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AIR BUBBLE BREAKWATER

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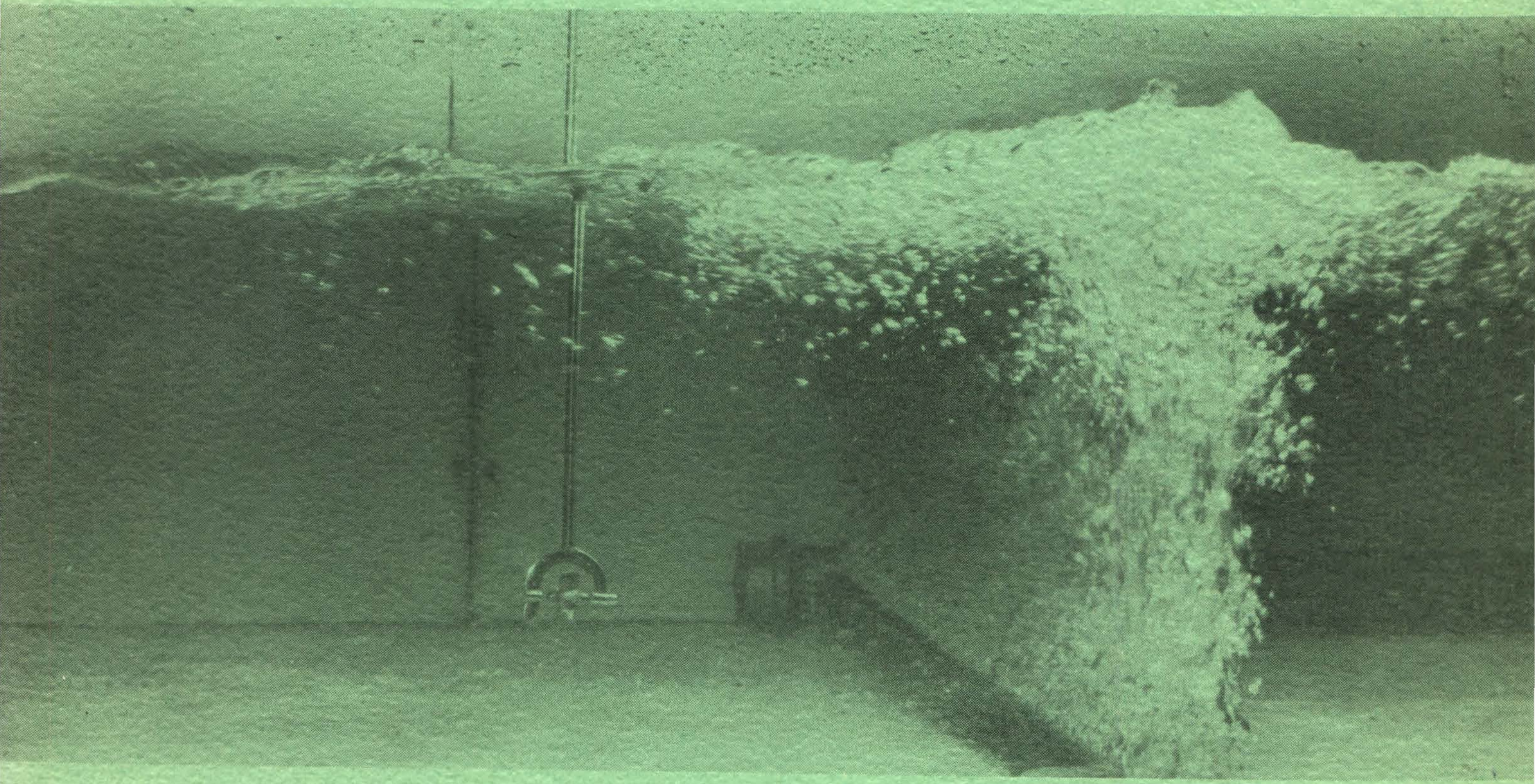

HYDRODYNAMICS LABORATORIES

California Institute of Technology

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AIR BUBBLE BREAKWATER

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This paper considers one aspect of the effectiveness of a single or repeated air bubble screen as a breakwater for gravitational waves in shallow water. The aspect considered arises from the change in density and compressibility of the bubbly water as compared with normal water outside the screen. The effects of currents produced by the mass of rising bubbles will be discussed elsewhere. Use is made here of the notation and some results from another paper entitled 'Gravitational Waves in a Shallow Compressible Liquid;' equations from that paper are denoted by primes. The properties of bubbly water are considered first, then the transmission of waves through a single bubble screen, and finally the transmission through a series of equally spaced screens.

PROPERTIES OF BUBBLY WATER

It is assumed that N bubbles are released at the bottom (y = h) per unit area and time. N is supposed to be small enough so that the bubbles move upward independently of each other. At a depth y, the radius of each is a (y); the upward speed corresponding to equilibrium motion is $U(y) = \rho_l g a^{2/3} \nu$,[1] where ρ_l is the density of the normal liquid (water) and ν is the coefficient of viscosity. If the expansion of the bubble as it rises is slow enough to be isothermal, then the radius at any depth is given by

$$p_o(y)\, a(y)^3 = p_{oo} a_o^3 = p_{oh} a_h^3, \qquad (1)$$

where p_{oo} and p_{oh} are the surface (atmospheric) and bottom pressures, and a_o and a_h the surface and bottom radii. The pressure at depth y is given by Eq. (12'):

$$p_o(y) = p_{oo} + g \int_y^o \rho_o(y)\, dy \qquad (2)$$

in terms of the density $\rho_o(y)$. If the density of the air is neglected, the density of the bubbly water can also be expressed in terms of $p_o(y)$:

$$\rho_o(y) = \rho_l \left(1 - \frac{N}{U}, \frac{4\pi a^3}{3}\right) = \rho_l - \frac{A}{p_o^{1/3}}, \quad A \equiv \frac{4\pi n \nu a_o p_{oo}{}^{1/3}}{g} \quad . \qquad (3)$$

The speed c(y) of pure elastic waves in the liquid at depth y is given by (see Eq. (4')):

$$\frac{1}{c^2} = \frac{d\rho_o}{dp_o}. \qquad (4)$$

The way in which the derivative in (4) is calculated depends on the rate at which the pressure changes due to waves traveling through the liquid. For slow variations, both a and U change, and

[1] Lamb, "Hydrodynamics", p. 601 (Dover, New York, 1945).

$$\frac{1}{c^2} = \frac{A}{3p_o^{4/3}} .$$

For fast variations (but not so fast that the air ceases to be isothermal), a will change without U, and

$$\frac{1}{c^2} = \frac{A}{p_o^{4/3}} .$$

In general, we put

$$\frac{1}{c^2} = \frac{\beta A}{p_o^{4/3}} , \tag{5}$$

where β is between 1/3 and 1. In practical cases, it seems likely to be near the lower limit. The compressibility of the water has been neglected in arriving at Eq. (5).

The density and pressure should be found by solving Eqs. (2) and (3) simultaneously. However, the density of the bubbly water probably cannot be made much different from that of normal water. Partly because of this, and partly because of the complications that would be entailed and the further approximations that are made below, we use normal water density to calculate the pressure from Eq. (2):

$$p_o(y) = p_{oo} + \rho_l g \, |y| \quad .$$

Eq. (23') should be used to calculate the wave speed c_w in shallow bubbly water. This is sufficiently complicated so that it is worthwhile using the simpler expression (24') instead, at least for the purpose of a preliminary study such as the present one. This formula:

$$c_w^{\,2} = c^2\left(1 - e^{-\frac{gh}{c^2}}\right) \tag{6}$$

assumes that c is independent of depth, so that the value of c at some average or typical depth must be used. Then the magnitude of the ratio gh/c^2 determines whether the waves are predominantly gravitational (c_w close to $\sqrt{gh}$) or elastic (c_w close to c) in character. With the help of Eq. (5) we obtain

$$\frac{gh}{c^2} = \frac{\beta g h A}{p_o^{4/3}} = \frac{\epsilon \beta \rho_l g h}{p_o} , \tag{7}$$

where $\epsilon = 1 - (\rho_o/\rho_l)$ is the volume fraction of air in the water at the typical depth. As a numerical example, we can assume that $\epsilon = 0.2$, $\beta = 1/3$, h = 100 ft, $p_o/\rho_l g$ = 30 ft. (one atmosphere) + 0.8x50 ft (half depth at density 0.8) = 70 ft. so that the ratio (7) is approximately equal to 0.1. This shows that the waves are predominantly gravitational in character. Expansion of the exponential in (6) gives:

$$c_w \cong \sqrt{gh}\left(1 - \frac{gh}{4c^2}\right) . \tag{8}$$

The interface between normal and bubbly water can be considered in the following way. As in the last section of the paper cited above, the turbulence and currents in the transition region between the two liquids are ignored. It is assumed that the proper boundary conditions for wave propagation consist in matching the pressure change and the horizontal velocity component that are associated with the wave motion at some typical point A (see Fig. 1). The static pressures are more nearly the same at the bottom than at the point A, where the upward current maintains a dynamic equilibrium. The pressure changes due to the wave motion are superposed on this and must be nearly the same on both sides of the interface.

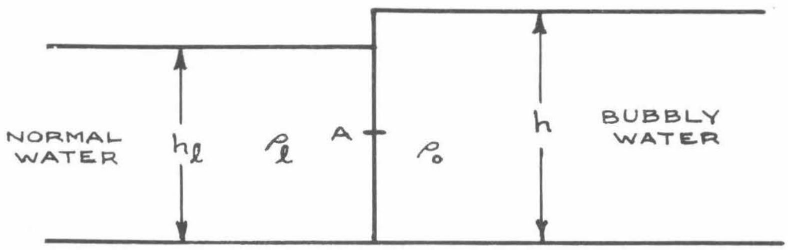

FIG. 1

The wave speed in normal water is $c_w^l = \sqrt{gh_l}$, so that the ratio of wave speeds is, from (8):

$$\frac{c_w}{c_w^l} = \sqrt{\frac{h}{h_l}}\left(1 - \frac{gh}{4c^2}\right). \tag{9}$$

In similar fashion, the impedance ratio is

$$\frac{Z}{Z_l} = \frac{\rho_0 c_w}{\rho_l c_w^l} = \frac{\rho_0}{\rho_l}\sqrt{\frac{h}{h_l}}\left(1 - \frac{gh}{4c^2}\right) \tag{10}$$

We can then write approximately, if the matching point A is at depth h/2:

$$\rho_0 = (1 - \epsilon)\rho_l;$$

$$\rho_0 h \cong \rho_l h_l, \quad \text{or:} \quad h_l \cong (1 - \epsilon)\, h;$$

$$\frac{gh}{c^2} \cong \frac{\epsilon \beta h}{30 + \left(\frac{1 - \epsilon}{2}\right)h}, \quad h \text{ in ft };$$

the last relation is derived from (7). Eqs. (9) and (10) can then be written:

$$\frac{c_w}{c_w^l} \cong \frac{1}{\sqrt{1 - \epsilon}}\left[1 - \frac{\epsilon}{6(1 - \epsilon)} \cdot \frac{h_l}{h_l + 60}\right], \tag{11}$$

$$\frac{Z}{Z_l} \cong \sqrt{1 - \epsilon}\left[1 - \frac{\epsilon}{6(1 - \epsilon)} \cdot \frac{h_l}{h_l + 60}\right], \tag{12}$$

where β has been set equal to 1/3 and h_l is measured in feet. The second terms in the brackets of (11) and (12) are usually small corrections, about 2½ per cent for the numerical example given above. Thus the difference in density between normal and bubbly water is of much greater importance than the difference in compressibility.

TRANSMISSION THROUGH A SINGLE SCREEN

The transmission problems considered in the remainder of this paper are most readily handled with the help of the complex number notation used in the theory of alternating current circuits. We assume that a wave of angular frequency ω (ordinary or circular frequency = $\omega/2\pi$) is incident from the left on a bubble screen that extends from $x = 0$ to $x = b$. The propagation number $k = \omega/c_w^l$ in the normal water is equal to 2π divided by the normal wave length, while the similar quantity for the bubbly water is $\alpha = \omega/c_w$. A typical sinusoidal wave propagating to the right in normal water has the form $Ae^{i(kx - \omega t)}$; since the real part of an expression like this is always implied, the actual wave is $|A| \cos(kx - \omega t + \phi)$, where the complex quantity A is equal to its magnitude $|A|$ multiplied by $e^{i\phi}$. The complex form has the advantage that the amplitude ($|A|$) and phase (ϕ) are combined in a single complex quantity (A). Then since all waves have the common time factor $e^{-i\omega t}$, it can be omitted throughout, and the wave represented simply by Ae^{ikx}. In similar fashion, a wave moving to the left can be represented by Be^{-ikx}.

The pressure at the typical depth A in Fig. 1 can then be written

$$\begin{aligned} p_1 &= Ae^{ikx} + Be^{-ikx}, & x &\leq 0; \\ p_1 &= Ce^{i\alpha x} + De^{-i\alpha x}, & 0 \leq x &\leq b; \\ p_1 &= Ee^{ikx}, & b &\leq x. \end{aligned} \tag{13}$$

A and B are the (complex) amplitudes of the incident and reflected waves to the left of the screen, C and D the amplitudes of the waves traveling to the left and right within the screen, and E the amplitude of the transmitted wave beyond the screen. The boundary conditions to be applied at the two interfaces $x = 0$ and $x = b$ are that the pressure and the horizontal velocity component be continuous. These are the same as requiring that p_1 and $(1/\rho_a)(\partial p_1/\partial x)$ be continuous, where ρ_a is equal to ρ_l in normal water and to ρ_o in bubbly water. Thus there are four equations, from which the ratios of B, C, D, and E to A can be calculated. The result for the ratio of the transmitted to the incident amplitude is

$$\left|\frac{E}{A}\right| = \frac{1}{\sqrt{1 + \dfrac{\left(Z^2 - Z_l^2\right)^2}{4Z^2 Z_l^2} \sin^2 \alpha b}}, \tag{14}$$

where the ratio of impedances is given in equation (12).

The attenuation produced by the screen is greatest when the screen is half a wave length thick, so that $\sin \alpha b = 1$. While this would probably be extravagant in practice, it nevertheless shows the best that can be done with a single screen. In this case, (14) reduces to

$$\left|\frac{E}{A}\right| = \frac{2\ Z\ Z_l}{Z^2 + Z_l^2} = \frac{2\left(\frac{Z}{Z_l}\right)}{1 + \left(\frac{Z}{Z_l}\right)^2} \tag{15}$$

According to equation (12), Z/Z_l is close to unity, so we can put $Z/Z_l = 1 - \delta$, where δ is small in comparison with unity. Then (15) can be written approximately

$$\left|\frac{E}{A}\right| = 1 - \tfrac{1}{2}\delta^2. \tag{16}$$

The numerical example given above leads to $\delta \cong 0.1$, when (16) shows that the amplitude decreases only by about ½ per cent. Thus a single bubble screen is not expected to make an effective breakwater, at least so far as the effect considered here is concerned.

TRANSMISSION THROUGH A SERIES OF SCREENS

It is well known that the transmission and reflection of waves can be greatly affected by weakly scattering elements (such as the bubble screen analyzed above), provided that the elements are so arranged that the waves scattered from them interfere with each other. In the present situation, one would expect that a series of bubble screens, spaced about half a wave length apart in the direction of wave propagation, would scatter waves backwards that would interfere constructively with each other. In this way, the reflected wave is built up, and correspondingly the transmitted wave is diminished.

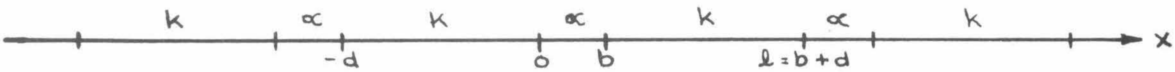

FIG. 2

We assume for the present that there is an infinite series of bubble screens, each of thickness b, in which the propagation number is α and the impedance Z. Between these screens are stretches of normal water of thickness d, in which the propagation number is k and the impedance Z_l. The repetition distance for the whole arrangement is $l = b + d$ (see Fig. 2). The theory presented below is very similar to that already worked out in connection with the electronic theory of crystals,[2] and applied to many other problems of wave propagation.[3]

It can be shown in a very general way[3] that there are wave solutions in a periodic lattice, such as that illustrated in Fig. 2, which are themselves repetitive except that the complex amplitude is multiplied by a constant μ in going from any point x to the point $x + l$. Thus the entire problem can be solved by considering only the solution between $x = -d$ and $x = b$, fitting boundary conditions at $x = 0$, and requiring that the values of pressure and horizontal velocity component at $x = b$ be equal to μ times the corresponding values at $x = -d$. We thus have, in analogy with Eqs. (13):

[2] Kronig and Penney, Proc. Roy. Soc., A130, 499 (1931).

[3] Brillouin, "Wave Propagation in Periodic Structures" (McGraw-Hill, New York, 1948).

$$p_1 = Ae^{ikx} + Be^{-ikx}, \quad -d \leqq x \leqq 0;$$
$$p_1 = Ce^{i\alpha x} + De^{-i\alpha x}, \quad 0 \leqq x \leqq b. \tag{17}$$

Application of the boundary conditions leads to four simultaneous homogeneous equations for the four unknowns A, B, C, and D. These equations have a nonvanishing solution only if the determinant of their coefficients is zero. Since these coefficients depend on μ, the vanishing of the determinant gives an equation from which μ can be calculated. This equation is a quadratic:

$$\mu^2 - 2\gamma\mu + 1 = 0,$$
$$\gamma \equiv \cos kd \cos \alpha b - \frac{1}{2}\left(\frac{Z}{Z_l} + \frac{Z_l}{Z}\right) \sin kd \sin \alpha b. \tag{18}$$

The two solutions of Eq. (18) are

$$\mu = \gamma \pm \sqrt{\gamma^2 - 1}. \tag{19}$$

It is apparent from (19) that μ is real if $\gamma \geqq 1$ or $\gamma \leqq -1$, and is complex if γ lies between -1 and 1. In the former case, the two real values of μ are reciprocals of each other, so that one value corresponds to a wave that decreases exponentially in amplitude going to the right, and the other to a wave that decreases in amplitude at the same exponential rate going to the left. In the latter case, the magnitude of the complex number μ is equal to unity, so that waves are propagated in both directions without attenuation, but with a change of phase from one screen to the next.

While the foregoing results are strictly valid only for an infinitely long series of bubble screens, they provide a useful approximation for the attenuation produced by a moderate number (say five or more) such screens.

Eqs. (18) and (19) give the condition that waves be attenuated and the amount of the attenuation. The latter is conveniently expressed in terms of an exponential function. The μ value that is less than unity corresponds to waves propagated to the right; since each screen multiplies the amplitude by μ, the ratio of transmitted to incident amplitude after n screens is

$$\mu^n = e^{-n \ln\left(\frac{1}{\mu}\right)}. \tag{20}$$

The expression (18) for γ can be rewritten in the form

$$\gamma = G \cos(kd + \psi),$$
$$G \equiv \sqrt{1 + \left[\frac{1}{4}\left(\frac{Z}{Z_l} + \frac{Z_l}{Z}\right)^2 - 1\right] \sin^2 \alpha b}, \tag{21}$$
$$\tan\psi \equiv \frac{1}{2}\left(\frac{Z}{Z_l} + \frac{Z_l}{Z}\right) \tan \alpha b.$$

We now put $Z/Z_l = 1 - \delta$, where δ is small in practical cases, and remember that α is nearly equal to k; then if the screen thickness is small in comparison with the wave length, we can approximate:

$$\frac{Z}{Z_l} + \frac{Z_l}{Z} \cong 2 + \delta^2 ,$$

$$\sin \alpha b \cong \tan \alpha b \cong kb.$$

Then ψ is approximately equal to kb, and (21) becomes:

$$\gamma \cong (1 + \tfrac{1}{2}\delta^2 k^2 b^2) \cos kl. \tag{22}$$

The attenuation is greatest when the magnitude of γ (regardless of its sign) is greatest. Apart from this requirement, l should be as small as possible so that the screens occupy a minimum amount of space. We therefore choose $l = \tfrac{1}{2}\lambda_o$, where λ_o is the wave length for which the greatest attenuation is desired. For any wave length that differs from λ_o by a relatively small amount, we can put:

$$\lambda = \lambda_o + \Delta\lambda,$$

$$k \cong \frac{2\pi}{\lambda_o}\left(1 - \frac{\Delta\lambda}{\lambda_o}\right) ,$$

$$kl \cong \pi\left(1 - \frac{\Delta\lambda}{2l}\right),$$

$$\cos kl \cong -1 + \frac{\pi^2(\Delta\lambda)^2}{8l^2} ,$$

$$\gamma \cong -1 - \frac{\pi^2\delta^2 b^2}{2l^2} + \frac{\pi^2(\Delta\lambda)^2}{8l^2}$$

Substitution of this expression for γ into (19) gives an approximate formula for μ:

$$\mu \cong -1 \pm \sqrt{\frac{\pi^2\delta^2 b^2}{l^2} - \frac{\pi^2(\Delta\lambda)^2}{4l^2}} .$$

The minus sign here indicates that the phase changes by 180° from one screen to the next. Then from (20), the ratio of transmitted to incident amplitude after n screens becomes, to the same approximation:

$$e^{\frac{-n\pi}{l}\sqrt{(\delta b)^2 - \left(\frac{\Delta\lambda}{2}\right)^2}} \tag{23}$$

Eq. (23) shows that the attenuation is greatest when $\Delta\lambda = 0$, and disappears when $\Delta\lambda = \pm 2\ \delta\ b$. Thus the maximum attenuation is

$$e^{\frac{-n\pi\delta b}{l}}, \tag{24}$$

and occurs when the wave length is twice the spacing between screens. With n = 10, $\delta = 0.1$, and $b/l = 0.1$, for example, (24) is equal to 0.73, so that the wave amplitude is decreased by about 27 per cent.

It is interesting to note that this effect is far greater than the ½ per cent decrease obtained in the last section for a single screen of the same total thickness (half a wave length). On the other hand, the single screen will attenuate a wide range of wave lengths almost equally well, because of the relatively slow variation of the sin αb term in (14), while according to (23) the series of thinner screens will attenuate only a narrow range of wave lengths that lie within 1 per cent on either side of λ_0. Thus the series of screens is useful only if it can be 'tuned' to a narrow band of wave lengths which is all that must be attenuated.