Transactions Briefs

The Doubly Terminated Lossless Digital Two-Pair in Digital Filtering

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***Abstract* — A digital lossless two-pair terminated at both ends with "passive" multipliers is studied. Conditions for low sensitivity of the transfer-function magnitude with respect to the digital multiplier coefficients are derived. It is shown that low sensitivity property can be achieved by forcing certain "matching" conditions, at the terminations. The application of these results to the understanding of some well-known digital filter structures is outlined. In particular, it is shown that the coupled-form biquad can be interpreted as a doubly terminated lossless digital two pair, and that it satisfies the "termination matching conditions" for almost all pole locations. All results derived in the paper are based on independent z-domain arguments.**

I. Introduction

The design of digital filter structures with low passband sensitivity with respect to multiplier coefficients has received considerable attention. In [1], [2] a general procedure is developed for the synthesis of low sensitivity ladder-type digital filter structures, that are independent of any continuous time prototypes. Well-known structures such as the wave digital filters [5], [6], the Gray and Markel lattice structures [8], and the coupled form biquad [7], can be looked upon as special cases of the structures developed in [1], [2].

In this paper we deal with the properties of a "doubly terminated" or "doubly constrained" digital two-pair (Fig. 1). We deal with the specific case where the two-pair is "lossless" (to be defined). In Section II we derive conditions that are to be satisfied by the structure so that the transfer-function magnitude exhibits low passband sensitivity. We show that for achieving low sensitivity property, the terminations m and n should be "matched" to the two-pair "input functions" in a certain sense. We deal with passband sensitivity with respect to the multipliers within the two-pair as well as the terminating multipliers m and n. It is shown that the "matching conditions" automatically ensure low sensitivity with respect to terminations. In Section III we apply these results for the understanding of well-known structures. In addition, we show that the coupled form structure satisfies the conditions laid down in Section II, for all practically useful pole radii and pole angles.

A digital two-pair [3] is a two-input two-output structure (Fig. 2) that can be described by a transfer matrix $\mathcal{T}(z)=[T_{ij}(z)]$:

$$\begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix} = \begin{bmatrix} T_{11}(z) & T_{12}(z) \\ T_{21}(z) & T_{22}(z) \end{bmatrix} \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} \tag{1}$$

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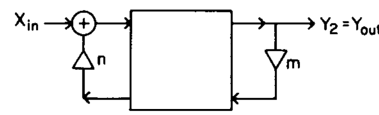

Fig. 1. The doubly constrained digital two-pair.

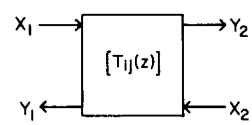

Fig. 2. A digital two-pair.

or equivalently by the Chain parameters A, B, C, D:

$$\begin{bmatrix} X_1(z) \\ Y_1(z) \end{bmatrix} = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} \begin{bmatrix} Y_2(z) \\ X_2(z) \end{bmatrix}. \tag{2}$$

The two descriptions are related as

$$A = 1/T_{21}, \quad B = -T_{22}/T_{21}, \quad C = T_{11}/T_{21}, \quad D = -(\det T)/T_{21}. \tag{3}$$

A digital filter structure (multi-input multi-output, in general) described by the transfer matrix $\mathcal{T}(z)$ is called LBR (lossless bounded real) if $\mathcal{T}(z)$ is real for real z, $\mathcal{T}(z)$ is stable and $\mathcal{T}^t(z^{-1})\mathcal{T}(z) = I$ for all z. For such a structure, the total input energy is equal to the total output energy. For an LBR two-pair, the following condition holds:

$$|X_1(e^{j\omega})|^2 + |X_2(e^{j\omega})|^2 = |Y_1(e^{j\omega})|^2 + |Y_2(e^{j\omega})|^2 \tag{4}$$

for all ω. It can be shown that a stable digital two pair is LBR if and only if

$$A\tilde{A} = 1 + B\tilde{B}, \quad B\tilde{B} = C\tilde{C}, \quad \tilde{C}(AD - BC) = B \tag{5}$$

where "tilde" denotes replacement of z with z^{-1}.

II. Conditions for Low Sensitivity of a Terminated LBR Two-Pair

Fig. 3 shows an LBR two-pair terminated at the right end with a real multiplier m, where $|m|<1$. In view of LBR property, (4) holds. The multiplier m simply constrains $X_2(z)$ to be equal to $mY_2(z)$. We thus obtain

$$|H(e^{j\omega})|^2 = \left|\frac{Y_2(e^{j\omega})}{X_1(e^{j\omega})}\right|^2 = \frac{1-|G_{\text{in}}(e^{j\omega})|^2}{1-m^2} \tag{6}$$

where $G_{\text{in}}(z) = Y_1(z)/X_1(z)$ is called the "input function" of the terminated two-pair. Thus $|H(e^{j\omega})|^2$ is bounded above by $M = 1/(1-m^2)$. This bound is attained at a frequency ω_k if $G_{\text{in}}(e^{j\omega_k}) = 0$. Now consider Fig. 4 which shows a typical plot of $|H(e^{j\omega})|^2$ which attains a maximum value of $H_{\max}$ at frequencies $\omega_1, \omega_2, \cdots$ in the passband.

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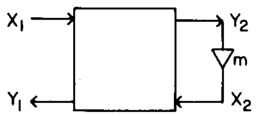

Fig. 3. Singly terminated LBR digital two-pair.

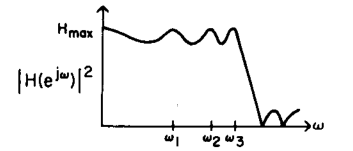

Fig. 4. A typical plot of $|H(e^{j\omega})|^2$.

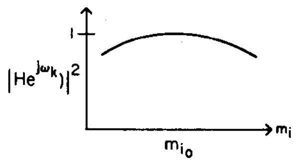

Fig. 5. Illustrating the low sensitivity property.

If the structure in Fig. 3 is such that the bound M is precisely equal to $H_{\max}$, then such an implementation has low passband sensitivity. This is because when $\omega = \omega_k$, $|H(e^{j\omega_k})|^2$ cannot increase as a result of quantization of multipliers inside the two-pair, and as a result, a plot of $|H(e^{j\omega_k})|^2$ against an internal multiplier m_i is as sketched in Fig. 5. Thus at frequencies ω_k, the first order sensitivity is zero. If we have a number of maxima in the passband, the overall passband sensitivity is, therefore, excellent. The structures advanced in [2] fall under the class described here. Certain wave-digital filters [5], [6] and orthogonal filters [9] belong to the class advanced in [2]. These structures have been verified to have low passband sensitivity. A number of simulation examples are included in [2].

Now, the input function $G_{\text{in}}(z)$ can be written in terms of the two-pair chain parameters as:

$$G_{\text{in}}(z) = \frac{C(z) + mD(z)}{A(z) + mB(z)} \tag{7}$$

and, therefore, if the bound M is attained by $|H(e^{j\omega})|$ at ω_k this implies

$$m = -C(e^{j\omega_k})/D(e^{j\omega_k}). \tag{8}$$

From (3), (5), and (8) we get[1]

$$m = T_{22}^*(e^{j\omega_k}). \tag{9}$$

Thus the attainment of the bound implies a "matching" of the constraining multiplier to the conjugate of T_{22}, which is defined as

$$T_{22}(z) = Y_2(z)/X_2(z)|_{X_1(z)=0}. \tag{10}$$

Next consider the doubly terminated LBR structure of Fig. 6. Both m and n are assumed real. Let us define the functions

$$H_{\text{in}}(z) = Y_2(z)/X_2(z)|_{X_{\text{in}}(z)=0} \tag{11}$$

$$G_{\text{in}}(z) = Y_1(z)/X_1(z) \tag{12}$$

Clearly $H_{\text{in}}(z)$ ("input function" looking from the right) is independent of m, and $G_{\text{in}}(z)$ is independent of n by definition. In order to analyze this structure it is convenient to consider a "derived structure" as follows: consider the two-pair shown in Fig. 7. This has a transfer matrix:

[1]Remembering that m is real, it is clear that if (9) holds then $T_{22}(e^{j\omega_k})$ has to be real, hence no conjugate symbol is necessary.

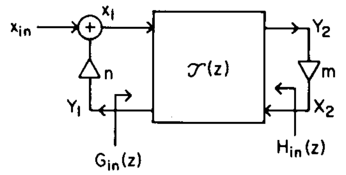

Fig. 6. The doubly constrained LBR two-pair.

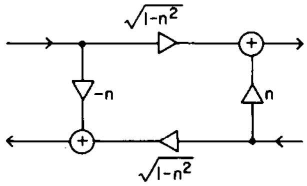

Fig. 7. A zero-order digital LBR two-pair.

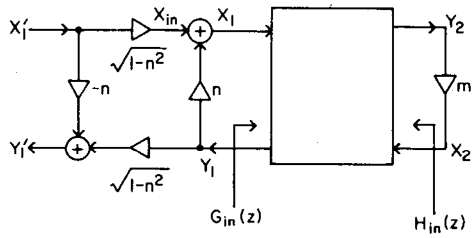

Fig. 8. The "derived structure."

$$\mathscr{T}_1 = \begin{bmatrix} -n & \sqrt{1-n^2} \\ \sqrt{1-n^2} & n \end{bmatrix} \tag{13}$$

which is LBR, i.e., $\mathscr{T}_1^t \mathscr{T}_1 = I$. If this is cascaded to the LBR two-pair of Fig. 6, we get an overall LBR two-pair that is singly terminated. This "derived structure" is shown in Fig. 8. Note that $G_{\text{in}}(z)$ and $H_{\text{in}}(z)$ are not affected by the cascading operation. Applying (6) to the modified circuit, we get

$$\left|\frac{Y_2(e^{j\omega})}{X_1'(e^{j\omega})}\right|^2 = \frac{1 - \left|\frac{Y_1'(e^{j\omega})}{X_1'(e^{j\omega})}\right|^2}{1-m^2} \tag{14}$$

i.e.,

$$|H(e^{j\omega})|^2 = \left|\frac{Y_2(e^{j\omega})}{X_{\text{in}}(e^{j\omega})}\right|^2 = \frac{(1-|G'(e^{j\omega})|^2)}{(1-n^2)(1-m^2)} \tag{15}$$

where $G'(z) = Y_1'(z)/X_1'(z)$. The upper bound on (15) is given by $M = 1/[(1-n^2)(1-m^2)]$ and is attained at $\omega = \omega_k$ if $G'(e^{j\omega_k}) = 0$. We have already seen that this also implies $H_{\text{in}}(e^{j\omega_k}) = m$. We now observe that $G'(e^{j\omega_k}) = 0$ implies

$$\sqrt{1-n^2}\, Y_1(e^{j\omega_k}) = nX_1'(e^{j\omega_k}) = n\frac{X_{\text{in}}(e^{j\omega_k})}{\sqrt{1-n^2}} \tag{16}$$

which leads to

$$Y_1(e^{j\omega_k}) = \frac{n}{1-n^2}[X_1(e^{j\omega_k}) - nY_1(e^{j\omega_k})] \tag{17}$$

i.e.,

$$G_{\text{in}}(e^{j\omega_k}) = Y_1(e^{j\omega_k})/X_1(e^{j\omega_k}) = n. \tag{18}$$

Thus if the bound M has to be attained, both the terminations should be matched to the respective "input functions."

We next note that for any doubly terminated LBR two-pair (Fig. 6), matching at one end automatically implies matching at the other. To show this let us assume $G_{in}(e^{j\omega_k}) = n$, and verify that $H_{in}(e^{j\omega_k})$ is indeed equal to m. In terms of the chain parameters of the two-pair we can write

$$H_{in}(z) = [-B(z) + nD(z)]/[A(z) - nC(z)] \quad (19)$$

With $G_{in}(e^{j\omega_k}) = n$ we get from (7)

$$m\left[D(e^{j\omega_k}) - nB(e^{j\omega_k})\right] = nA(e^{j\omega_k}) - C(e^{j\omega_k}). \quad (20)$$

By applying the properties (5) of an LBR two-pair it can be verified that (20) implies

$$m\left[A(e^{-j\omega_k}) - nC(e^{-j\omega_k})\right] = nD(e^{-j\omega_k}) - B(e^{-j\omega_k}). \quad (21)$$

Complex conjugation then leads to:

$$m = \left[-B(e^{j\omega_k}) + nD(e^{j\omega_k})\right]/\left[A(e^{j\omega_k}) - nC(e^{j\omega_k})\right] \quad (22)$$

which proves $H_{in}(e^{j\omega_k}) = m$. Thus if the structure is designed such that $n = G_{in}(e^{j\omega_k})$ at the maximal points in the passband, then conditions for low sensitivity are automatically satisfied.

Sensitivity with Respect to Terminations

Consider again Fig. 6. Assume that the structure is designed such that

$$H_{\max} \triangleq |H(e^{j\omega})|^2_{\max} = M \triangleq 1/\left[(1-n^2)(1-m^2)\right]. \quad (23)$$

In other words, the maximum value of the transfer function magnitude $H_{\max}$ is equal to the maximum gain M that can *ever be achieved* with the structure. Such realizations, in which $H_{\max}$ is equal to M, will be called "properly terminated." (For example, if $H_{\max} = 1$, then a "properly terminated" realization has $n = m = 0$, i.e., it is in fact a simple unterminated LBR two-pair). From (15) we have

$$\frac{\partial |H(e^{j\omega})|^2}{\partial n} = \frac{2n}{(1-m^2)(1-n^2)^2} - \frac{2|G'(e^{j\omega})|}{(1-m^2)(1-n^2)} \cdot \left[\frac{n|G'(e^{j\omega})|}{1-n^2} + \frac{\partial |G'(e^{j\omega})|}{\partial n}\right]. \quad (24)$$

For a "properly designed" structure, the second term is zero for all ω_k and thus the right-hand side of (24) has same value for all ω_k. Moreover, G' is very nearly zero almost everywhere in the passband of a filter with very small passband error (or passband ripple), and so $\partial|H(e^{j\omega})|^2/\partial n$ is nearly constant.[2] In other words, perturbation of n affects the passband almost uniformly, in the form of a constant error. This is clearly not harmful. From (15) it is clear that similar arguments can be made with respect to m. This situation is similar to the interesting phenomenon pointed out by Orchard [4, p. 294] for doubly terminated LC networks in the continuous time domain. In summary, therefore, a properly doubly terminated LBR two-pair displays low passband sensitivity with respect to multipliers, including the constraining multipliers m and n.

III. Relation to Other Known Structures

Wave digital filters [5], [6] are derived from doubly terminated lossless continuous-time filters by using the bilinear transformation on the transform variable, and wave-transformations on the "flow-variables." With proper scaling, the wave-digital structures can be represented as in Fig. 6. If the prototype continuous-time filter is designed so that the source transfers "maximum available power" at the maxima of transfer function magnitude, then the resulting digital structure is "properly terminated" in the sense we described earlier. So, the low sensitivity properties are thus inherited, and preserved.

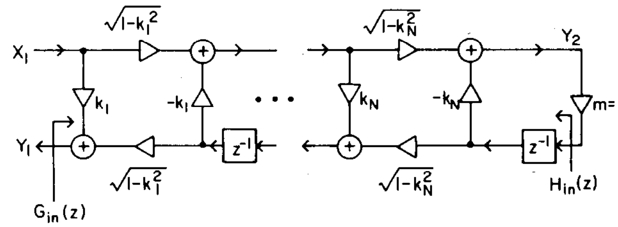

Fig. 9. The Gray and Markel normalized cascaded lattice structure.

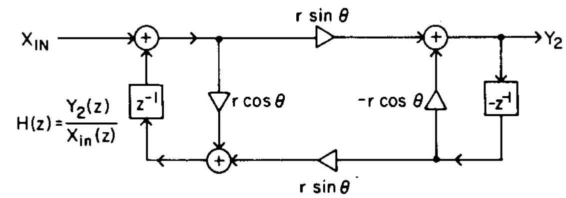

Fig. 10. The coupled form structure.

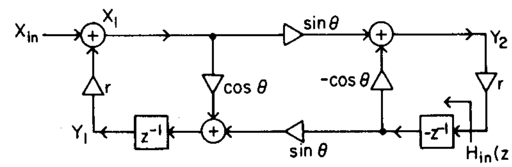

Fig. 11. An implementation of the coupled form structure using an LBR two-pair.

Next consider the Gray and Markel cascaded-lattice structure [8] realizing an all pole function, as shown in Fig. 9. Here, $n = 0$ and $m = 1$, and the two-pair in between the terminations is LBR. However, the upper bound on $|H(e^{j\omega})|^2$ is infinite because m is equal to unity, and is, therefore, unattainable. This can also be seen by noting that $G_{in}(z)$ is an allpass function with $|G_{in}(e^{j\omega})| = 1$ for all ω, and can never match "n" which is zero. In addition, the all pole function $H(z)$ does not have a plot that resembles a reasonably flat passband as in Fig. 4, and it is generally not of interest to "properly design" the structure anyway. (In order to obtain transfer function magnitudes of the form illustrated in Fig. 4, tap coefficients are needed in a lattice structure.)

The LBR-based structures presented in [2] are special cases of Fig. 6 with $n = 0$, and $0 \leqslant |m| < 1$. These are designed in such a way that the matching conditions are automatically satisfied.

Next consider the second-order "coupled form" section, which is known for low sensitivity properties [7]. We wish to point out an important property of these structures in relation to "terminal matching." Fig. 10 shows a redrawn version of the coupled form structure with poles at $z = re^{\pm j\theta}$. We can rearrange the multipliers and obtain the structure of Fig. 11 which is equivalent to Fig. 10 (except for an overall scale factor in $H(z)$). This is precisely of the form in Fig. 6 where the two-pair has transfer matrix

$$\mathcal{T}(z) = \begin{bmatrix} z^{-1}\cos\theta & -z^{-2}\sin\theta \\ \sin\theta & z^{-1}\cos\theta \end{bmatrix} \quad (25)$$

[2] The assumption that the passband ripple is very small is reasonable because it is generally these filters that need to be implemented in a "low-sensitive" manner.

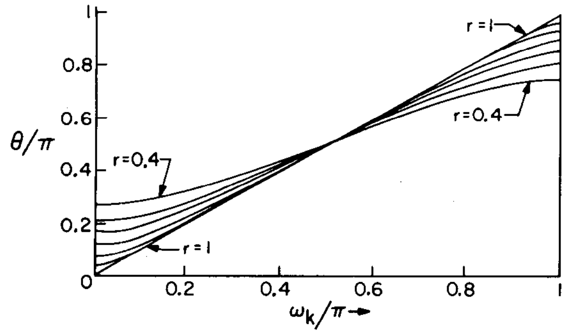

Fig. 12. Plot demonstrating that (27) approximates a straight line for $r \to 1$.

which is easily seen to be LBR. This version of the coupled form can be implemented with the help of a (lossless) cordic processor, terminated with the multiplier "r" on either side. Let us now see if this structure can be made to satisfy the terminal matching conditions. From Fig. 11 we have

$$H_{\text{in}}(z) = z^{-1}\left[\frac{\cos\theta - rz^{-1}}{1 - rz^{-1}\cos\theta}\right]. \tag{26}$$

The condition $H_{\text{in}}(e^{j\omega_k}) = r$ implies:

$$(1 + r^2)\cos\theta = 2r\cos\omega_k. \tag{27}$$

If this is satisfied, the value of $|H(e^{j\omega_k})|^2$ is equal to the bound $1/(1 - r^2)^2$. For values of r close to unity, θ is approximately the value of ω for which $|H(e^{j\omega})|$ maximizes. Thus for poles close to the unit circle, we have $2 \simeq 1 + r^2$, and (27) is nearly satisfied in the "passband," close to the peak. Fig. (12) shows plots of ω_k/π against θ/π for various r, where ω_k is calculated from (27). This plot shows that, for values of r that are not very small, the "matching" condition is satisfied at a frequency ω_k in the passband, almost equal to θ.

Concluding Remarks

In general, an LBR two-pair can always be "properly terminated" in order to achieve the bound M, so that the low-sensitivity requirement can be satisfied. The only constraint on m and n is that they should be strictly bounded by unity, i.e., $|m| < 1$ and $|n| < 1$. If $|m|$ or $|n|$ becomes equal to unity, then the bound on $|H(e^{j\omega})|$ is not achievable by a stable transfer function.

Given a stable transfer function $H(z)$, one can first scale it so that its maximum magnitude on the unit circle ($z = e^{j\omega}$) is unity. Then the transfer function can be realized using the procedure outlined in [2] in the form of Fig. 6, with $n = 0$ (m may or may not turn out to be zero, but will satisfy "$|m| < 1$"). Thus even an unterminated or singly terminated digital LBR two-pair has the potentiality to be a low-sensitive structure.

The results obtained in this paper are based entirely on z-domain arguments and digital filter signal flow graphs. Similar results can, however, also be obtained from continuous-time doubly terminated filter theory, by transforming voltage and current variables into wave variables (i.e., by going into the "scattering domain") and then applying the bilinear transformation. (Note that in the continuous-time world, properly doubly terminated lossless (LC) networks have very low passband sensitivity but singly terminated networks do not necessarily have this property. This is not surprising because an unterminated "port" in the continuous time corresponds in the wave-variable domain to an equivalent "port" terminated in a multiplier of magnitude equal to unity.) The analysis of a general doubly terminated digital LBR two-pair circuit in Section II, however, enables us to obtain a better understanding of other well-known digital filter circuits, such as the coupled form, wave filters, and cascaded lattice structures.

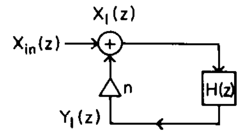

Fig. 13. A power transfer approach.

Appendix

The results of this paper show that, even though our derivations are based on an entirely z-domain argument, we have certain analogies with well-known results in continuous time filter theory. In this Appendix, we wish to point out a further instance of this analogy.

Consider the digital signal flow graph of Fig. 13. Let us define the "net power" delivered to the "load" $H(z) = H_r(z) + jH_i(z)$ at a frequency ω to be

$$P = |X_1(e^{j\omega})|^2 - |Y_1(e^{j\omega})|^2. \tag{A.1}$$

It is easy to verify that

$$P = \frac{1 - H_r^2(e^{j\omega}) - H_i^2(e^{j\omega})}{\left[1 - nH_r(e^{j\omega})\right]^2 + n^2H_i^2(e^{j\omega})}. \tag{A.2}$$

Maximizing P with respect to H_r and H_i leads to the following solutions:

Either

$$H_i(e^{j\omega_0}) = 0 \quad \text{and} \quad H_r(e^{j\omega_0}) = n \tag{A.3}$$

or

$$H_i(e^{j\omega_0}) = 0 \quad \text{and} \quad H_r(e^{j\omega_0}) = 1/n. \tag{A.4}$$

Assuming that both n and $H(z)$ are passive (i.e., bounded real), we have $|n| < 1$ and $|H(e^{j\omega})| < 1$ and therefore the only acceptable solution for passive digital networks is (A.3). Thus "maximum power" is transferred to $H(z)$ at a frequency ω_0 where $H^*(e^{j\omega})$ matches n. This is analogous to the well-known maximum power-transfer theorem of continuous-time network theory.

Acknowledgments

The author wishes to thank Dr. S. K. Mitra, of the University of California, Santa Barbara, for discussions which provoked the author to include the Appendix.

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