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1. Introduction

In this paper we consider the functional

$$Q_V[u] = \int_{\mathbb{R}^d} \left(|\nabla u|^p - V\, |u|^p \right) dx, \qquad u \in W^{1,p}(\mathbb{R}^d), \quad p > 1, \tag{1.1}$$

with a given function $V : \mathbb{R}^d \to \mathbb{R}$ which is assumed to vanish at infinity in a sense to be made precise. We are interested in the minimization problem

$$\lambda(V) = \inf_{u \in W^{1,p}(\mathbb{R}^d)} \frac{Q_V[u]}{\int_{\mathbb{R}^d} |u|^p\, dx}\,. \tag{1.2}$$

If (1.2) admits a minimizer u, then the latter satisfies in the weak sense the non-linear eigenvalue equation

$$-\Delta_p(u) - V|u|^{p-2}\, u = \lambda(V)\, |u|^{p-2}\, u\,, \tag{1.3}$$

where $-\Delta_p(u) := -\nabla \cdot (|\nabla u|^{p-2}\, \nabla u)$ is the p-Laplacian. Equation (1.3) is a particular case of a quasilinear differential problem and we refer to the monographs [LU, PS] and to [S1, S2, Tr] for the general theory of such equations. The p-Laplacian equation with a zero-th order term V has attracted particular attention. Existence of positive solutions to the equation $-\Delta_p(u) = V|u|^{p-2}\, u$ and related regularity questions were studied in [PoSh, PT2, TT, To, PT1]. For the discussion of maximum and comparison principles and positive Liouville theorems, see [GS, PTT].

In the present paper we are going to study the behaviour of $\lambda(\alpha V)$ for small values of α. It is not difficult to see that $\lambda(\alpha V) \to 0$ as $\alpha \to 0$ for all sufficiently regular and decaying V. Our goal here is to find the correct asymptotic order and the correct asymptotic coefficient.

Key words and phrases. p-Laplacian, weak coupling, Sobolev inequality.

It turns out that the asymptotic order depends essentially on the relation between the values of the exponent p and the dimension d. If $p < d$, then by the Hardy inequality [OK] we have

$$\int_{\mathbb{R}^d} |\nabla u|^p \, dx \, \geq \Big(\frac{d-p}{p}\Big)^p \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \, dx, \qquad u \in W^{1,p}(\mathbb{R}^d), \quad d > p \, .$$

Therefore, if $|V(x)| \leq C\, |x|^{-p}$ for some $C > 0$, then $\lambda(\alpha V) = 0$ for all α small enough. However, if $p \geq d$ and $\int_{\mathbb{R}^d} V > 0$, then we have $\lambda(\alpha V) < 0$ for any $\alpha > 0$. The latter is easily verified by a suitable choice of test functions. Moreover, if V is bounded and compactly supported, then $\lambda(\alpha V) < 0$ for any $\alpha > 0$ even when $\int_{\mathbb{R}^d} V = 0$, see [PT1, Prop. 4.5]. Consequently, we will always assume that $p \geq p$.

The question about the asymptotic behavior of $\lambda(\alpha V)$ for small α was intensively studied in the linear case $p = 2$ (see, e.g., [BGS, Kl1, KS, Si]), where equation (1.3) defines the ground state energy of the Schrödinger operator $-\Delta - V$. In particular, it turns out that for sufficiently fast decaying V we have

$$\sqrt{-\lambda(\alpha V)} \, = \, \frac{1}{2}\, \alpha \int_{\mathbb{R}} V \, dx - c\, \alpha^2 + o(\alpha^2), \quad \alpha \to 0, \qquad d = 1, \; p = 2, \tag{1.4}$$

with an explicit constant c depending on V, see [Si]. The proof of (1.4) is based on the Birman-Schwinger principle and on the explicit knowledge of the unperturbed Green function. With suitable modifications, this method was applied also to Schrödinger operators with long-range potentials, [BGS, Kl2], and even to higher order and fractional Schrödinger operators [AZ1, AZ2, Ha].

Much less is known about the non-linear case $p \neq 2$ where the operator-theoretic methods developed for $p = 2$ cannot be used. We will therefore apply a different, purely variational technique which allows us to analyze the asymptotic behaviour of $\lambda(\alpha V)$ for all $p > 1$. A similar variational approach has already been used in a linear problem in [FMV], but here we take it much further into the quasi-linear realm (where, for instance the symmetry reduction that we crucial in [FMV] is no longer available).

We will present our main results separately for $p > d$, see Theorem 2.1, and for $p = d$, see Theorem 2.2. In the case $p > d$ we shall show, in particular, that there is a close relation between the asymptotic behaviour of $\lambda(\alpha V)$ and the Sobolev interpolation inequality (see, e.g., [Ad, Thm 5.9])

$$\|u\|_\infty^p \leq \, \mathcal{S}_{d,p} \, \|\nabla u\|_p^d \, \|u\|_p^{p-d} \, , \qquad u \in W^{1,p}(\mathbb{R}^d) \, , \quad d < p \, . \tag{1.5}$$

By convention $\mathcal{S}_{d,p}$ will always denote the optimal (that is, smallest possible) constant in (1.5). On one hand, the constant $\mathcal{S}_{d,p}$ enters into the asymptotic coefficient in the expansion of $\lambda(\alpha V)$, see equation (2.1). On the other hand, minimizers of problem (1.2), when suitably rescaled and normalised, converge (up to a subsequence) locally uniformly to a minimizer of the Sobolev inequality (1.5) as $\alpha \to 0$, see Proposition 3.7.

The case $p = d$ is much more delicate and requires (slightly) more regularity of the potential V since functions in $W^{1,d}(\mathbb{R}^d)$, which appear in (1.2), are not necessarily bounded. While the case $p > d$ can be dealt with by energy methods (i.e. on the $W^{1,p}(\mathbb{R}^d)$ level of regularity), heavier PDE technics (Harnack's inequality, Hölder continuity bounds) are necessary to deal with $p = d$. The subtly of the case $p = d$ can also be seen in the asymptotic order: while $\lambda(\alpha V)$ vanishes algebraically as $\alpha \to 0$ for $p > d$, it vanishes exponentially fast for $p = d$, see equation (2.2).

As we shall see, the asymptotic coefficient will depend on V only through $\int_{\mathbb{R}^d} V\,dx$. We emphasize here that we do *not* impose a sign condition on V. Thus, the positive and the negative parts of V contribute both to the asymptotic coefficient and there will be cancellations. This is one of main difficulties that we overcome. In fact, if V is non-negative, then the proof is considerably simpler.

A common feature of both Theorems 2.1 and Theorem 2.2 is that their proofs rely, among other things, on the fact that minimizers u_α of (1.2), suitably normalized, converge locally uniformly to a constant. While in the case $d < p$ this follows from Morrey's Sobolev inequality and energy considerations, for $d = p$ we have to employ a regularity argument related to the Hölder continuity of u_α, see Lemma 4.6, with explicit dependence on the coefficients of the equation.

2. Main results

Our main results describe the asymptotics of the infimum $\lambda(\alpha V)$ of the functional $Q_{\alpha V}[u]$ as $\alpha \to 0$, see (1.1) and (1.2). Our first theorem concerns the subcritical case $p > d$.

Theorem 2.1. *Let $p > d \geq 1$. Let $V \in L^1(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} V(x)\,dx > 0$. Then*

$$\lim_{\alpha\to 0+} \alpha^{-\frac{p}{p-d}}\ \lambda(\alpha V) \;=\; -\frac{p-d}{p}\ \left(\frac{d}{p}\right)^{\frac{d}{p-d}}\ \left(\mathcal{S}_{d,p}\int_{\mathbb{R}^d} V(x)\,dx\right)^{\frac{p}{p-d}}\ , \tag{2.1}$$

where $\mathcal{S}_{d,p}$ is the sharp constant in the Sobolev inequality (1.5).

We also have a theorem that describes the asymptotics of the minimizers of the functional $Q_{\alpha V}[u]$; see Proposition 3.7.

In the endpoint case $d = p$ we have

Theorem 2.2. *Let $p = d > 1$. Suppose that $V \in L^q(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ for some $q > 1$ and that $\int_{\mathbb{R}^d} V(x)\,dx > 0$. Then*

$$\lim_{\alpha\to 0+} \alpha^{\frac{1}{d-1}}\ \log\frac{1}{|\lambda(\alpha V)|} = d\ \omega_d^{\frac{1}{d-1}}\ \left(\int_{\mathbb{R}^d} V(x)\,dx\right)^{-\frac{1}{d-1}}\ , \tag{2.2}$$

where ω_d denotes the surface area of the unit sphere in $\mathbb{R}^d$.

Remark 2.3. Let us compare the assumptions on V in Theorems 2.1 and 2.2. If $p > d$ and $V_+ \notin L^1(\mathbb{R}^d)$, $V_- \in L^1(\mathbb{R}^d)$, then Theorem 2.1 easily implies that

$$\lim_{\alpha\to 0+} \alpha^{-\frac{p}{p-d}}\ \lambda(\alpha V) = -\infty\,.$$

Thus, at least under the additional hypothesis $V_- \in L^1(\mathbb{R}^d)$, the condition $V_+ \in L^1(\mathbb{R}^d)$ is necessary and sufficient for finite asymptotics of $\alpha^{-\frac{p}{p-d}}\ \lambda(\alpha V)$. This is not true for the asymptotics of $\alpha^{\frac{1}{d-1}}\ \log|\lambda(\alpha V)|^{-1}$ in the case $p = d$, and this is the reason for the additional assumption $V \in L^q(\mathbb{R}^d)$ for some $q > 1$. Indeed, we claim that there are $0 \leq V \in L^1(\mathbb{R}^d)$ such that $\lambda(\alpha V) = -\infty$ for any $\alpha > 0$. To see this, choose $\sigma \in (1, d)$ and consider $V(x) = |x|^{-d}|\log|x||^{-\sigma}$ for $|x| \leq e^{-1}$ and $V(x) = 0$ for $|x| > e^{-1}$. Then $\sigma > 1$ implies $V \in L^1(\mathbb{R}^d)$. Since $\sigma < d$ we can choose a $\rho \in [(\sigma-1)/d, (d-1)/d)$ and define $u(x) = |\ln|x||^{\rho}\zeta(x)$, where the function $\zeta \in C_0^\infty(\mathbb{R}^d)$ equals one in a neighborhood of the origin. Then $\rho < (d-1)/d$ implies that $u \in W^{1,d}(\mathbb{R}^d)$, whereas $\rho \geq (\sigma-1)/d$ implies that $\int_{\mathbb{R}^d} V|u|^d\,dx = \infty$. Thus, $Q_{\alpha V}[u] = -\infty$ for any $\alpha > 0$.

Remark 2.4. In the quadratic case $p = 2$, Theorems 2.1 and 2.2 recover the asymptotics originally found in [Si] using a different, operator theoretic approach. Both (2.1) and (2.2) were originally proved in [Si] under more restictive conditions on V. For $d = 1$ these restrictions were later removed in [Kl1, Sec.4]; note also that according to Lemma 3.3 below we have $\mathcal{S}_{1,2} = 1$ for $p = 2$ and $d = 1$.

While our theorems give a complete answer in the case $V \in L^1(\mathbb{R}^d)$ (plus additional assumptions if $p = d$) with $\int_{\mathbb{R}^d} V\,dx > 0$, the following questions, which we consider interesting, remain open:

(1) What happens if $V \in L^1(\mathbb{R}^d)$ (plus some additional assumptions), but $\int_{\mathbb{R}^d} V\,dx = 0$? For results in the case $p = 2$, see [Si, Kl1, BCEZ].
(2) What happens if $V \notin L^1(\mathbb{R}^d)$, but $V(x) = |x|^{-\sigma}(1+o(1))$ as $|x| \to \infty$ with $0 < \sigma \leq d$? For results in the case $p = 2$, see [Kl2].

The proofs of Theorems 2.1 and 2.2 are given in Sections 3 and 4 respectively.

Notation. Given $r > 0$ and a point $x \in \mathbb{R}^d$ we denote by $B(r,x) \subset \mathbb{R}^d$ the open ball with radius r centred in x. If $x = 0$, then we write B_r instead of $B(r,0)$. Furthermore, given a set $\Omega \subset \mathbb{R}^d$ we denote by Ω^c its complement in $\mathbb{R}^d$. The L^q norm of a function u in Ω will be denoted by $\|u\|_{L^q(\Omega)}$ if $\Omega \neq \mathbb{R}^d$ and by $\|u\|_q$ if $\Omega = \mathbb{R}^d$.

3. Case $d < p$

Before we proceed with the proof of Theorem 2.1 we give some preliminary results concerning Sobolev inequality (1.5) and the properties of the functional $Q_V[u]$.

3.1. **Sobolev inequality.** We recall that $\mathcal{S}_{d,p}$ denotes the optimal constant in the Sobolev interpolation inequality (1.5). In this subsection we discuss a closely related (and, in fact, equivalent, as we shall show) minimization problem which depends on a parameter $v > 0$ in addition to an exponent $q > d \geq 1$. We define

$$E(v) = \inf_{\|u\|_p = 1} \left(\|\nabla u\|_p^p - v|u(0)|^p \right). \tag{3.1}$$

(Note that by the Sobolev embedding theorem any function in $W^{1,q}(\mathbb{R}^d)$, $q > d$, has a continuous representative and therefore $u(0)$ is unambiguously defined. The following lemma shows, in particular, that $E(v) > -\infty$.

Lemma 3.1. *Let $p > d \geq 1$ and $v > 0$. Then*

$$E(v) = -\frac{p-d}{p} \left(\frac{d}{p}\right)^{\frac{d}{p-d}} (\mathcal{S}_{d,p} v)^{\frac{p}{p-d}} .$$

Moreover, the infimum is attained by a non-negative, symmetric decreasing function. Finally, any minimizing sequence is relatively compact in $W^{1,p}(\mathbb{R}^d)$.

We include a proof of this lemma for the sake of completeness.

Proof. By the Sobolev inequality (1.5) we have

$$|u(0)|^p \leq \|u\|_\infty^p \leq \mathcal{S}_{d,p} \|\nabla u\|_p^d \|u\|_p^{p-d}$$

and, therefore, if $\|u\|_p = 1$,

$$\begin{aligned} \|\nabla u\|_p^p - v|u(0)|^p \geq \|\nabla u\|_p^p - v\mathcal{S}_{d,p}\|\nabla u\|_p^d &\geq \inf_{X\geq 0} \left(X^p - v\mathcal{S}_{d,p} X^d \right) \\ &= -\frac{p-d}{p} \left(\frac{d}{p}\right)^{\frac{d}{p-d}} (\mathcal{S}_{d,p} v)^{\frac{p}{p-d}} \,. \end{aligned}$$

This shows that $E(v) \geq -\frac{p-d}{p} \left(\frac{d}{p}\right)^{\frac{d}{p-d}} (\mathcal{S}_{d,p} v)^{\frac{p}{p-d}}$. In particular, $E(v) > -\infty$.

To prove the reverse inequality, we first note that, by scaling,

$$E(v) = E(1)\, v^{\frac{p}{p-d}} \,.$$

(To see this, write u in the form $u(x) = v^{\frac{d}{p(p-d)}} w(v^{\frac{1}{p-d}} x)$.) We note also that $E(v) < 0$. (Indeed, for a fixed $u \in W^{1,p}(\mathbb{R}^d)$ with $\|u\|_p = 1$ and $u(0) \neq 0$ we clearly have $\|\nabla u\|_p^p - v|u(0)|^p \to -\infty$ as $v \to \infty$ and therefore $E(v) < 0$ for all sufficiently large v. By the scaling law, this implies that $E(v) < 0$ for any v.)

Now let $u \in W^{1,p}(\mathbb{R}^d)$. Then, by the Sobolev embedding theorem u can be assumed to be continuous and vanishing at infinity, so there is an $a \in \mathbb{R}^d$ such that $|u(a)| = \|u\|_\infty$. Let $\tilde{u}(x) = u(x+a)/\|u\|_p$. Then, by the definition of $E(v)$,

$$\|\nabla \tilde{u}\|_p^p - v|\tilde{u}(0)|^p \geq E(v) \,,$$

i.e.,

$$\|\nabla u\|_p^p \geq v\|u\|_\infty^p + E(v)\|u\|_p^p = v\|u\|_\infty^p + E(1)\ v^{\frac{p}{p-d}} \|u\|_p^p \,.$$

Since this is true for any $v > 0$ we have

$$\begin{aligned} \|\nabla u\|_p^p \geq v\|u\|_\infty^p + E(v)\|u\|_p^p &\geq \sup_{v>0} \left(v\|u\|_\infty^p + E(1)\ v^{\frac{p}{p-d}} \|u\|_p^p \right) \\ &= \|u\|_\infty^{\frac{p^2}{d}} \|u\|_p^{-\frac{p(p-d)}{d}} |E(1)|^{-\frac{p-d}{d}} \left(\frac{p-d}{p}\right)^{\frac{p-d}{d}} \frac{d}{p} \,. \end{aligned}$$

This proves that $\mathcal{S}_{d,p} \leq |E(1)|^{\frac{p-d}{p}} \left(\frac{p-d}{p}\right)^{-\frac{p-d}{p}} \left(\frac{d}{p}\right)^{\frac{d}{p}}$.

We next prove that any minimizing sequence is relatively compact in $W^{1,p}(\mathbb{R}^d)$. Let $(u_n) \subset W^{1,p}(\mathbb{R}^d)$ be a minimizing sequence for $E(v)$. Using the bounds in the first part of the proof it is easy to see that (u_n) is bounded in $W^{1,p}(\mathbb{R}^d)$ and therefore, after passing to a subsequence if necessary, we may assume that u_n converges weakly in $W^{1,p}(\mathbb{R}^d)$ to some $u \in W^{1,p}(\mathbb{R}^d)$. By weak convergence,

$$\liminf_{n\to\infty} \|\nabla u_n\|_p^p \geq \|\nabla u\|_p^p \,, \qquad 1 \geq \liminf_{n\to\infty} \|u_n\|_p^p \geq \|u\|_p^p \,, \tag{3.2}$$

and, by the Rellich–Kondrashov theorem (see, e.g., [LL, Thm. 8.9]), $u_n(0) \to u(0)$. We conclude that

$$0 > E(v) = \lim_{n\to\infty} \left(\|\nabla u_n\|_p^p - v|u_n(0)|^p \right) \geq \|\nabla u\|_p^p - v|u(0)|^p \geq E(v)\|u\|_p^p \,.$$

This, together with the second assertion in (3.2) implies that $\|u\|_p = 1$. Together with the first assertion in (3.2) and the convergence of $u_n(0)$ it also implies that $\|\nabla u_n\|_p \to \|\nabla u\|_p$. Thus, u_n converges in fact strongly to u in $W^{1,p}(\mathbb{R}^d)$.

Thus, we have shown that there is a minimizer. In view of the rearrangement inequalities $\|\nabla u^*\|_p \leq \|\nabla u\|_p$, $\|u^*\|_p = \|u\|_p$ and $|u^*(0)| \geq |u(0)|$ (see, e.g., [Ta] and [LL, Thm. 3.4]) we see that among the minimizers there is a non-negative, symmetric decreasing function. This concludes the proof. □

Remark 3.2. It is easy to see that

$$E(v) = \inf_{\|u\|_p=1} \left(\|\nabla u\|_p^p - v\|u\|_\infty^p \right).$$

This will be useful in the following.

In one dimension we can compute the value of the sharp constant $\mathcal{S}_{d,p}$ in (1.5).

Lemma 3.3. *If $d = 1$, then $\mathcal{S}_{1,p} = \frac{p}{2}$ for any $p > 1$.*

Proof. Let u be the (symmetric decreasing) optimizer for $E(v)$. The Euler–Lagrange equation reads

$$(p-1)\, u''(x)\,(-u'(x))^{p-2} = \lambda u(x)^{p-1} \qquad \text{in } (0,\infty)\,, \tag{3.3}$$

together with the boundary condition

$$2(-u'(0+))^{p-1} = v u(0)^{p-1}\,.$$

Multiplying (3.3) by u' we obtain

$$\left((p-1)(-u')^p - \lambda u^p\right)' = 0 \qquad \text{in } (0,\infty)\,.$$

Since $u \in W^{1,p}(\mathbb{R}^d)$ we have $u(x) \to 0$ as $x \to \infty$. Since $(p-1)(-u')^p - \lambda u^p$ is constant, $\lim_{x\to\infty} u'(x)$ exists as well and, therefore, needs to be zero. Thus

$$(p-1)(-u')^p - \lambda u^p = 0 \qquad \text{in } (0,\infty)\,. \tag{3.4}$$

Note that this shows that $\lambda > 0$. Moreover, we obtain

$$-u' = \left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}} u \qquad \text{in } (0,\infty)\,,$$

and, thus,

$$u(x) = u(0)\exp\left(-\left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}} x\right) \qquad \text{in } (0,\infty)\,.$$

The boundary condition implies that $\lambda = (p-1)(v/2)^{p/(p-1)}$. We conclude that

$$E(v) = \frac{2\int_0^\infty |u'|^p\,dx - v u(0)^p}{2\int_0^\infty u^p\,dx} = -(p-1)\left(\frac{v}{2}\right)^{\frac{p}{p-1}}.$$

By Lemma 3.1 this implies the assertion. □

3.2. **Preliminaries.**

Lemma 3.4. *Let $p > d$ and assume that $V \in L^1(\mathbb{R}^d)$. Then for any $u \in W^{1,p}(\mathbb{R}^d)$,*

$$Q_V[u] \,\geq\, -\frac{p-d}{p}\, \left(\frac{d}{p}\right)^{\frac{d}{p-d}}\, \left(\mathcal{S}_{d,p} \int_{\mathbb{R}^d} V_+ \, dx\right)^{\frac{p}{p-d}}\, \|u\|_p^p. \tag{3.5}$$

Moreover, $Q_V[u]$ is weakly lower semi-continuous in $W^{1,p}(\mathbb{R}^d)$.

Proof. For any $u \in W^{1,p}(\mathbb{R}^d)$,

$$Q_V[u] \geq \|\nabla u\|_p^p - \int_{\mathbb{R}^d} V_+ \, dx \, \|u\|_\infty^p \geq E\left(\int_{\mathbb{R}^d} V_+ \, dx\right).$$

The second inequality used Remark 3.2. The first assertion now follows from Lemma 3.1.

To prove weak lower semi-continuity assume that (u_j) converges weakly in $W^{1,p}(\mathbb{R}^d)$ to some u. Then the sequence (u_j) is bounded in $W^{1,p}(\mathbb{R}^d)$ and hence, by (1.5), in $L^\infty(\mathbb{R}^d)$. We have

$$\Big| \int_{\mathbb{R}^d} V(|u_j|^p - |u|^p)\, dx \Big| \leq \|u_j - u\|_{L^\infty(B_R)} \|f_j\|_\infty \|V\|_1 + 2\left(\sup_j \|u_j\|_\infty^p\right) \|V\|_{L^1(B_R^c)}, \tag{3.6}$$

where $f_j := (|u_j|^p - |u|^p)/(|u_j| - |u|)$ satisfies $|f_j| \leq p \max\{|u_j|^{p-1}, |u|^{p-1}\}$ and is therefore bounded. Since the sequence (u_j) is bounded in $W^{1,p}(\mathbb{R}^d)$, inequality (1.5) implies that $\|f_j\|_\infty$ is bounded uniformly with respect to j. On the other hand, the Rellich-Kondrashov theorem (see, e.g., [LL, Thm.8.9]) says that (u_j) converges to u uniformly on compact subsets of $\mathbb{R}^d$. Hence, sending first $j \to \infty$ and then $R \to \infty$ in (3.6) shows that the functional $\int_{\mathbb{R}^d} V\, |u|^p\, dx$ is weakly continuous on $W^{1,p}(\mathbb{R}^d)$. Since $\|\nabla u\|_p^p$ is weakly lower semi-continuous, due to the fact that $p > 1$, the same is true for $Q_V[u]$. □

Remark 3.5. Note that inequality (3.5) yields the lower bound in (2.1) in the case $V \geq 0$.

Corollary 3.6. *Let $V \in L^1(\mathbb{R}^d)$ and $p > d$. Assume that $\lambda(V) < 0$. Then there is a non-negative function $u \in W^{1,p}(\mathbb{R}^d)$ such that*

$$\lambda(V) = \frac{Q_V[u]}{\|u\|_p^p}. \tag{3.7}$$

Proof. Let (u_j) be a minimizing sequence for Q_V, normalized such that $\|u_j\|_p = 1$ for any $j \in \mathbb{N}$. Since $\lambda(V) < 0$, we may assume without loss of generality that $Q_V[u_j] < 0$ for any $j \in \mathbb{N}$. Hence with the help of (1.5) we get

$$\|\nabla u_j\|_p^p < \int_{\mathbb{R}^d} V_+ \, |u_j|^p \, dx \,\leq\, \|V_+\|_1 \, \|u_j\|_\infty^p \leq \mathcal{S}_{d,p} \, \|V_+\|_1 \, \|\nabla u_j\|_p^d. \tag{3.8}$$

Since $p > d$, it follows that the sequence (u_j) is bounded in $W^{1,p}(\mathbb{R}^d)$ and, after passing to a subsequence if necessary, we may assume that (u_j) converges weakly in $W^{1,p}(\mathbb{R}^d)$ to some $u \in W^{1,p}(\mathbb{R}^d)$. The weak convergence implies

$$\|u\|_p \leq \liminf_{j\to\infty} \|u_j\|_p = 1.$$

Since $Q_V[u]$ is weakly lower semicontinuous by Lemma 3.4, the above inequality implies

$$0 > \lambda(V) = \lim_{j\to\infty} Q_V[u_j] \geq Q_V[u] \geq \lambda(V)\, \|u\|_p^p \geq \lambda(V).$$

This implies that $Q_V[u] = \lambda(V)$ and $\|u\|_p = 1$, i.e., u is a minimizer for the problem (1.2).

Since $u \in W^{1,p}(\mathbb{R}^d)$ implies $|u| \in W^{1,p}(\mathbb{R}^d)$ with $|\nabla|u|| = |\nabla u|$ almost everywhere (see, e.g., [LL, Thm. 6.17]), we may choose u non-negative. □

3.3. **Proof of Theorem 2.1. Upper bound.** For any fixed function $\varphi \in W^{1,p}(\mathbb{R}^d)$ with $\|\varphi\|_p = 1$ we define

$$v_\alpha(x) := \alpha^{\frac{d}{p(p-d)}}\, \varphi(\alpha^{\frac{1}{p-d}} x), \quad \alpha > 0\,.$$

Then $\|v_\alpha\|_p = 1$ for all $\alpha > 0$ and

$$\lambda(\alpha V) \leq Q_{\alpha V}[v_\alpha] = \alpha^{\frac{p}{p-d}} \Big(\|\nabla \varphi\|_p^p - \int_{\mathbb{R}^d} V(x) |\varphi(\alpha^{\frac{1}{p-d}} x)|^p \, dx \Big)\,.$$

Since $\varphi \in W^{1,p}(\mathbb{R}^d)$, the Sobolev embedding implies that $\varphi \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and therefore, by dominated convergence,

$$\int_{\mathbb{R}^d} V(x) |\varphi(\alpha^{\frac{1}{p-d}} x)|^p \, dx \to \int_{\mathbb{R}^d} V \, dx \; |\varphi(0)|^p \qquad \text{as } \alpha \to 0\,.$$

Since φ is arbitrary, we have shown that

$$\limsup_{\alpha \to 0+} \alpha^{\frac{p}{d-p}} \lambda(\alpha V) = \inf_{\|\varphi\|_p = 1} \Big(\|\nabla \varphi\|_p^p - \int_{\mathbb{R}^d} V \, dx \; |\varphi(0)|^p \Big) = E \Big(\int_{\mathbb{R}^d} V \, dx \Big)\,.$$

The upper bound in Theorem 2.1 now follows from Lemma 3.1.

3.4. **Proof of Theorem 2.1. Lower bound.** It follows from the proof of the upper bound that $\lambda(\alpha V) < 0$ for all sufficiently small $\alpha > 0$ and hence, by Corollary 3.6, for all such α there is a non-negative minimizer u_α of the problem (1.2). (It is easy to see that, in fact, $\lambda(\alpha V) < 0$ for *all* $\alpha > 0$. Indeed, $\alpha^{-1} Q_{\alpha V}[u]$ is non-increasing for every $u \in W^{1,p}(\mathbb{R}^d)$ and therefore $\alpha^{-1}\lambda(\alpha V)$ is non-increasing. Thus, if it is negative for some $\alpha > 0$, it is negative for all larger α's.)

We normalize u_α so that $\|u_\alpha\|_p = 1$. The key step in the proof is to show that

$$\lim_{\alpha \to 0+} \alpha^{-\frac{d}{p-d}} \int_{\mathbb{R}^d} V(x) \, \left(u_\alpha(x)^p - u_\alpha(0)^p\right) \, dx = 0\,. \tag{3.9}$$

Assuming this for the moment, let us complete the proof. We define

$$f_\alpha(x) = \alpha^{-\frac{d}{p(p-d)}} \, u_\alpha\Big(x \, \alpha^{-\frac{1}{p-d}} \Big) \tag{3.10}$$

and observe that $\|f_\alpha\|_p = 1$ and

$$\|\nabla f_\alpha\|_p^p - \int_{\mathbb{R}^d} V_\alpha(x) \, f_\alpha(x)^p \, dx = \alpha^{-\frac{p}{p-d}} \; Q_{\alpha V}[u_\alpha]\,,$$

where $V_\alpha(x) = \alpha^{-d/(p-d)} V(x \, \alpha^{-1/(p-d)})$. Since (3.9) can be rewritten as

$$\lim_{\alpha \to 0} \Big(\int_{\mathbb{R}^d} V_\alpha(x) \, f_\alpha(x)^p \, dx - \int_{\mathbb{R}^d} V \, dx \; f_\alpha(0)^p \Big) = 0\,,$$

we obtain

$$
\begin{aligned}
\liminf_{\alpha\to 0+} \alpha^{-\frac{p}{p-d}}\ \lambda(\alpha V) &= \liminf_{\alpha\to 0+} \alpha^{-\frac{p}{p-d}}\ Q_{\alpha V}[u_\alpha] \\
&= \liminf_{\alpha\to 0+} \left(\|\nabla f_\alpha\|_p^p - \int_{\mathbb{R}^d} V\,dx\, f_\alpha(0)^p \right) \\
&\geq E\left(\int_{\mathbb{R}^d} V\,dx \right) \\
&= -\frac{p-d}{p} \left(\frac{d}{p}\right)^{\frac{d}{p-d}} \left(\mathcal{S}_{d,p} \int_{\mathbb{R}^d} V(x)\,dx \right)^{\frac{p}{p-d}} . \qquad (3.11)
\end{aligned}
$$

The last equality comes from Lemma 3.1. This is the lower bound claimed in Theorem 2.1.

It remains to prove (3.9). Arguing as in (3.8) we obtain $\|\nabla u_\alpha\|_p^p \leq \alpha \mathcal{S}_{d,p} \|V_+\|_1 \|\nabla u_\alpha\|_p^d$, and therefore

$$
\|\nabla u_\alpha\|_p \leq C \alpha^{\frac{1}{p-d}} . \qquad (3.12)
$$

According to (1.5) this also implies

$$
\|u_\alpha\|_\infty^p \leq C' \alpha^{\frac{d}{p-d}} . \qquad (3.13)
$$

By Morrey's Sobolev inequality there is a constant $\mathcal{M} = \mathcal{M}_{d,p}$ such that for all $v \in W^{1,p}(\mathbb{R}^d)$ and all $x, y \in \mathbb{R}^d$ one has

$$
|v(x) - v(y)| \leq \mathcal{M} |x-y|^{(p-d)/p} \|\nabla v\|_p . \qquad (3.14)
$$

We now fix $R > 0$ and use Morrey's inequality (3.14) together with (3.12) to get for all $x \in B_R$

$$
|u_\alpha(x) - u_\alpha(0)| \leq \mathcal{M} R^{\frac{p-d}{p}} \|\nabla u_\alpha\|_p \leq C_R\ \alpha^{\frac{1}{p-d}}
$$

This, together with (3.13), yields for all $x \in B_R$

$$
|u_\alpha(x)^p - u_\alpha(0)^p| \leq p\,|u_\alpha(x) - u_\alpha(0)|\ \max\{u_\alpha(x)^{p-1}, u_\alpha(0)^{p-1}\} \leq C_R'\ \alpha^{\frac{p+d(p-1)}{p(p-d)}}
$$

Thus,

$$
\begin{aligned}
&\alpha^{-\frac{d}{p-d}} \left| \int_{\mathbb{R}^d} V(x)\ (u_\alpha(x)^p - u_\alpha(0)^p)\ dx \right| \\
&\quad \leq \alpha^{-\frac{d}{p-d}} \|V\|_1 \sup_{B_R} |u_\alpha^p - u_\alpha(0)^p| + \alpha^{-\frac{d}{p-d}}\, 2\, \|u_\alpha\|_\infty^p \int_{B_R^c} |V|\,dx \\
&\quad \leq \alpha^{\frac{1}{p}} C_R' \|V\|_1 + 2C' \int_{B_R^c} |V|\,dx .
\end{aligned}
$$

Letting first $\alpha \to 0$ and then $R \to \infty$ we obtain (3.9). This completes the proof.

3.5. **Convergence of minimizers.** The following theorem about the behavior of the u_α is an (almost) immediate consequence of Lemma 3.1 and Theorem 2.1 and its proof.

Proposition 3.7. *Let $p > d$ and let $V \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} V(x)dx > 0$. For $\alpha > 0$ let u_α be a non-negative minimizer of $Q_{\alpha V}[\cdot]$ with $\|u_\alpha\|_p = 1$ and define f_α by (3.10). Then for any sequence $(\alpha_n) \subset (0,\infty)$ converging to zero there is a subsequence (α_{n_k}) and an $f_0 \in W^{1,p}(\mathbb{R}^d)$ such that $f_{\alpha_{n_k}} \to f_0$ in $W^{1,p}(\mathbb{R}^d)$. Moreover, f_0 is a minimizer of (3.1) with $v = \int_{\mathbb{R}^d} V\,dx$.*

We recall that, by the Sobolev embedding theorem and the Rellich–Kondrachov theorem, convergence in $W^{1,p}(\mathbb{R}^d)$ for $p > d$ implies convergence in $L^\infty(\mathbb{R}^d)$ and in $C^{0,(p-d)/p}(\mathbb{R}^d)$.

We also note that if the minimizer of the Sobolev inequality (1.5) is unique (up to translations, dilations and multiplication by constants), then Proposition 3.7 implies that f_α converges as $\alpha \to 0$ (without passing to a subsequence).

Proof. It follows from (3.11) together with the upper bound in Theorem 2.1 that (f_α) is a minimizing sequence for problem (3.1) with $v = \int_{\mathbb{R}^d} V\,dx$. Therefore, the assertion follows from the relative compactness asserted in Lemma 3.1. □

4. Case $d = p$

Throughout this section we suppose that $p = d$. Similarly as in the case $d < p$ we start with a couple of preliminary lemmas which which will be used to ensure existence of a minimizer of problem (1.2).

4.1. **Preliminary results.**

Lemma 4.1. *Assume that $V \in L^q(\mathbb{R}^d)$ with some $q > 1$. Then $Q_V[u]/\|u\|_d^d$ is bounded from below and $Q_V[\cdot]$ is weakly lower semi-continuous in $W^{1,p}(\mathbb{R}^d)$.*

Recall that by Sobolev inequalities, see, e.g., [Ad], for every $r \in [d, \infty)$ there is a constant $\tilde{\mathcal{S}}_{d,r}$ such that

$$\|u\|_r \leq \tilde{\mathcal{S}}_{d,r}\, \|\nabla u\|_d^\theta\, \|u\|_d^{1-\theta}\,, \qquad \text{for all } u \in W^{1,d}(\mathbb{R}^d)\,. \tag{4.1}$$

Here $0 \leq \theta < 1$ is defined by $\frac{d}{r} = 1 - \theta$.

Proof. Hölder's inequality and (4.1) with $r = dq/(q-1)$ imply that

$$\int_{\mathbb{R}^d} V|u|^d\,dx \leq \|V_+\|_q \|u\|_r^d \leq \|V_+\|_q\, \tilde{\mathcal{S}}_{d,r}\, \|\nabla u\|_d^{d\theta}\, \|u\|_d^{d(1-\theta)}\,.$$

Thus,

$$\begin{aligned} Q_V[u] &\geq \|\nabla u\|_d^d - \|V_+\|_q\, \tilde{\mathcal{S}}_{d,r}\, \|\nabla u\|_d^{d\theta}\, \|u\|_d^{d(1-\theta)} \\ &\geq \inf_{X \geq 0} \Big(X - \|V_+\|_q\, \tilde{\mathcal{S}}_{d,r}\, X^\theta\, \|u\|_d^{d(1-\theta)} \Big) \\ &\geq -C\, \|V_+\|_q^{\frac{1}{1-\theta}}\, \|u\|_d^d \end{aligned}$$

where $C > 0$ depends only on d and q (through r). This proves lower boundedness.

Now let us prove weak lower semi-continuity of $Q_V[u]$. As in the proof of Lemma 3.4 it suffices to show that $\int_{\mathbb{R}^d} V|u|^p\,dx$ is weakly continuous on $W^{1,d}(\mathbb{R}^d)$. Assume that (u_j) converges weakly in $W^{1,d}(\mathbb{R}^d)$ to some u. Given $\delta > 0$ define $\Omega_\delta = \{x \in \mathbb{R}^d : |V(x)| > \delta\}$. Since (u_j) is bounded in $L^d(\mathbb{R}^d)$, we have

$$\Big| \int_{\Omega_\delta^c} V(|u|^d - |u_j|^d)\,dx \Big| \leq C\,\delta \tag{4.2}$$

with C independent of j. Moreover, the Sobolev inequality (4.1) implies that u_j is uniformly bounded in $L^r(\mathbb{R}^d)$ for every $r \in [d,\infty)$. Hence by Hölder inequality

$$\Big|\int_{\Omega_\delta} V(|u|^d - |u_j|^d)\,dx\Big| \le \|V\|_q \left(\int_{\Omega_\delta} ||u|^d - |u_j|^d|^{\frac{q}{q-1}}\,dx\right)^{\frac{q-1}{q}}$$
$$= \|V\|_q \left(\int_{\Omega_\delta} |(|u| - |u_j|)\,\varphi_j|^{\frac{q}{q-1}}\,dx\right)^{\frac{q-1}{q}},$$

where for every $r \in [d,\infty)$ there is a C_r such that $\|\varphi_j\|_r \le C_r$ for all j. Since Ω_δ has finite measure, $u_j \to u$ in $L^r(\Omega_\delta)$ for any $r < \infty$ by the Rellich–Kondrashov theorem. (For instance, in [LL, Thm. 8.9], the Rellich–Kondrashov theorem is only stated for bounded sets. However, for any $\varepsilon > 0$ we can find a bounded set $\omega \subset \Omega_\delta$ such that $|\Omega_\delta \setminus \omega| < \varepsilon$. Then $u_j \to u$ in $L^r(\omega)$ by the bounded Rellich-Kondrashov theorem and, since (u_j) is bounded in $L^s(\Omega_\delta)$ for some $s > r$, by Hölder $\|u_j\|_{L^r(\Omega_\delta\setminus\omega)} \le \|u_j\|_{L^s(\Omega_\delta)}\varepsilon^{(s-r)/s}$. Thus, $u_j \to u$ in $L^r(\Omega_\delta)$, as claimed.)

We thus conclude, again with $r = 2q/(q-1)$, that

$$\int_{\Omega_\delta} |(|u| - |u_j|)\,\varphi_j|^{\frac{q}{q-1}}\,dx \;\le\; C_r^{\frac{q}{q-1}} \left(\int_{\Omega_\delta} |u - u_j|^{\frac{2q}{q-1}}\,dx\right)^{1/2} \to 0 \quad \text{as } j \to \infty.$$

This in combination with (4.2) proves the claimed weak continuity. □

4.2. **Proof of Theorem 2.2. Upper bound.**

Proposition 4.2. *Let $V \in L^1(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} V(x)\,dx > 0$. Then*

$$\limsup_{\alpha\to0+} \alpha^{\frac{1}{d-1}} \log\frac{1}{|\lambda(\alpha V)|} \;\le\; d\,\omega_d^{\frac{1}{d-1}} \Big(\int_{\mathbb{R}^d} V(x)\,dx\Big)^{-\frac{1}{d-1}}. \tag{4.3}$$

Proof. Let $\beta > 1$ and consider the family of test functions v_β defined by

$$v_\beta(x) = 1 \quad \text{if} \quad |x| \le 1, \qquad v_\beta(x) = \left(1 - \frac{\log|x|}{\log\beta}\right)_+ \quad \text{if } |x| > 1\,. \tag{4.4}$$

Then $v_\beta \in W^{1,d}(\mathbb{R}^d)$ and, since $0 \le v_\beta \le \chi_{\{|\cdot|<\beta\}}$, we have

$$\|v_\beta\|_d^d \;\le\; c\,\beta^d$$

for all $\beta > 1$ with a constant $c > 0$ depending only on d. Moreover,

$$Q_{\alpha V}[v_\beta] \le \omega_d\,(\log\beta)^{1-d} - \alpha \int_{\mathbb{R}^d} V(x)\,dx + \alpha R_\beta$$

with

$$R_\beta = \int_{\{|x|>1\}} V_+ \left(1 - \left(1 - \frac{\log|x|}{\log\beta}\right)_+\right) dx\,.$$

By dominated convergence, $R_\beta \to 0$ as $\beta \to \infty$.

Let $\varepsilon > 0$ be given and choose $\beta_\varepsilon > 1$ such that

$$R_\beta \le \varepsilon \int_{\mathbb{R}^d} V\,dx \qquad \text{for all } \beta \ge \beta_\varepsilon\,.$$

Now, for any α, define

$$\beta(\alpha) = \exp\left(\left(\frac{\omega_d}{\alpha(1-\varepsilon)\int_{\mathbb{R}^d} V\,dx}\right)^{1/(d-1)}\right).$$

Note that $\beta(\alpha) > 1$ and that

$$\frac{\omega_d}{(\log\beta(\alpha))^{d-1}} - \alpha(1-\varepsilon)\int_{\mathbb{R}^d} V\,dx = 0\,.$$

Define $\alpha_\varepsilon > 0$ by $\beta(\alpha_\varepsilon) = \beta_\varepsilon$. Then for $\alpha \le \alpha_\varepsilon$ our upper bound on $Q_{\alpha V}[v_\beta]$ is non-positive and therefore

$$\begin{aligned}
\lambda(\alpha V) &\le \frac{Q_{\alpha V}[v_{\beta(\alpha)}]}{\|u_{\beta(\alpha)}\|_d^d} \\
&\le c^{-1}\,\beta(\alpha)^{-d}\left(\omega_d\,(\log\beta(\alpha))^{1-d} - \alpha\int_{\mathbb{R}^d} V(x)\,dx + \alpha R_\beta\right) \\
&= -c^{-1}\alpha\left(\varepsilon\int_{\mathbb{R}^d} V\,dx - R_{\beta(\alpha)}\right)\exp\left(-d\left(\frac{\omega_d}{\alpha(1-\varepsilon)\int_{\mathbb{R}^d} V\,dx}\right)^{1/(d-1)}\right).
\end{aligned} \tag{4.5}$$

This implies

$$\limsup_{\alpha\to 0+} \alpha^{\frac{1}{d-1}}\,\log\frac{1}{|\lambda(\alpha V)|} \le d\,\omega_d^{\frac{1}{d-1}}\left((1-\varepsilon)\int_{\mathbb{R}^d} V(x)\,dx\right)^{-\frac{1}{d-1}}.$$

By letting $\varepsilon \to 0$ we arrive at (4.3). □

Corollary 4.3. *Let V satisfy assumptions of Lemma 4.1. Then for every $\alpha > 0$ there exists a locally bounded positive function $u_\alpha \in W^{1,d}(\mathbb{R}^d)$ such that $\lambda(\alpha V)\,\|u_\alpha\|_d^d = Q_{\alpha V}[u_\alpha]$.*

Proof. Inequality (4.5) with β large enough shows that $\lambda(\alpha V) < 0$ for all $\alpha > 0$. Hence the existence of a non-negative minimizer u_α follows from Lemma 4.1 in the same way as in the case $d < p$. Since u_α is a non-negative weak solution of (1.3), the Harnack inequality [S1, Thm. 6] implies that u_α is locally bounded and positive. □

4.3. **Proof of Theorem 2.2. Lower bound.**

The case of positive V.

Proposition 4.4. *Assume that $0 \le V \in L^q(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ for some $q > 1$ with $V \not\equiv 0$. Then there are $\alpha_0 > 0$ and $C > 0$ such that for all $0 < \alpha \le \alpha_0$ we have*

$$\lambda(\alpha V) \ge -C\,\alpha^{-1}\,\exp\left[-\left(\frac{d^{d-1}\,\omega_d}{\alpha\int_{\mathbb{R}^d} V\,dx}\right)^{\frac{1}{d-1}}\right]. \tag{4.6}$$

Proof. Let V^* be the symmetric decreasing rearrangement of V. Since $\int_{\mathbb{R}^d} V\,dx = \int_{\mathbb{R}^d} V^*\,dx$, $\int_{\mathbb{R}^d} V^q\,dx = \int_{\mathbb{R}^d} (V^*)^q\,dx$ and, by rearrangement inequalities (see, e.g., [Ta] and [LL, Thm. 3.4]),

$$\lambda(\alpha V) \ge \lambda(\alpha V^*)\,,$$

we may and will assume in the following that $V = V^*$.

By Corollary 4.3 there is a minimizer u_α of $Q_{\alpha V}[u]/\|u\|_d^d$. Again, by rearrangement inequalities, we may assume that u_α is a radially symmetric function which is non-increasing with respect to the radius. Let $\rho > 0$ be an arbitrary parameter. (In this proof there is no loss in assuming that $\rho = 1$, but in the proof of Proposition 4.5 we will repeat the argument with a general ρ.) We normalize u_α such that

$$u_\alpha(x) = u_\alpha(|x|) = 1, \qquad \text{for all } x \in \mathbb{R}^d \text{ with } |x| = \rho\,.$$

Let $R \geq 2\rho$ be a parameter to be specified later and let χ be defined by

$$\chi(r) = 1 \quad \text{if} \ \ 0 \leq r \leq \rho, \qquad \chi(r) = \Big(1 - \frac{r-\rho}{R-\rho}\Big)_+ \quad \text{if} \ \ r > \rho\,.$$

Then for any $\varepsilon \in (0,1]$ we have

$$\begin{aligned} \|\nabla(\chi\, u_\alpha)\|_d^d &\leq (1+\varepsilon)\|\chi\, \nabla u_\alpha\|_d^d + c\, \varepsilon^{1-d}\, \|u_\alpha\, \nabla \chi\|_d^d \\ &\leq (1+\varepsilon)\|\nabla u_\alpha\|_d^d + c'\, \varepsilon^{1-d}\, R^{-d}\, \|u_\alpha\|_d^d\,, \end{aligned}$$

and therefore

$$\|\nabla u_\alpha\|_d^d \geq \|\nabla(\chi\, u_\alpha)\|_d^d/(1+\varepsilon) - c''\, \varepsilon^{1-d}\, R^{-d}\, \|u_\alpha\|_d^d\,. \tag{4.7}$$

Since $\chi\, u_\alpha$ has support in the ball of radius of radius R and is bounded from below by one on the ball of radius ρ, the formula for the capacity of two nested balls [M, Sec. 2.2.4] gives

$$\|\nabla u_\alpha\|_d^d \geq \frac{\omega_d\, (\log(R/\rho))^{1-d}}{1+\varepsilon} - c''\, \varepsilon^{1-d}\, R^{-d}\, \|u_\alpha\|_d^d\,. \tag{4.8}$$

Moreover, since $|u_\alpha(x)| \leq 1$ for $|x| > 1$, we obtain

$$\lambda(\alpha V) \geq \frac{\omega_d\, (\log(R/\rho))^{1-d} - (1+\varepsilon)\, \alpha \left(\int_{B_1} V u_\alpha^d\, dx + \int_{B_1^c} V\, dx\right)}{(1+\varepsilon)\, \|u_\alpha\|_d^d} - \frac{c''}{\varepsilon^{d-1}\, R^d}\,. \tag{4.9}$$

We next claim that there are constants $C >$ and $\alpha_0 > 0$ such that for all $0 < \alpha \leq \alpha_0$,

$$\sup_{B_\rho} \left(u_\alpha^d - 1\right) \leq C\alpha^{\frac{1}{d-1}}\,. \tag{4.10}$$

Accepting this for the moment and returning to (4.9) we obtain

$$\lambda(\alpha V) \geq \frac{\omega_d\, (\log(R/\rho))^{1-d} - (1+\varepsilon)\left(1 + C\alpha^{\frac{1}{d-1}}\right)\, \alpha \int_{\mathbb{R}^d} V\, dx}{(1+\varepsilon)\, \|u_\alpha\|_d^d} - \frac{c''}{\varepsilon^{d-1}\, R^d}\,.$$

For given $0 < \varepsilon \leq 1$ and $0 < \alpha \leq \alpha_0$ we now choose

$$R = \rho \exp\left(\left(\frac{\omega_d}{(1+\varepsilon)\left(1 + C\alpha^{\frac{1}{d-1}}\right)\, \alpha \int_{\mathbb{R}^d} V\, dx}\right)^{\frac{1}{d-1}}\right)$$

so that

$$\lambda(\alpha V) \geq -\frac{c''}{\varepsilon^{d-1}\rho^d}\, \exp\left(-d\left(\frac{\omega_d}{(1+\varepsilon)\left(1 + C\alpha^{\frac{1}{d-1}}\right)\, \alpha \int_{\mathbb{R}^d} V\, dx}\right)^{\frac{1}{d-1}}\right)\,.$$

Finally, we choose $\varepsilon = C\alpha^{\frac{1}{d-1}}$ to obtain

$$\lambda(\alpha V) \geq -\frac{c'''}{\alpha}\, \exp\left(-d\left(\frac{\omega_d}{\left(1 + C'\alpha^{\frac{1}{d-1}}\right)\, \alpha \int_{\mathbb{R}^d} V\, dx}\right)^{\frac{1}{d-1}}\right)\,. \tag{4.11}$$

Up to increasing c''' this implies the statement of the proposition.

Thus, it remains to prove (4.10). For simplicity we give the proof only for $\rho = 1$ (which is enough for the proof of the proposition). We apply Alvino's version of the Moser–Trudinger inequality [Al] to the function $u_\alpha - 1$ and obtain

$$0 \,<\, u_\alpha(r) - 1 \,\leq\, C\, \|\nabla u_\alpha\|_{L^d(B_1)}\, |\log r|^{\frac{d-1}{d}}\,, \qquad r \leq 1. \tag{4.12}$$

Using this upper bound on u_α we arrive at

$$\begin{aligned}
\|\nabla u_\alpha\|^d_{L^d(B_1)} &\leq \|\nabla u_\alpha\|_d^d \\
&\leq \alpha \int_{\mathbb{R}^d} V |u_\alpha|^d \, dx \\
&\leq \alpha 2^{d-1} \left(\|V\|_{L^1(B_1)} + C \|\nabla u_\alpha\|^d_{L^d(B_1)} \omega_d \int_0^1 V(r) |\log r|^{d-1} r^{d-1} \, dr \right).
\end{aligned}$$

The assumption $V \in L^q(\mathbb{R}^d)$ for some $q > 1$ implies that $V \in L^1(B_1, |\log|x||^{d-1}\, dx)$, and therefore there is a $C' > 0$ and an $\alpha_0 > 0$ such that for all $0 < \alpha \leq \alpha_0$

$$\|\nabla u_\alpha\|^d_{L^d(B_1)} \leq C'\, \alpha^{1/d}\,.$$

Re-inserting this into (4.12), we find for all $0 < \alpha \leq \alpha_0$

$$0 \,<\, u_\alpha(r) - 1 \,\leq\, C''\, \alpha^{1/d}\, |\log r|^{\frac{d-1}{d}}\,, \qquad r \leq 1. \tag{4.13}$$

Hence the minimizer u_α satisfies for all $0 < r \leq 1$,

$$\begin{aligned}
((-r\, u_\alpha'(r))^{d-1})' &= \alpha\, V(r)\, u_\alpha(r)^{d-1}\, r^{d-1} + \lambda(\alpha)\, u_\alpha(r)^{d-1} r^{d-1} \\
&\leq \alpha\, V(r)\, r^{d-1} \left(1 + C'' \alpha^{\frac{1}{d}}\, |\log r|^{\frac{d-1}{d}}\right)^{d-1}
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
((-r\, u_\alpha'(r))^{d-1})' &= \alpha\, V(r)\, u_\alpha(r)^{d-1}\, r^{d-1} + \lambda(\alpha)\, u_\alpha(r)^{d-1} r^{d-1} \\
&\geq \lambda(\alpha)\, r^{d-1} \left(1 + C'' \alpha^{\frac{1}{d}}\, |\log r|^{\frac{d-1}{d}}\right)^{d-1}.
\end{aligned} \tag{4.15}$$

Since the right hand sides of (4.14) and (4.15) are integrable with respect to r (for (4.14) we use here again the assumption that $V \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ for some $q > 1$), the function $(-r\, u_\alpha'(r))^{d-1}$ has a finite limit as $r \to 0$. Since $u_\alpha \in W^{1,d}(\mathbb{R}^d)$, it follows that this limit must be zero. Thus, from (4.14) we get for all $0 < r \leq 1$

$$\begin{aligned}
(-r\, u_\alpha'(r))^{d-1} &\leq \alpha \int_0^r V(s)\, s^{d-1} \left(1 + C'' \alpha^{\frac{1}{d}}\, |\log s|^{\frac{d-1}{d}}\right)^{d-1} ds \\
&\leq \alpha \|V\|_{L^q(B_1)} \left(\int_0^r s^{d-1} \left(1 + C'' \alpha^{\frac{1}{d}}\, |\log s|^{\frac{d-1}{d}}\right)^{q'(d-1)} ds \right)^{1/q'} \\
&\leq C''' \alpha \|V\|_{L^q(B_1)} r^{d/q'} \left(1 + |\log r|\right)^{\frac{(d-1)^2}{d}}\,.
\end{aligned}$$

Finally, this implies that

$$\begin{aligned}
u_\alpha(r) - 1 &= - \int_r^1 u_\alpha'(s)\, ds \\
&\leq \left(C''' \alpha \|V\|_{L^q(B_1)}\right)^{\frac{1}{d-1}} \int_r^1 s^{\frac{d}{q'(d-1)}} \left(1 + |\log s|\right)^{\frac{(d-1)}{d}} \frac{ds}{s}\,.
\end{aligned}$$

Since the integral on the right side converges, we have shown (4.10). This completes the proof of the lemma. □

The case of compactly supported V.

Proposition 4.5. *Let V be a function with compact support, $\int_{\mathbb{R}^d} V(x)\,dx > 0$ and $V \in L^q(\mathbb{R}^d)$ for some $q > 1$. Then there are $\alpha_0 > 0$ and $C > 0$ such that for all $0 < \alpha \leq \alpha_0$ we have*

$$\lambda(\alpha V) \geq -\exp\left[-\left(\frac{d^{d-1}\,\omega_d}{\alpha\,\int_{\mathbb{R}^d} V\,dx\,\,(1+C\alpha^{\frac{1}{d}})}\right)^{\frac{1}{d-1}}\right]. \tag{4.16}$$

Similarly as in the case $d < p$ a key ingredient in the proof is to show that minimizers, when suitably normalised, converge locally to a constant function. In the case $d < p$ we deduced this from Morrey's inequality. Here the argument is considerably more complicated and based on Harnack's inequality for quasi-linear equations. We shall prove

Lemma 4.6. *For each $d \in \mathbb{N}$, $q > 1$ and $M > 0$ there are constants $C > 0$ and $\beta \in (0,1)$ with the following property. Let $\rho > 0$ and assume that $W \in L^q_{\rm loc}(\mathbb{R}^d)$ with $W \leq 0$ in $B^c_{5\rho}$ and $\rho^{d-\frac{d}{q}}\|W\|_{L^q(B_{15\rho})} \leq M$. Then, if $u \in W^{1,d}(\mathbb{R}^d)$ is a positive, weak solution of the equation $-\Delta_d(u) = W u^{d-1}$ in $\mathbb{R}^d$ satisfying $\inf_{B_{5\rho}} u \leq 1$ and if $y \in \mathbb{R}^d$ and $r > 0$ are so that $B(3r,y) \subset B_{3\rho}$, we have*

$$\sup_{B(r,y)} u - \inf_{B(r,y)} u \leq C\,\|W\|^{1/d}_{L^q(B_{5\rho})}\,\rho^{1-\frac{1}{q}-\beta}\,r^\beta\,. \tag{4.17}$$

The point of this lemma is that the dependence of W enters explicitly on the right side of (4.17). In our application, we will have $\|W\|_{L^q(B_{5\rho})} \to 0$, and therefore Lemma 4.6 shows that the oscillations of u vanish with an explicit rate.

We recall that u is a weak solution of $-\Delta_d(u) = W|u|^{d-2}u$ in $\mathbb{R}^d$ if

$$\int_{\mathbb{R}^d} |\nabla u|^{d-2}\,\nabla u\cdot\nabla\varphi\,dx = \int_{\mathbb{R}^d} W|u|^{d-2}\,u\,\varphi\,dx \tag{4.18}$$

for any $\varphi \in W^{1,d}(\mathbb{R}^d)$.

The following lemma, whose proof can be found, for instance, in [Mo1, Mo2] or [LU, Lem. 2.4.1], plays a key role in the proof of Lemma 4.6.

Lemma 4.7. *Let $\Omega \subseteq \mathbb{R}^d$ be open and assume that $u \in W^{1,d}(\Omega)$ is such that there are constants $K > 0$ and $\beta > 0$ such that for all $y \in \Omega$ and $r > 0$ with $B(r,y) \subset \Omega$ one has*

$$\int_{B(r,y)} |\nabla u|^d\,dx \,\leq\, K\,r^{\beta d}\,. \tag{4.19}$$

Then for all $y \in \Omega$ and $r > 0$ such that $B(3r/2,y) \subset \Omega$ we have

$$\sup_{B(r/2,y)} u - \inf_{B(r/2,y)} u \,\leq\, \frac{4}{\beta}\left(\frac{K}{\omega_d}\right)^{\frac{1}{d}} r^\beta. \tag{4.20}$$

Proof of Lemma 4.6. By the Harnack inequality [S1, Thm.6] there is a constant C_1, which depends only on d, q and an upper bound on $\rho^{d-\frac{d}{q}}\,\|W\|_{L^q(B_{15\rho})}$ such that

$$\sup_{B_{5\rho}} u \leq\, C_1\,\inf_{B_{5\rho}} u\,.$$

Since $\inf_{B_{5\rho}} u(x) \leq 1$, we conclude that

$$\sup_{B_{5\rho}} u(x) \leq C_1 . \tag{4.21}$$

Our goal is to apply Lemma 4.7 with $\Omega = B_{3\rho}$. We have to verify condition (4.19) for some K and β. First, note that

$$\int_{\mathbb{R}^d} |\nabla u|^d \, dx = \int_{\mathbb{R}^d} W u^d \, dx \leq \int_{B_{5\rho}} W \, u^d \, dx \leq \omega_d^{1-\frac{1}{q}} (5\rho)^{d-\frac{d}{q}} \|W\|_{L^q(B_{5\rho})} \, C_1^d = c_1 \mathcal{N} , \tag{4.22}$$

where we have set $c_1 = \omega_d^{1-\frac{1}{q}} 5^{d-\frac{d}{q}}$ and

$$\mathcal{N} = \rho^{d-\frac{d}{q}} \|W\|_{L^q(B_{5\rho})} \, C_1^d . \tag{4.23}$$

Hence, for any $\beta > 0$, (4.19) holds for any ball $B(r,y) \subset B_{3\rho}$ with $r \geq \rho$ provided we choose the constan K at least as big as $c_1 \mathcal{N} \rho^{-\beta d}$.

Thus, it remains to verify (4.19) for $r < \rho$. Let $0 \leq \zeta \leq 1$ be a radial function with support in $\overline{B_2}$ which is $\equiv 1$ on B_1 and satisfies $|\nabla \zeta| \leq 1$. Let y and s be such that $B(2s,y) \subset B_{5\rho}$. We choose the test function $\varphi(x) = \zeta(|x-y|/s)(u(x)-a)$ in (4.18), where the parameter a will be specified later. This gives the inequality

$$\begin{aligned} \int_{B(s,y)} |\nabla u|^d \, dx &\leq \int_{\mathbb{R}^d} \zeta(|x-y|/s) |\nabla u|^d \, dx \\ &\leq \int_{B(2s,y)} |W| \, u^{d-1} \, |u-a| \, dx + s^{-1} \int_{A(s,y)} |\nabla u|^{d-1} |u-a| \, dx . \end{aligned} \tag{4.24}$$

with $A(s,y) = B(2s,y) \setminus B(s,y)$. Now we set $a = \frac{1}{|A(s,y)|} \int_{A(s,y)} u \, dx$, where $|A(s,y)|$ denotes the Lebesgue measure of $A(s,y)$. By the Hölder and Poincaré inequalities,

$$\begin{aligned} \int_{A(s,y)} |\nabla u|^{d-1} |u-a| \, dx &\leq \Big(\int_{A(s,y)} |\nabla u|^d \, dx \Big)^{\frac{d-1}{d}} \Big(\int_{A(s,y)} |u-a|^d \, dx \Big)^{\frac{1}{d}} \\ &\leq C^{\mathrm{P}} \, s \int_{A(s,y)} |\nabla u|^d \, dx , \end{aligned}$$

where C^{P} is the constant in the Poincaré inequality in $A(1,0)$. By scaling one easily sees that the Poincaré constant in $A(s,y)$ is given by $C^{\mathrm{P}} s$. This fact was used in the previous bound.

Let us bound the first term on the right side of (4.24). Since both u and $|a|$ are bounded from above by C_1 on $B(2s,y)$, see (4.21), we have

$$\int_{B(2s,y)} |W| \, u^{d-1} \, |u-a| \, dx \leq \|W\|_{L^1(B(2s,y))} 2 C_1^p \leq c_2 \, \mathcal{N} \, (s/\rho)^{d-\frac{d}{q}} ,$$

where $c_2 = \omega_d^{1-\frac{1}{q}} \, 2^{d+1-\frac{d}{q}}$.

Thus, (4.24) implies

$$\int_{B(s,y)} |\nabla u|^d \, dx \leq \, c_2 \, \mathcal{N} \, (s/\rho)^{d-\frac{d}{q}} + C^{\mathrm{P}} \int_{A(s,y)} |\nabla u|^d \, dx,$$

where $c_1 = 2^{d+1-\frac{d}{q}}\,\omega_d^{1-\frac{1}{q}}$. Adding $C^{\rm P}\int_{B(s,y)}|\nabla u|^d\,dx$ to both sides of the above inequality we arrive at

$$\int_{B(s,y)}|\nabla u|^d\,dx \le c_3\,\mathcal{N}\,(s/\rho)^{d-\frac{d}{q}}+\kappa\int_{B(2s,y)}|\nabla u|^d\,dx, \tag{4.25}$$

with $c_3 = c_2/(1+C^{\rm P})$ and

$$\kappa = \frac{C^{\rm P}}{1+C^{\rm P}} < 1.$$

To simplify the notation, we introduce the shorthand $D(s) = \int_{B(s,y)}|\nabla u|^d\,dx$. Iterating inequality (4.25) gives

$$D(2^{-n}s) \le c_3\,\mathcal{N}\,(s/\rho)^{d-\frac{d}{q}}\,2^{n(\frac{d}{q}-d)}\sum_{j=0}^{n-1}\big(\kappa\,2^{d-\frac{d}{q}}\big)^j+\kappa^n\,D(s)$$

for all $n\in\mathbb{N}$ and every $s>0$ such that $B(s,y)\subset B_{5\rho}$. Next, we sum the geometric series on the right side and obtain a c_4 and a $\mu<1$ (both depending only on d and q) such that

$$2^{n(\frac{d}{q}-d)}\sum_{j=0}^{n-1}\big(\kappa\,2^{d-\frac{d}{q}}\big)^j \le c_4\,\mu^n \qquad \text{for all } n\in\mathbb{N}\,.$$

Thus, recalling (4.22),

$$D(2^{-n}s) \le \Big(c_3c_4\,(s/\rho)^{d-\frac{d}{q}}+c_1\Big)\,\mathcal{N}\,\max\{\mu^n,\kappa^n\} \tag{4.26}$$

for all $n\in\mathbb{N}$ and all s such that $B(s,y)\subset B_{5\rho}$.

Now let $B(r,y)\subset B_{3\rho}$ with $r<\rho$. There are $k\in\mathbb{N}$ and $t\in[1,2)$ such that $2^{-k-1}\,t\rho < r \le 2^{-k}\,t\rho$. Since $B(t\rho,y)\subset B_{5\rho}$ we may apply inequality (4.26) with $k=n$ and $s=t\rho$ to get

$$\begin{aligned}\int_{B(r,y)}|\nabla u|^d\,dx &\le D(2^{-k}t\rho)\\ &\le \Big(c_3c_4\,t^{d-\frac{d}{q}}+c_1\Big)\,\mathcal{N}\,\max\{\mu^k,\kappa^k\}\\ &\le \Big(c_3c_4\,2^{d-\frac{d}{q}}+c_1\Big)\,\mathcal{N}\left(\frac{2r}{\rho}\right)^{\beta d} \quad \text{with } \beta = -\frac{\log\max\{\mu,\kappa\}}{d\,\log 2}>0\,.\end{aligned}$$

To summarize, we have shown that (4.19) holds for any $B(r,y)\subset B_{3\rho}$ with the above choice of β and with

$$K = \max\Big\{c_1,\Big(c_3c_4\,2^{d-\frac{d}{q}}+c_1\Big)\,2^{\beta d}\Big\}\,\mathcal{N}\rho^{-\beta d}\,.$$

Here c_1, c_3 and c_4 depend only on d and q, and $\mathcal{N}$ was defined in (4.23). In view of Lemma 4.7 this proves (4.17). □

Proof of Proposition 4.5. The beginning of the proof is identical to that of Proposition 4.4. Let $\rho>0$ be such that the support of V is contained in $\overline{B_{5\rho}}$. We let again u_α be a minimizer of $Q_{\alpha V}[u]/\|u\|_d^d$. From Corollary 4.3 we know that u_α can be chosen strictly positive and therefore we may normalize u_α by $\inf_{B_\rho}u_\alpha = 1$. Arguing exactly as before we arrive at the following variant of (4.9),

$$\lambda(\alpha V) \ge \frac{\omega_d\,(\log(R/\rho))^{1-d}-(1+\varepsilon)\,\alpha\int_{\mathbb{R}^d}V\,|u_\alpha|^d\,dx}{(1+\varepsilon)\,\|u_\alpha\|_d^d}-c''\,\varepsilon^{1-d}\,R^{-d}\,. \tag{4.27}$$

We now claim that there is a constant $C>0$ (depending on d, q, V, but not on α) such that

$$|u_\alpha(x)-1| \leq C\,\alpha^{\frac{1}{d}} \qquad \text{for all } x\in B_\rho\,. \tag{4.28}$$

Indeed, this follows from Lemma 4.6 applied to $W=\alpha V+\lambda(\alpha V)$ and $u=u_\alpha$ with $B(r,y)=B_\rho$. Note that we indeed have $\inf_{B_{5\rho}} u_\alpha \leq \inf_{B_\rho} u_\alpha = 1$. Moreover, we use the fact that $\lambda(\alpha V)\geq -C\alpha$, which follows easily from the bounds in Lemma 4.1.

With a similar choice as in Lemma 4.4 for R we obtain

$$\lambda(\alpha V) \geq -\frac{c''}{\varepsilon^{d-1}\rho^d}\exp\left(-d\left(\frac{\omega_d}{(1+\varepsilon)\left(1+C\,\alpha^{\frac{1}{d}}\right)\alpha\int_{\mathbb{R}^d} V\,dx}\right)^{\frac{1}{d-1}}\right).$$

Choosing $\varepsilon = C\alpha^{\frac{1}{d}}$ we obtain

$$\lambda(\alpha V) \geq -\frac{c'''}{\alpha^{\frac{d-1}{d}}}\exp\left(-d\left(\frac{\omega_d}{\left(1+C'\,\alpha^{\frac{1}{d}}\right)\alpha\int_{\mathbb{R}^d} V\,dx}\right)^{\frac{1}{d-1}}\right).$$

This implies the statement of the proposition. □

The general case. We can finally give the

Proof of Theorem 2.2. We use an approximation argument and fix $\varepsilon\in(0,1)$ and $R>0$. Define $V_< = V\chi_{\{|\cdot|<R\}}$ and $V_> = V_+\chi_{\{|\cdot|\geq R\}}$. Then the inequality

$$Q_{\alpha V}[u] \geq (1-\varepsilon)Q_{(1-\varepsilon)^{-1}\alpha V_<}[u] + \varepsilon Q_{\varepsilon^{-1}\alpha V_>}[u]$$

for every $u\in W^{1,d}(\mathbb{R}^d)$ implies

$$\lambda(\alpha V) \geq (1-\varepsilon)\lambda\left(\frac{\alpha}{1-\varepsilon}\,V_<\right) + \varepsilon\lambda\left(\frac{\alpha}{\varepsilon}V_>\right).$$

Thus,

$$\begin{aligned}\log\frac{1}{|\lambda(\alpha V)|} &\geq \log\frac{1}{(1-\varepsilon)\;|\lambda((1-\varepsilon)^{-1}\alpha V_<)|} - \log\left(1+\frac{\varepsilon\;|\lambda(\varepsilon^{-1}\alpha V_>)|}{(1-\varepsilon)\;|\lambda((1-\varepsilon)^{-1}\alpha V_<)|}\right)\\ &\geq \log\frac{1}{(1-\varepsilon)\;|\lambda((1-\varepsilon)^{-1}\alpha V_<)|} - \frac{\varepsilon\;|\lambda(\varepsilon^{-1}\alpha V_>)|}{(1-\varepsilon)\;|\lambda((1-\varepsilon)^{-1}\alpha V_<)|}\,.\end{aligned}$$

From now on we consider R so large that $\int_{B_R} V\,dx>0$. It then follows from Proposition 4.5 that

$$\liminf_{\alpha\to 0+}\alpha^{\frac{1}{d-1}}\log\frac{1}{(1-\varepsilon)|\lambda((1-\varepsilon)^{-1}\alpha V_<)|} \geq (1-\varepsilon)^{\frac{1}{d-1}} d\;\omega_d^{\frac{1}{d-1}}\;\left(\int_{B_R} V(x)\,dx\right)^{-\frac{1}{d-1}}.$$

On the other hand, we recall from Proposition 4.6 that there are constants $C>0$ and $\alpha_0>0$ such that for all $0<\alpha\leq\alpha_0\varepsilon$,

$$\lambda(\varepsilon^{-1}\alpha V_>) \geq -C\varepsilon\alpha^{-1}\exp\left(-\left(\frac{\varepsilon d^{d-1}\,\omega_d}{\alpha\int_{B_R^c} V_+\,dx}\right)^{\frac{1}{d-1}}\right)$$

Moreover, we recall from Proposition 4.2 that for every $\delta \in (0,1)$ there are constants $C_\delta > 0$ and α_δ such that for all $0 < \alpha \leq \alpha_\delta(1-\varepsilon)$,

$$\lambda((1-\varepsilon)^{-1}\alpha V_<) \leq -(1-\varepsilon)^{-1}\alpha\, C_\delta \exp\left(-\left(\frac{(1-\varepsilon)d^{d-1}\omega_d}{\alpha(1-\delta)\int_{B_R} V\,dx}\right)^{\frac{1}{d-1}}\right). \tag{4.29}$$

Thus, for $\alpha \leq \min\{\alpha_0\varepsilon, \alpha_\delta(1-\varepsilon)\}$,

$$\frac{|\lambda(\varepsilon^{-1}\alpha V_>)|}{|\lambda((1-\varepsilon)^{-1}\alpha V_<)|} \leq \frac{C\varepsilon(1-\varepsilon)}{C_\delta \alpha^2} \exp\left(-\left(\frac{\varepsilon d^{d-1}\,\omega_d}{\alpha\int_{B_R^c} V_+\,dx}\right)^{\frac{1}{d-1}} + \left(\frac{(1-\varepsilon)d^{d-1}\omega_d}{\alpha(1-\delta)\int_{B_R} V\,dx}\right)^{\frac{1}{d-1}}\right)$$

For every fixed ε and δ there is an $R_0 > 0$ such that for all $R > R_0$,

$$\frac{\varepsilon}{\int_{B_R^c} V_+\,dx} > \frac{1-\varepsilon}{(1-\delta)\int_{B_R} V\,dx}.$$

Thus, for all $R > R_0$ we have

$$\lim_{\alpha\to 0} \frac{|\lambda(\varepsilon^{-1}\alpha V_>)|}{|\lambda((1-\varepsilon)^{-1}\alpha V_<)|} = 0.$$

To summarize, we have shown that for all $\varepsilon \in (0,1)$ and for all $R > R_0$,

$$\liminf_{\alpha\to 0+} \alpha^{\frac{1}{d-1}} \log\frac{1}{|\lambda(\alpha V)|} \geq (1-\varepsilon)^{\frac{1}{d-1}} d\, \omega_d^{\frac{1}{d-1}} \left(\int_{B_R} V(x)\,dx\right)^{-\frac{1}{d-1}}.$$

Letting $\varepsilon \to 0$ and $R \to \infty$ we obtain the theorem. □

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