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Dimensional Hausdorff Properties of Singular Continuous Spectra

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We present an extension of the Gilbert-Pearson theory of subordinacy, which relates dimensional Hausdorff spectral properties of one-dimensional Schrödinger operators to the behavior of solutions of the corresponding Schrödinger equation. We use this theory to analyze these properties for several examples having the singular-continuous spectrum, including sparse barrier potentials, the almost Mathieu operator and the Fibonacci Hamiltonian.

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Singular continuous spectra have been extensively studied recently. Our interest here is in the classification and decomposition of such spectra with respect to dimensional Hausdorff measures. The measure-theoretical aspect of this point of view goes back to Rogers and Taylor [1], and it has been studied recently within spectral theory by Last [2] and by del Rio *et al.* [3] who have shown that the singular-continuous spectrum which is produced by localized rank-one perturbations of Anderson-model Hamiltonians in the localized regime [4] must be purely zero dimensional—in the sense that the associated spectral measures are supported on a set of zero Hausdorff dimension.

The main purpose of this paper is to report a general method for spectral analysis of one-dimensional Schrödinger operators from this point of view. It is a natural extension of the Gilbert-Pearson theory of subordinacy [5,6], and it allows us to analyze the dimensional Hausdorff properties for a number of examples with the singular-continuous spectrum. Below we describe the main ideas of our study and some of the main results. Mathematically complete proofs of these results will be given elsewhere [7].

Most of our discussion will be restricted to one-dimensional discrete (tight-binding) Schrödinger operators of the form

$$(H\psi)(n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n). \quad (1)$$

We shall consider two kinds of such operators: "line" operators acting on $\ell^2(\mathbb{Z})$ $(-\infty < n < \infty)$, and "half-line" operators acting on $\ell^2(\mathbb{Z}^+)$ $(n > 0)$, which are considered with a phase boundary condition of the form

$$\psi(0)\cos\theta + \psi(1)\sin\theta = 0, \quad (2)$$

where $-\pi/2 < \theta < \pi/2$.

Before formulating our main result, which would require some definitions, we would like to describe some of its applications. We stress at this point that the dimensional Hausdorff properties which we study are those which are associated with the spectral measures of the corresponding operators. The spectra themselves, as sets, are closed sets, and their dimensions may be larger than those which are associated with the spectral measures. A description of the precise spectral-theoretic scheme which underlies our study is given below.

We start with a somewhat artificial example of half-line operators with sparse barrier potentials. More specifically, we consider potentials which vanish for all n's outside a sparse (fastly growing) sequence of points $\{L_n\}_{n=1}^{\infty}$ where $|V(L_n)| \to \infty$ as $n \to \infty$. Simon and Spencer [8] have shown that the Schrödinger operators corresponding to such potentials have no absolutely continuous spectrum, and Gordon [9] has shown that if the $|V(L_n)|$'s grow sufficiently fast (compared to the growth of the L_n's) then for (Lebesgue) almost every (a.e.) boundary phase

θ the corresponding operators have a pure-point spectrum with exponentially decaying eigenfunctions. It is easy to see [10], however, that if the L_n's grow sufficiently fast [compared to the growth of the $|V(L_n)|$'s] then, for every boundary phase θ, the spectrum in $(-2,2)$ is purely singular continuous, and Simon [11] has recently shown that if the growth is even faster then the spectrum in $(-2,2)$ is purely one dimensional, in the sense that the spectral measure does not give weight to sets of Hausdorff dimension less than 1. By applying theorem 1 below, we have shown the following.

Theorem 2.—Let $\alpha \in (0,1)$. Let $L_n = 2^{(n^n)}$ and define a potential $V(k)$ for $k > 0$ by $V(L_n) = L_n^{(1-\alpha)/(2\alpha)}$; $V(k) = 0$ if $k \notin \{L\}_{n=1}^{\infty}$. Then the following hold: (i) For every boundary phase θ, the spectrum of the corresponding half-line discrete Schrödinger operator consists of the interval $[-2,2]$ (which is the essential spectrum) along with some discrete point spectrum outside this interval. (ii) For every θ, the Hausdorff dimensionality of the spectrum in $(-2,2)$ is bounded between dimensions α and $\beta \equiv 2\alpha/(1+\alpha)$, in the sense that the restriction of the spectral measure to $(-2,2)$ is supported on a set of Hausdorff dimension β and does not give weight to sets of Hausdorff dimension less than α. (iii) For Lebesgue a.e. θ, the spectrum in $[-2,2]$ is of exact dimension α, namely, the restriction of the spectral measure to $[-2,2]$ is supported on a set of Hausdorff dimension α and does not give weight to sets of Hausdorff dimension less than α.

Remark.—The result only requires the L_n's to be sufficiently sparse (namely, to grow sufficiently fast). $L_n = 2^{(n^n)}$ is a particular choice for which the sufficient sparseness is easy to show.

Next we consider two examples of line operators with quasiperiodic potentials. The first is the almost Mathieu (also called Harper) operator $H_{\beta,\lambda,\theta}$, which is the operator of the form (1) on $\ell^2(\mathbb{Z})$ with potential $V(n) = V_{\beta,\lambda,\theta}(n) = \lambda\cos(2\pi\beta n + \theta)$, where λ, θ are any real numbers, and β is an irrational. Aubry and Andre [12] have conjectured that $H_{\beta,\lambda,\theta}$ has a purely absolutely continuous spectrum whenever $|\lambda| < 2$, and a purely point spectrum (with exponentially localized eigenfunctions) whenever $|\lambda| > 2$. While the $|\lambda| < 2$ part of this conjecture may be correct (so far, the existence of the absolutely continuous spectrum [13] and the absence of the point spectrum [14] have been established rigorously), the $|\lambda| > 2$ case turned out to be more delicate: The absolutely continuous spectrum is absent [15], but both pure-point and singular-continuous spectra occur, depending on arithmetical properties of both β and θ [16]. It turns out, though, that if we concentrate on the dimensional Hausdorff properties of the spectral measures, rather than distinguishing between pure-point and singular-continuous spectra, the situation becomes simpler.

Theorem 3.—For $|\lambda| > 2$, every irrational β, and every θ, $H_{\beta,\lambda,\theta}$ has a purely zero-dimensional spectrum, in the sense that its spectral measures are all supported on a set of zero Hausdorff dimension.

Remarks.—(i) The spectrum of $H_{\beta,\lambda,\theta}$, as a set, is known in this case ($|\lambda| > 2$) to have positive Lebesgue measure [17]. (ii) The result extends to potentials of the form $V(n) = f(2\pi\beta n + \theta)$, where $f(x) \equiv \sum_{k=1}^{N} \lambda_k \cos(kx)$, in which case we prove that the spectrum is purely zero dimensional whenever $|\lambda_N| > 2$.

Our second line example is the Fibonacci Hamiltonian H_λ, which is the operator of the form (1) on $\ell^2(\mathbb{Z})$ with potential $V(n) = \lambda\{[(n+1)\omega] - [n\omega]\}$, where $\omega = (\sqrt{5}-1)/2$ is the golden mean, and $[x] \equiv \max\{m \in \mathbb{Z} \mid m \leq x\}$. H_λ is the most studied of all one-dimensional quasicrystal models. It is known [18] that for every $\lambda \neq 0$ it has a purely singular-continuous spectrum, and, moreover, its spectrum (as a set) is a Cantor set of zero Lebesgue measure. We have shown the following.

Theorem 4.—For every λ there exists an $\alpha > 0$ such that H_λ has a purely α-continuous spectrum, namely, its spectral measures do not give weight to sets of Hausdorff dimension less than α.

Remark.—There exists strong numerical evidence [19] that the spectrum of H_λ (as a set) has Hausdorff dimension strictly less than 1 (for every $\lambda \neq 0$), and this would imply that its spectrum must also be β singular (see below) for some $\beta < 1$.

Let us now describe the spectral-theoretic scheme in the context of which the above results should be understood.

Consider a separable Hilbert space $\mathcal{H}$, and a self-adjoint operator H. Recall [20] that for each $\psi \in \mathcal{H}$, the spectral measure μ_ψ (also known to physicists as the local spectral density) is the unique Borel measure obeying $\langle\psi, f(H)\psi\rangle = \int f(x)\, d\mu_\psi(x)$ for any measurable function f. By Lebesgue's decomposition theorem, every Borel measure μ decomposes uniquely as $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}} + \mu_{\mathrm{pp}}$. The absolutely continuous part, μ_{ac}, gives zero weight to sets of zero Lebesgue measure. The pure-point part, μ_{pp}, is a countable sum of atomic (Dirac) measures. The singular-continuous part, μ_{sc}, gives zero weight to countable sets and is supported on some set of zero Lebesgue measure [we say that a measure μ is supported on a set S if $\mu(\mathbb{R} \setminus S) = 0$]. Letting $\mathcal{H}_{\mathrm{ac}} \equiv \{\psi \mid \mu_\psi \text{ is purely absolutely continuous}\}$, $\mathcal{H}_{\mathrm{sc}} \equiv \{\psi \mid \mu_\psi \text{ is purely singular continuous}\}$, and $\mathcal{H}_{\mathrm{pp}} \equiv \{\psi \mid \mu_\psi \text{ is purely pure point}\}$, one obtains a decomposition, $\mathcal{H} = \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}} \oplus \mathcal{H}_{\mathrm{pp}}$. $\mathcal{H}_{\mathrm{ac}}$, $\mathcal{H}_{\mathrm{sc}}$, and $\mathcal{H}_{\mathrm{pp}}$ are closed (in norm), mutually orthogonal subspaces, which are invariant under H. The absolutely continuous spectrum (σ_{ac}), singular-continuous spectrum (σ_{sc}), and pure-point spectrum (σ_{pp}) are defined as the spectra of the restrictions of H to the corresponding subspaces, and $\mathrm{Spec}\,(H) \equiv \sigma = \sigma_{\mathrm{ac}} \cup \sigma_{\mathrm{sc}} \cup \sigma_{\mathrm{pp}}$.

The above standard scheme of spectral theory can be extended to further decompose the singular-continuous subspace by using Hausdorff measures. Recall that for any subset S of $\mathbb{R}$ and $\alpha \in [0,1]$, the α-dimensional Hausdorff measure, h^{α}, is given by

$$h^{\alpha}(S) \equiv \lim_{\delta \to 0} \inf_{\delta-\text{covers}} \sum_{\nu=1}^{\infty} |b_{\nu}|^{\alpha}, \tag{3}$$

where a δ-cover is a cover of S by a countable collection of intervals, $S \subset \bigcup_{\nu=1}^{\infty} b_{\nu}$, such that for each ν the length of b_{ν} is at most δ. h^1 coincides with Lebesgue measure, and h^0 is the counting measure (assigning to each set the number of points in it). Given any $\varnothing \neq S \subseteq \mathbb{R}$, there exists a unique $\alpha(S) \in [0,1]$ such that $h^{\alpha}(S) = 0$ for any $\alpha > \alpha(S)$, and $h^{\alpha}(S) = \infty$ for any $\alpha < \alpha(S)$. This unique $\alpha(S)$ is called the Hausdorff dimension of S. A rich theory of decomposing measures with respect to Hausdorff measures has been developed by Rogers and Taylor [1]. Below we discuss only a small part of it. A much more detailed description has been given by Last [2].

Given $\alpha \in [0,1]$, a measure μ is called α continuous if $\mu(S) = 0$ for every set S with $h^{\alpha}(S) = 0$. It is called α singular if it is supported on some set S with $h^{\alpha}(S) = 0$. We say that μ is one dimensional if it is α continuous for every $\alpha < 1$. We say that it is zero dimensional if it is α singular for every $\alpha > 0$. μ is said to have exact dimension α if, for every $\epsilon > 0$, it is both $(\alpha - \epsilon)$ continuous and $(\alpha + \epsilon)$ singular.

Given a (positive, finite) measure μ and $\alpha \in [0,1]$, we define

$$D_{\mu}^{\alpha}(x) \equiv \limsup_{\epsilon \to 0} \frac{\mu(x-\epsilon, x+\epsilon)}{(2\epsilon)^{\alpha}} \tag{4}$$

and $T_{\infty} \equiv \{x \mid D_{\mu}^{\alpha}(x) = \infty\}$. The restriction $\mu(T_{\infty} \cap \cdot) \equiv \mu_{\alpha s}$ is α singular, and $\mu((\mathbb{R} \setminus T_{\infty}) \cap \cdot) \equiv \mu_{\alpha c}$ is α continuous. Thus, each measure decomposes uniquely into an α-continuous part and an α-singular part, $\mu = \mu_{\alpha c} + \mu_{\alpha s}$. Moreover, an α-singular measure must have $D_{\mu}^{\alpha}(x) = \infty$ a.e. (with respect to it) and an α-continuous measure must have $D_{\mu}^{\alpha}(x) < \infty$ a.e. It is important to note that $D_{\mu}^{\alpha}(x)$ is defined with a lim sup. The corresponding limit need not exist.

We let $\mathcal{H}_{\alpha c} \equiv \{\psi \mid \mu_{\psi}$ is α continuous$\}$ and $\mathcal{H}_{\alpha s} \equiv \{\psi \mid \mu_{\psi}$ is α singular$\}$. $\mathcal{H}_{\alpha c}$ and $\mathcal{H}_{\alpha s}$ are mutually orthogonal closed subspaces which are invariant under H, and $\mathcal{H}$ decomposes as $\mathcal{H} = \mathcal{H}_{\alpha c} \oplus \mathcal{H}_{\alpha s}$. The α-continuous spectrum $(\sigma_{\alpha c})$ and α-singular spectrum $(\sigma_{\alpha s})$ are defined as the spectra of the restrictions of H to the corresponding subspaces, and $\sigma = \sigma_{\alpha c} \cup \sigma_{\alpha s}$. Note, in particular, that when we classify spectra as being α-singular, zero dimensional, of exact dimension α, etc., we always relate to the corresponding properties of the spectral measures.

The above scheme for spectral classification can be related to the dynamics of the underlying quantum systems. A detailed account of such relations has been given by Last [2].

It should be pointed out that certain fractal and multifractal studies of some operators with the singular-continuous spectrum (including some of the examples we discussed above) have been carried out by several authors [19,21]. While such studies are related to the above decomposition theory, the relations are generally far from trivial, and we believe that they are only partial. One should exercise extreme care when attempting to interpret the results of such studies within the framework of the scheme discussed above.

From here on we shall restrict our discussion to one-dimensional tight-binding Schrödinger operators of the form (1). While we discuss discrete operators, the subordinacy results we describe are equally valid for continuous Schrödinger operators of the form $-\frac{d^2}{dx^2} + V$.

Consider first half-line operators, defined with a phase boundary condition of the form (2). For such operators, it is well known that the spectral measures for lattice-point vectors δ_n, where $\delta_n(m) = \delta_{nm}$, are all mutually equivalent (namely, they have the same sets of zero measure). Thus, the spectral problem reduces to analyzing a single spectral measure, which we choose to be $\mu = \mu_{\delta_1}$. The Gilbert-Pearson theory of subordinacy [5] relates the pointwise behavior of the spectral measure μ at some energy E to the behavior of solutions of the corresponding Schrödinger equation

$$\psi(n+1) + \psi(n-1) + V(n)\psi(n) = E\psi(n). \tag{5}$$

Given a solution of (5), we let $\|\psi\|_L$ denote the norm of the solution ψ over length L. It is useful to consider the length L as a continuous variable (allowed to take any positive real value), and so we define

$$\|\psi\|_L = \left[\sum_{n=1}^{[L]} |\psi(n)|^2 + (L - [L])|\psi([L]+1)|^2 \right]^{1/2}, \tag{6}$$

where $[L]$ denotes the integer part of L. A (nontrivial) solution ψ of (5) is called a subordinate solution if for any other solution φ of (5), which is not a constant multiple of ψ, $\lim_{L \to \infty} \frac{\|\psi\|_L}{\|\varphi\|_L} = 0$. Note that a subordinate solution must be unique (up to constant multiples). The Gilbert-Pearson theory relates the decomposition of the spectral measure μ to subordinacy of solutions as follows: The absolutely continuous part of μ is supported on the set of energies for which (5) has no subordinate solutions. (In fact, this set of energies is, up to a set of both Lebesgue and spectral measure zero, the set where μ has a finite nonvanishing derivative.) The singular part of μ is supported on the set of energies for which the solutions which obey the appropriate boundary condition (2) are subordinate.

Let us now denote by ψ_1 the solution of (5) which obeys the boundary condition (2) and has normalization $|\psi_1(0)|^2 + |\psi_1(1)|^2 = 1$. Let us denote by ψ_2 the solution of (5) which obeys the orthogonal boundary condition to (2), namely, $\psi_2(0)\sin\theta - \psi_2(1)\cos\theta = 0$, and has normalization $|\psi_2(0)|^2 + |\psi_2(1)|^2 = 1$. Our main result is the following.

Theorem 1.—For any $\alpha \in (0,1)$ and every $E \in \mathbb{R}$, $D_\mu^\alpha(E) = \infty$ if and only if

$$\liminf_{L\to\infty} \frac{\|\psi_1\|_L}{\|\psi_2\|_L^\beta} = 0,$$

where $\beta = \alpha/(2 - \alpha)$.

Remark.—Theorem 1 is proven with the same ideas used by Gilbert and Pearson, but it requires some optimization of their analysis. As a by-product, we also get a simplified proof of their original results. A key observation is to assign to each $\epsilon > 0$ a length $L(\epsilon)$ via the equality $\|\psi_1\|_{L(\epsilon)}\|\psi_2\|_{L(\epsilon)} = 1/(2\epsilon)$, for which we prove the explicit inequality

$$\frac{5 - \sqrt{24}}{|m(E + i\epsilon)|} < \frac{\|\psi_1\|_{L(\epsilon)}}{\|\psi_2\|_{L(\epsilon)}} < \frac{5 + \sqrt{24}}{|m(E + i\epsilon)|},$$

where $m(z)$ is the Weyl-Titchmarsh function [22].

For spectral analysis, theorem 1 can be combined with the existence of generalized eigenfunctions [22], from which one can show that for a.e. E with respect to the spectral measure μ, the solution ψ_1 must obey $\limsup_{L\to\infty} \frac{\|\psi_1\|_L}{L^{1/2}\ln L} < \infty$ and $\liminf_{L\to\infty} \frac{\|\psi_1\|_L}{L^{1/2}} < \infty$. Another useful fact is the constancy of the Wronskian $\psi_1(n + 1)\psi_2(n) - \psi_2(n + 1)\psi_1(n)$, which implies $\|\psi_1\|_L\|\psi_2\|_L \geq (L - 1)/2$.

We now discuss briefly line operators. The spectral measures of a line operator can be constructed from those of corresponding two half-line operators (a left and a right), and while the relations are not completely trivial, they do allow an extension of the subordinacy theory to this case. Gilbert [6] has shown that the absolutely continuous part of the spectral measures of a line operator is supported on the set of energies for which at least one of the half-line problems has no subordinate solution. The singular part is supported on the set of energies for which (5) has a solution which is subordinate both to the right and to the left. The probing of dimensional Hausdorff properties is somewhat more delicate in this case since it involves a lim inf rather than a limit. Nevertheless, in concrete settings, such as the ones discussed in theorems 3 and 4, the required control can be obtained.

In conclusion we would like to remark the following: The classification of spectra with respect to dimensional Hausdorff measures extends the usual spectral classification in a natural way, and provides a useful way of distinguishing between different kinds of singular-continuous spectra. The subordinacy theory extends to this point of view in a natural way, and allows one to answer the relevant spectral questions whenever the nature of the solutions of the corresponding Schrödinger equation is sufficiently well understood. We note, in particular, that the singular-continuous spectrum which occurs in "close neighborhood" to Anderson localization (as in the case of the strongly coupled almost Mathieu operator or the rank-one perturbed Anderson model) tends to be purely zero dimensional; while the singular-continuous spectrum of the Fibonacci Hamiltonian, which has been identified as having "critical states" in physics literature [19], is α continuous for some positive α.

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Note added.—As we were completing this paper we learned of a preprint by Remling [23] which obtains a restricted version of theorem 1.

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