Its theoretical Fourier series coefficients are the following:

$$h(0)=\frac{0.6}{\pi}-0.2$$

$$h(2)=0.15$$

$$h(n)=-\frac{2.4}{\pi}\frac{\cos\left(\frac{\pi n}{4}\right)}{n^2-4},\qquad n\neq 0,\ n\neq 2. \tag{16}$$

Fig. 5(b) shows $h(n)$, $n\in[0,63]$. We sample $X(f)$ at 32 points over one period in equispaced intervals. Fig. 5(c) shows the results of $\hat{h}(n)$, $n=0,1,2,\cdots,63$, computed through the direct IFFT. Fig. 5(d) shows the results of $\tilde{h}(n)$, $n=0,1,\cdots,63$, computed through the first method. Fig. 5(e) shows the results of $\tilde{\tilde{h}}(n)$, $n=0,1,2,\cdots,63$, computed through the second method. We define the computation errors as the following:

$$\hat{e}(n)=|\hat{h}(n)-h(n)|,\qquad 0\leqslant n\leqslant 15$$

$$\tilde{e}(n)=|\tilde{h}(n)-h(n)|,\qquad 0\leqslant n\leqslant 63$$

$$\tilde{\tilde{e}}(n)=|\tilde{\tilde{h}}(n)-h(n)|,\qquad 0\leqslant n\leqslant 63. \tag{17}$$

$\hat{e}(n)$, $\tilde{e}(n)$ and $\tilde{\tilde{e}}(n)$ are shown in Fig. 5(f), (g), and (h), respectively.

From Fig. 5(f), (g), and (h), we can see that $\tilde{h}(n)$ and $\tilde{\tilde{h}}(n)$ approximate $h(n)$ very well from $n=0$ to $n=63$, but $\hat{h}(n)$ approximate $h(n)$ only from $n=0$ to $n=15$; $\tilde{h}(n)$ are more accurate than $\hat{h}(n)$ except a few points at the beginning, $\tilde{\tilde{h}}(n)$ are the most accurate one of them.

In order to make the comparison clearer, we compute the average errors as the following:

$$\hat{e}_a=\frac{1}{16}\sum_{n=0}^{15}\hat{e}(n)$$

$$\tilde{e}_a=\frac{1}{64}\sum_{n=0}^{63}\tilde{e}(n)$$

$$\tilde{\tilde{e}}_a=\frac{1}{64}\sum_{n=0}^{63}\tilde{\tilde{e}}(n) \tag{18}$$

and the maximum errors as the following:

$$\hat{e}_M=\max\{\hat{e}(n)\},\qquad 0\leqslant n\leqslant 15$$

$$\tilde{e}_M=\max\{\tilde{e}(n)\},\qquad 0\leqslant n\leqslant 63$$

$$\tilde{\tilde{e}}_M=\max\{\tilde{\tilde{e}}(n)\},\qquad 0\leqslant n\leqslant 63. \tag{19}$$

The results are shown in the following table:

method	IDFT	first method	second method
extent of coefficients	16	64	64
average error (10^{-4})	17.6	2.4	0.6
maximum error (10^{-4})	33.4	24.6	3.5

Based on the previous analysis and computation results, the following conclusions can be drawn:

(1) $\hat{h}(n)$, computed through the IDFT, is an aliased version of $h(n)$; so that it is periodic and approximates $h(n)$ only when $|n|<M/2$.

(2) $\tilde{h}(n)$, computed through the first method, is a weighted $\hat{h}(n)$, so that it is not periodic and approximates $h(n)$ better than $\hat{h}(n)$ within a large extent of n's, even when $|n|\gg M$.

(3) $\tilde{\tilde{h}}(n)$, computed through the second method, is generally more accurate than $\tilde{h}(n)$ within a large extent of n's, but requires a more complex computation.

(4) All three $\hat{h}(n)$, $\tilde{h}(n)$, and $\tilde{\tilde{h}}(n)$ can be computed through the IFFT. $\hat{h}(n)$ and $\tilde{h}(n)$ need M multiplications more than the FFT, while $\tilde{\tilde{h}}(n)$ needs $2M$ multiplications more than the FFT.

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On Power-Complementary FIR Filters

P. P. VAIDYANATHAN

Abstract—**Conditions are derived, under which two linear-phase FIR filter transfer functions $H(z)$ and $G(z)$ have the power-complementary property, i.e., $|H(e^{j\omega})|^2+|G(e^{j\omega})|^2=1$. It is shown that, the constraint of linear phase on the transfer functions strongly restricts the class of frequency responses that can be realized by a power-complementary pair.**

There exist a number of applications [1]–[3] where it is required to have two transfer functions $H(z)$ and $G(z)$ such that they are *power-complementary*, i.e., satisfy the relation

$$|H(e^{j\omega})|^2+|G(e^{j\omega})|^2=1. \tag{1}$$

Typical examples are in the design of filter banks for audio systems [3], and in the design of decimation-interpolation type of filter banks [1], [2]. A natural question that arises in this connection, when the filters are restricted to have finite-duration impulse response (FIR filters) is the following: is it possible to have two FIR transfer functions $G(z)$ and $H(z)$ satisfying (1), with *both* $H(z)$ and $G(z)$ having linear phase? If so, what are the additional restrictions on such FIR transfer functions?

Basically, given *any* linear-phase FIR transfer function $H(z)$ (scaled so that $|H(e^{j\omega})|\leqslant 1$), we can always find an FIR spectral factor $G(e^{j\omega})$ of the positive function $1-|H(e^{j\omega})|^2$, i.e., $G(z)$ satisfying[1]

$$\tilde{H}(z)H(z)+\tilde{G}(z)G(z)=1. \tag{2}$$

The question of interest is, under what conditions does there exist a spectral factor $G(z)$ that *also* has linear phase? The answer to this problem turns out to be simple, but it does not seem to have made explicit appearance in the literature—hence this correspondence.

It is clear that a pair of transfer functions $H(z)$ and $G(z)$ satisfy (2) if and only if $z^{-K}H(z)$ and $z^{-L}G(z)$ satisfy the same property for arbitrary integers K and L. Accordingly, without loss of generality, let us assume $H(z)$ and $G(z)$ to be of the form

$$H(z)=h_0+h_1z^{-1}+\cdots+h_{N-1}z^{-(N-1)} \tag{3}$$

$$G(z)=g_0+g_1z^{-1}+\cdots+g_{N-1}z^{-(N-1)} \tag{4}$$

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[1] $\tilde{G}(z)$ stands for $G(z^{-1})$ and so on.

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where h_n is either symmetric or antisymmetric [4] with respect to $(N-1)/2$ (which may be nonintegral) and so is g_n. If $H(z)$ and $G(z)$ satisfy (2) then, in particular, the following is true:

$$h_0 h_{N-1} + g_0 g_{N-1} = 0. \tag{5}$$

We should, therefore, have $g_0 \neq 0$ and $g_{N-1} \neq 0$ if h_0 and h_{N-1} are nonzero. We assume h_n and g_n in (3), (4) to be real numbers, which is the most common situation. Now, if $G(z)$ and $H(z)$ are to have linear phase, three cases can be distinguished, corresponding to the following three situations:

$$h_n = h_{N-1-n}, \qquad g_n = -g_{N-1-n} \tag{6.1}$$

$$h_n = h_{N-1-n}, \qquad g_n = g_{N-1-n} \tag{6.2}$$

$$h_n = -h_{N-1-n}, \qquad g_n = -g_{N-1-n} \tag{6.3}$$

for all n such that $0 \leqslant n \leqslant N-1$. These will be called Case 1, Case 2, and Case 3, respectively.

First consider Case 1. Since h_n and g_n are, respectively, symmetric and antisymmetric, we have

$$H(z) = z^{-(N-1)}\tilde{H}(z) \tag{7}$$

$$G(z) = -z^{-(N-1)}\tilde{G}(z). \tag{8}$$

If the power-complementary property of (2) holds, then (7) and (8) imply

$$H^2(z) - G^2(z) = z^{-(N-1)} \tag{9}$$

i.e.,

$$[H(z)+G(z)][H(z)-G(z)] = z^{-(N-1)}. \tag{10}$$

The polynomials on the left side of (10) are therefore constrained to be of the form

$$H(z)+G(z) = \alpha z^{-n_1} \tag{11}$$

$$H(z)-G(z) = \frac{1}{\alpha} z^{-n_2} \tag{12}$$

where α is real-valued, and $N-1 = n_1 + n_2$. Thus for $n \neq n_1, n_2$, we have $h_n = g_n = 0$. If $n_1 \neq n_2$, then we should have, in addition:

$$h_{n_1} + g_{n_1} = \alpha, \quad h_{n_1} - g_{n_1} = 0 \tag{13}$$

$$h_{n_2} + g_{n_2} = 0, \quad h_{n_2} - g_{n_2} = \frac{1}{\alpha} \tag{14}$$

which imply

$$h_{n_1} = g_{n_1} = \alpha/2, \quad h_{n_2} = -g_{n_2} = \frac{1}{2\alpha}. \tag{15}$$

If $n_1 = n_2$, then it can be verified that, $g_n = 0$ for all n, and h_n is nonzero ($=1$) only for $n = n_1 = (N-1)/2$. In summary, if $G(z)$ and $H(z)$ fall under Case 1 (i.e., linear phase filters satisfying (6.1)) then they are restricted to be of the form

$$H(z) = \frac{1}{2}(z^{-n_1} + z^{-n_2}) \tag{16}$$

$$G(z) = \frac{1}{2}(z^{-n_1} - z^{-n_2}). \tag{17}$$

Next consider Case 2. Here, because of (6.2), we have

$$G(z) = z^{-(N-1)}\tilde{G}(z) \tag{18}$$

whereas $H(z)$ satisfies (7). Accordingly (2) now implies

$$H^2(z) + G^2(z) = z^{-(N-1)} \tag{19}$$

which leads to

$$H(z) + jG(z) = \alpha z^{-n_1} \tag{20}$$

$$H(z) - jG(z) = \alpha^* z^{-n_2} \tag{21}$$

where $N-1 = n_1 + n_2$ and $\alpha\alpha^* = 1$. Thus

$$h_n + jg_n = 0, \quad n \neq n_1 \tag{22}$$

$$h_n - jg_n = 0, \quad n \neq n_2 \tag{23}$$

For $n \neq n_1, n_2$ we therefore have $h_n = g_n = 0$. If $n_1 \neq n_2$ then

$$h_{n_1} + jg_{n_1} = \alpha, \quad h_{n_1} - jg_{n_1} = 0 \tag{24}$$

$$h_{n_2} + jg_{n_2} = 0, \quad h_{n_2} - jg_{n_2} = \alpha^* \tag{25}$$

which is not possible for real valued h_n, g_n. Thus we must have $n_1 = n_2 = (N-1)/2$, in which case,

$$h_{(N-1)/2} + jg_{(N-1)/2} = \alpha, \qquad \alpha\alpha^* = 1 \tag{26}$$

which restricts $H(z)$ and $G(z)$ to have the form

$$H(z) = (\cos\theta)\, z^{-(N-1)/2}, \quad G(z) = (\sin\theta)\, z^{-(N-1)/2}. \tag{27}$$

Finally consider Case 3, where both $H(z)$ and $G(z)$ have antisymmetric impulse responses. Then

$$H(z) = -z^{-(N-1)/2}\tilde{H}(z), \quad G(z) = -z^{-(N-1)/2}\tilde{G}(z) \tag{28}$$

whence (2) implies

$$H^2(z) + G^2(z) = -z^{-(N-1)}. \tag{29}$$

An easy way to see that Case 3 is impossible is to note that, for antisymmetric impulse responses [4], we necessarily have $H(e^{j\omega}) = G(e^{j\omega}) = 0$ for $\omega = 0$, hence it is not possible to satisfy (29) for $z = 1$ anyway. Another way to see this is that for positive-real z, the right side of (29) is strictly negative, whereas the left side is necessarily nonnegative, which is an impossible situation.

In summary we have the following result: Let $H(z)$ and $G(z)$ be two linear phase FIR transfer functions as in (3), (4) and satisfying the power-complementary condition of (2). If the impulse response symmetry is as in (6.1) then $H(z)$ and $G(z)$ are restricted to be as in (16), (17). If, on the other hand, (6.2) holds then $H(z)$ and $G(z)$ are as in (27). Finally, it is *not* possible for the impulse responses to be *both* antisymmetric (Eqn. (6.3)).

Note that, (16), (17) correspond to

$$|H(e^{j\omega})| = \left|\cos(n_1 - n_2)\frac{\omega}{2}\right|, \quad |G(e^{j\omega})| = \left|\sin(n_1 - n_2)\frac{\omega}{2}\right| \tag{30}$$

whereas, (27) corresponds to simple delays. Thus the linear-phase constraint on $H(z)$ and $G(z)$ satisfying (2) restricts them to be rather trivial transfer functions. Recall also that, in the case of IIR filters [3] with linear phase numerators, it is quite possible to obtain nontrivial transfer functions (for example, elliptic filters of odd order) satisfying (2).

A generalization of the linear-phase FIR problem discussed above is the following: suppose that we have a bank of M linear-phase FIR filters $G_0(z), G_1(z), \cdots G_{M-1}(z)$ satisfying the condition

$$|G_0(e^{j\omega})|^2 + |G_1(e^{j\omega})|^2 + \cdots |G_{M-1}(e^{j\omega})|^2 = 1. \tag{31}$$

Then what class of magnitude responses can be realized by $G_k(z)$? Once again, basically, given $M-1$ linear-phase FIR filters $G_0(z), G_1(z), \cdots G_{M-2}(z)$, scaled such that

$$\left|G_0(e^{j\omega})\right|^2+\left|G_1(e^{j\omega})\right|^2+\cdots\left|G_{M-2}(e^{j\omega})\right|^2\leqslant 1 \quad (32)$$

we can always find a spectral factor $G_{M-1}(z)$ of the non-negative function

$$1-\left(\left|G_0(e^{j\omega})\right|^2+\left|G_1(e^{j\omega})\right|^2+\cdots\left|G_{M-2}(e^{j\omega})\right|^2\right) \quad (33)$$

such that (31) holds. It is of interest to explore under what conditions a *linear phase* spectral factor $G_{M-1}(z)$ exists, and whether these conditions permit us to realize nontrivial magnitude responses $|G_k(e^{j\omega})|$ or not.

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A Counterexample to 2-D Lyapunov Equations by Mertzios

TAO LIN AND MASAYUKI KAWAMATA

***Abstract*—In [1], Mertzios introduced two-dimensional (2-D) Lyapunov equations for the covariance and noise matrices to analyze the output roundoff noise under l_2-norm scaling in 2-D digital filters described by the local state-space model. However, his 2-D Lyapunov equations include some crucial errors. This paper presents a counterexample to his 2-D Lyapunov equations, and shows that his 2-D Lyapunov equations cannot give the covariance and noise matrices in 2-D state-space digital filters.**

I. Introduction

The covariance and noise matrices are necessary to analyze and minimize the variance of roundoff noise under the l_2-norm scaling in state-space digital filtering. The covariance and noise matrices in 1-D state-space digital filters can be obtained as solutions of Lyapunov equations [2], [3]. Recently, Mertzios [1] introduced 2-D Lyapunov equations for the covariance and noise matrices to analyze the variance of roundoff noise in 2-D state-space digital filters. However, his 2-D Lyapunov equations include some crucial errors. This paper presents a counterexample to the 2-D Lyapunov equations, and discusses the error of his derivation for the 2-D Lyapunov equations.

II. State-Space Model

Consider the 2-D filter described in state-space as follows [1]:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \left[\begin{array}{c} b_1 \\ \hline b_2 \end{array}\right] u(i,j) \quad (1a)$$

and

$$y(i,j) = \left[\begin{array}{c|c} c_1 & c_2 \end{array}\right] \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + du(i,j) \quad (1b)$$

or more compactly

$$x' = Ax + bu \quad (2a)$$

$$y = cx + du \quad (2b)$$

where $x^h \in R_{n1}$ is the horizontal state-vector, $x^v \in R_{n2}$ is the vertical state-vector, u is the input, y is the output, and A, b, c, and d are constant matrices of appropriate dimensions.

III. A Counterexample to 2-D Lyapunov Equations

To analyze the variance of roundoff noise under l_2-norm scaling in 2-D state-space digital filters, the covariance matrix K and the noise matrix W are necessary just as in the 1-D case. We restrict ourselves to the covariance matrix K in this section. The 2-D covariance matrix K is defined as

$$K = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f(i,j)f^t(i,j) \quad (3)$$

where $f(i,j)$ is the unit pulse response sequence of the state due to a unit pulse sequence at the input under zero initial conditions $(x^h(0,j) = x^v(i,0) = 0,\ i,j \geqslant 0)$. Mertzios derived the 2-D covariance matrix K as the solution of the following 2-D Lyapunov equation [1]:

$$K = AKA^t + \left[\begin{array}{c|c} b_1b_1^t & 0 \\ \hline 0 & b_2b_2^t \end{array}\right]. \quad (4)$$

We show, using a counterexample, that the covariance matrix K cannot be obtained as the solution of the 2-D Lyapunov equation (4). Consider the following simple example:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \left[\begin{array}{c|c} 0 & a \\ \hline a & 0 \end{array}\right] \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \left[\begin{array}{c} 1 \\ \hline 1 \end{array}\right] u(i,j) \quad (5)$$

were $|a| < 1$ for stability. The solution K of the 2-D Lyapunov equation (4) is

$$K = \begin{bmatrix} 1/(1-a^2) & 0 \\ 0 & 1/(1-a^2) \end{bmatrix}. \quad (6)$$

We next obtain the covariance matrix K based on the definition (3). The unit impulse response sequence $f(i,j) = [f^h(i,j), f^v(i,j)]^t$ of the filter can be easily obtained as

$$f^h(m,n) = \begin{cases} 0, & m=n=0 \\ a^{2m-1}, & m=n>0 \\ a^{2m-2}, & m=n+1>0 \\ 0, & \text{otherwise} \end{cases} \quad (7)$$

$$f^v(m,n) = \begin{cases} 0, & m=n=0 \\ a^{2n-1}, & n=m>0 \\ a^{2n-2}, & n=m+1>0 \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

Substituting $f(i,j) = [f^h(i,j) f^v(i,j)]^t$ into the definition (3) of the covariance matrix K yields

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