Derivation of New and Existing Discrete-Time Kharitonov Theorems Based on Discrete-Time Reactances

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Abstract—**The stability of a continuous-time linear time-invariant system with an uncertain characteristic polynomial has been analyzed by Kharitonov and other researchers. An interesting proof of Kharitonov's continuous-time stability theorem, based on reactance functions, has recently been advanced. Some of Kharitonov's results have been partially extended to discrete-time systems recently. On the other hand, the use of discrete-time reactances in digital signal processing was noticed by Schuessler over a decade ago. In this paper we first use a discrete-time reactance approach to give a second proof of existing discrete-time Kharitonov-type results. We then use the same reactance language to derive a new discrete-time Kharitonov-type theorem which, in some sense, is a very close analog to the continuous-time case. We also point out the relation between discrete-time reactances and the technique of line-spectral pairs (LSP) used in speech compression.**

I. Introduction

In digital signal processing, it is often necessary to test the stability of a linear time-invariant system (such as an IIR digital filter). Well-established stability-test procedures are available for this, such as, for example, Jury's test [19], [17]. The use of discrete-time reactances in digital signal processing, for stability-test purposes, can be inferred from Schuessler's work [11].

In some applications, the coefficients of the denominator $D(z)$ of the transfer function $H(z)$ may be uncertain for one of several possible reasons. One such example is in adaptive IIR filtering for system identification [21], where the unknown system's denominator should be estimated in order to construct a rational function with certain passivity properties. Other situations of this type may arise due to quantization effects, or merely due to inherent uncertainty in the estimation of denominator coefficients.

In the continuous-time world, stability of linear systems with uncertain denominators has received considerable attention. One of the most interesting results in this direction is Kharitonov's theorem [1]. To be more specific, suppose we are given a polynomial

$$E(s) = e_0 + e_1 s + \cdots + e_N s^N. \quad (1)$$

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Assume that this is the denominator of the transfer function $H(s)$ of a continuous-time system. With $H(s)$ expressed in irreducible form, we know that $H(s)$ is asymptotically stable if and only if $E(s)$ is strictly Hurwitz (abbreviated SH), i.e., if and only if $E(s)$ has all zeros restricted to the open left half plane (i.e., Re $[s] < 0$). Consider the situation where the coefficients e_k (assumed to be real in this paper) are uncertain, each belonging to a known region:

$$e_{k,min} \leq e_k \leq e_{k,max}, \quad 0 \leq k \leq N. \quad (2)$$

Notice that the bounds on e_k and e_m are independent of each other for $k \neq m$. Let $\mathcal{S}$ denote the set of polynomials $E(s)$ which have coefficients belonging to the region $\mathcal{R}$ described by (2) in the $N + 1$-dimensional space. A fundamental result, proved by Kharitonov [1], says that every $E(s)$ belonging to $\mathcal{S}$ is SH, if and only if four specific polynomials belonging to $\mathcal{S}$ are SH. An elegant and clear reinterpretation of this result, based on the language of electrical reactances, was advanced in [10].

Interesting extensions of Kharitonov's results to the discrete-time case have been reported in recent years. Some authors [3], [5] have used a direct discrete-time approach for this purpose. (The use of bilinear transformation in this context has also been discussed in [3].) To put them in the perspective of this paper, let us define

$$D(z) = d_0 + d_1 z^{-1} + \cdots + d_N z^{-N}, \quad d_N \neq 0, \quad (3)$$

to be the Nth degree denominator of a discrete-time system $H(z)$. We say that $D(z)$ is SH if all its zeros are strictly inside the unit circle of the z-plane. Then $H(z)$ is asymptotically stable if and only if $D(z)$ is SH. We consider only real d_k in this paper. Assume now that d_k are uncertain, and belonging to intervals of the form

$$d_{k,min} \leq d_k \leq d_{k,max}, \quad 0 \leq k \leq N \quad (4)$$

where the bounds on d_k and d_m are independent for $k \neq m$. Once again, let $\mathcal{R}$ denote the region in the $N + 1$-dimensional space, described by (4), and let $\mathcal{S}$ denote the set of $D(z)$ satisfying (4). It turns out that there is no simple equivalent of Kharitonov's continuous-time result [1] for this setup.

If we permit only a subset of coefficients d_k to be uncertain, then results similar to Kharitonov's first theorem in [1] (which is weaker than his second theorem) can still be derived as done in [3]. More specifically, suppose that only the coefficients

$$d_k, \quad N/2 \leq k \leq N, \tag{5}$$

are uncertain. For example, with $N = 4$, d_2, d_3, and d_4 are uncertain, whereas with $N = 5$, we permit d_3, d_4, and d_5 to be uncertain. Let us define an "extreme polynomial" to be one which takes on some pattern of extreme values for these uncertain coefficients. For example, with $N = 4$, $D(z) = d_0 + d_1 z^{-1} + d_{2,min} z^{-2} + d_{3,min} z^{-3} + d_{4,max} z^{-4}$ is an extreme polynomial. Clearly, there are 2^K extreme polynomials where K is the number of uncertain coefficients. The results in [3] show that every $D(z)$ in $\mathcal{S}$ is SH if and only if these 2^K extreme polynomials are SH, provided that only the coefficients (5) are the uncertain ones. If a coefficient d_k with $k < N/2$ is uncertain, a similar result does not hold, as demonstrated by the counterexample in [3].

Schuessler [11] has recognized the importance of discrete-time reactances in digital signal processing, particularly in the context of stability testing. The purpose of this paper is twofold. We first reinterpret the recent discrete-time Kharitonov-type results in terms of the discrete-time reactance language used by Schuessler. This is done in Section II, which also includes a brief review of discrete-time reactances. The second purpose of the paper is to present a new theorem (Section III) for discrete-time systems, which we believe to be a closer analog of Kharitonov's continuous-time result. This result is obtained by rewriting (3) in terms of a new set of $N + 1$ coefficients. If all these coefficients are uncertain (with uncertainties independent of each other), then the set of all polynomials in the uncertain class are SH if and only if four extreme polynomials (to be defined in Section III) are SH. Notice that if these uncertainties are mapped to those of d_k, the bounds on d_k and d_m are not in general independent. This situation is similar to be the one that arises when one attempts to use the bilinear transformation to derive discrete-time Kharitonov results from the continuous-time domain [3]. The coefficients we use in Section III are, however, *not* related to bilinear transformation. The result and its derivation in Section III are directly in the discrete-time domain.

In Section IV we point out the close connection between discrete-time reactances and the concept of *line-spectral pairs* (LSP) which is used in speech compression [22]–[31]. Such a connection is precisely the reason why the LSP parameters can be quantized without violating the SH property of the denominator $D(z)$ of the all-pole speech model.

Notations Used in the Paper: The variables s and z are the transform variables for continuous-time and discrete-time cases, respectively. The corresponding steady-state frequencies are Ω and ω, respectively, so that $s = j\Omega$ and $z = e^{j\omega}$. A *real rational* function $G(s)$ or $H(z)$ is a ratio of two polynomials (in s or in z^{-1}, respectively) with real coefficients. In this paper, we shall deal only with real rational functions and real-coefficient polynomials. The abbreviation LBR stands for lossless bounded real and denotes a real rational transfer function $H(z)$ which is all-pass and asymptotically stable. The degree of a rational function $H(z)$ is the highest power of z^{-1} appearing in its expression after cancelling common factors between numerator and denominator.

II. Reinterpretation of Known Results

In the continuous-time world, a reactance $Z(s)$ is the driving point impedance of an LC network with positive valued inductors and capacitors. Such functions satisfy several beautiful mathematical properties which have far-reaching implications in the synthesis of lossless and passive networks [18]. The text by Balabanian and Bickart [13] is an excellent reference on reactances and their properties. In this section we shall make use of the concept of discrete-time reactances [11] to rederive some discrete-time Kharitonov-type results published recently [3].

A. Discrete-Time Reactances and Relation to Stability

A discrete-time reactance function $G(z)$ is, in principle, the bilinearly transformed version of a reactance $Z(s)$. All the properties of $Z(s)$ are thereby transformed into the z-domain appropriately. As mentioned in Section I, all our discussions are restricted to real rational functions only. The reactance can be independently defined in the z domain as follows.

Definition: A real rational $G(z)$ is said to be a reactance if $\text{Re}\,[G(e^{j\omega})] = 0$ and if $\text{Re}\,[G(z)] > 0$ for $|z| > 1$.

In other words, $G(z)$ is a reactance if it is purely imaginary on the unit circle, and has positive real part outside the unit circle. The above definition induces many properties on the reactance $G(z)$. Some of these, which are relevant for our discussion, are summarized here. See [11] for details.

Property 1—The Alternation Property: All the poles and zeros of $G(z)$ are *single*, lie on the unit circle, and interlace with each other (see Fig. 1). As a result, an Nth degree reactance has N poles and N zeros on the unit circle.

Property 2: The point $z = 1$ (i.e., $\omega = 0$) is necessarily a pole or a zero. The same is true of $z = -1$ (i.e., $\omega = \pi$).

Property 3—The Monotone Increasing Property: If we plot the imaginary part of $G(e^{j\omega})$ with respect to ω, the result is a monotone increasing function, separated by discontinuities caused by poles (see Fig. 2).

Property 4: A convex combination of two reactances $G_1(z)$ and $G_2(z)$ (i.e., a function of the form $\lambda G_1(z) + (1 - \lambda)\, G_2(z)$ with $0 \leq \lambda \leq 1$) is a reactance.

A number of useful theorems in connection with reactances are stated next. The proofs follow immediately from [11] and [13].

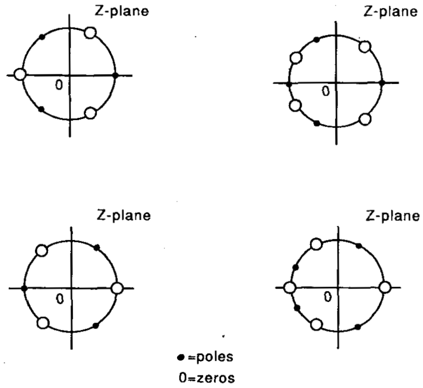

Fig. 1. Examples of pole-zero patterns for discrete-time reactance functions.

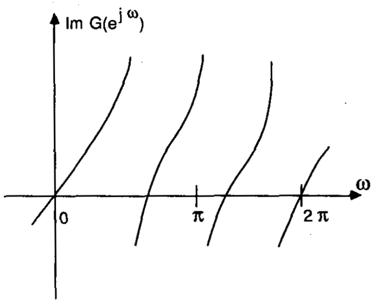

Fig. 2. A typical plot of the imaginary part of a reactance $G(e^{j\omega})$ of degree 3.

Theorem 1—The Alternation Theorem: A real rational reactance function $G(z)$ satisfies the alternation property. Conversely, if $G(z)$ satisfies the alternation property, then either $G(z)$ or $-G(z)$ is a reactance.

Ambiguity: When testing for the reactance property, the sign ambiguity in the above theorem can be resolved by evaluating $G(z)$ at some point, say $z = 2$, outside the unit circle. In proving theorems, the ambiguity can often be resolved by invoking the positive-slope condition seen in Fig. 2.

Theorem 2—The All-Pass Theorem: Let $G(z)$ and $H(z)$ be two real rational functions related by

$$G(z) = \frac{1 + H(z)}{1 - H(z)}. \tag{6}$$

Then $G(z)$ is an Nth degree reactance if and only if $H(z)$ is an Nth degree asymptotically stable all-pass function (i.e., if and only if $H(z)$ is Nth degree LBR).

Relation Between Reactances and Stability Test: Suppose $D(z)$ is a polynomial of degree N as in (3). Define a rational function

$$H(z) = \frac{\hat{D}(z)}{D(z)}, \tag{7}$$

where we have defined

$$\hat{D}(z) = d_N + d_{N-1}z^{-1} + \cdots + d_0 z^{-N}, \tag{8}$$

so that $\hat{D}(z)$ is the "flipped" polynomial. Clearly, $|H(e^{j\omega})| = 1$ so that $H(z)$ is all-pass. Now, any zero of $D(z)$ on the unit circle is also a zero of $\hat{D}(z)$, so that it cancels in the ratio $\hat{D}(z)/D(z)$. Consequently, $H(z)$ has degree N if and only if $D(z)$ has no zeros on the unit circle. Moreover, $H(z)$ is asymptotically stable (hence LBR) if and only if $D(z)$ has no zeros outside the unit circle. Combining these, we obtain the following result.

Lemma 1: The real-coefficient polynomial $D(z)$ is SH if and only if $H(z)$ in (7) is Nth degree LBR.

We can now combine this lemma with Theorem 2 to obtain more convenient stability results. With $H(z)$ as in (7), note that the function $G(z)$ in (6) becomes

$$G(z) = \frac{D(z) + \hat{D}(z)}{D(z) - \hat{D}(z)}. \tag{9}$$

Defining

$$S(z) = \frac{D(z) + \hat{D}(z)}{2}, \qquad A(z) = \frac{D(z) - \hat{D}(z)}{2}, \tag{10}$$

we see that $S(z)$ and $A(z)$ are the symmetric and antisymmetric parts of the polynomial $D(z)$, and that $D(z) = S(z) + A(z)$. From Theorem 2 we know that $G(z)$ in (9) is a reactance of degree N if and only if $H(z)$ is LBR of degree N. By combining this with the above observations, we have the following result.

Theorem 3: Let $D(z)$ be a real-coefficient Nth degree polynomial as in (3). Define the symmetric and antisymmetric parts as in (10) and let $G(z) = S(z)/A(z)$. Then $G(z)$ is a reactance of degree N if and only if $D(z)$ is SH.

In practice, in order to test whether $D(z)$ is SH, it is usually convenient to test whether $H(z)$ is LBR (i.e., by using Jury's test [17], [19]), rather than to test whether $G(z)$ is a reactance. The usefulness of the above theorem is primarily theoretical, as we shall see in future proofs.

A Modified Reactance Function: With $H(z)$ defined as in (7), suppose we define

$$H'(z) = z^{-L}H(z) \tag{11}$$

where L is a nonnegative integer. Clearly, $D(z)$ is SH if and only if $H'(z)$ is LBR of degree $N + L$. Define

$$G'(z) = \frac{S'(z)}{A'(z)} \tag{12}$$

where

$$S'(z) = \frac{D(z) + z^{-L}\hat{D}(z)}{2},$$

$$A'(z) = \frac{D(z) - z^{-L}\hat{D}(z)}{2}. \tag{13}$$

Clearly, $S'(z)$ and $A'(z)$ are again symmetric and antisymmetric polynomials. In a manner analogous to Theorem 3, the following result is also true.

Theorem 4: With $L \geq 0$, $G'(z)$ is a reactance of degree $N + L$ if and only if $D(z)$ is SH of degree N.

The proof is readily developable, and is therefore omitted here. With $L = 0$, this result is of course the same as Theorem 3. The usefulness of nonzero L will become evident soon.

B. Second Interpretation of Known Discrete-Time Kharitonov-Type Results

Let $D(z)$ be an Nth degree polynomial as in (3), and let the Mth coefficient d_M be the only uncertain coefficient, lying in the range

$$d_{M,min} \leq d_M \leq d_{M,max}. \tag{14}$$

Define the two Nth degree extreme polynomials $D_{min}(z)$ and $D_{max}(z)$ to be the polynomial $D(z)$ with d_M taking on the lower and upper bounds, respectively. Clearly, any d_M in the range (14) is a convex combination of the form $d_M = \lambda d_{M,min} + (1 - \lambda) d_{M,max}$ where $0 \leq \lambda \leq 1$. So any $D(z)$ with d_M satisfying (14) (and other coefficients fixed) can be written as a convex combination

$$D(z) = \lambda D_{min}(z) + (1 - \lambda) D_{max}(z), \qquad 0 \leq \lambda \leq 1. \tag{15}$$

For $M \geq N/2$, it has been shown in [3] that $D(z)$ is SH for all d_M in (14), if and only if $D_{min}(z)$ and $D_{max}(z)$ are SH. We shall now revisit the proof with the help of reactances.

If $D_{min}(z)$ and $D_{max}(z)$ are SH, we can associate degree $N + L$ reactances $G'_{min}(z)$ and $G'_{max}(z)$ with them, by use of Theorem 4. These reactances are

$$G'_{min}(z) = S'_{min}(z)/A'_{min}(z), \qquad G'_{max}(z) = S'_{max}(z)/A'_{max}(z) \tag{16}$$

where the symmetric and antisymmetric components are defined as

$$S'_{min}(z) = \frac{D_{min}(z) + z^{-L}\hat{D}_{min}(z)}{2}, \qquad A'_{min}(z) = \frac{D_{min}(z) - z^{-L}\hat{D}_{min}(z)}{2}, \tag{17}$$

and

$$S'_{max}(z) = \frac{D_{max}(z) + z^{-L}\hat{D}_{max}(z)}{2}, \qquad A'_{max}(z) = \frac{D_{max}(z) - z^{-L}\hat{D}_{max}(z)}{2}. \tag{18}$$

The purpose of the parameter L is the following. We can choose L (for a given M) in such a way that the centers of the S and A polynomials coincide with M. If we do so, then $A'_{min}(z)$ and $A'_{max}(z)$ are the same, which simplifies the proof.

More specifically, choose $L = 2M - N$, so that the four polynomials $A'_{min}(z)$, $A'_{max}(z)$, $S'_{min}(z)$, and $S'_{max}(z)$ have the same center. By the above definitions we see that $G'_{min}(z)$ and $G'_{max}(z)$ have the same denominators. Their numerators are also the same except for the center coefficient.

Now consider $D(z)$ with any d_M satisfying (14). The remaining coefficients d_k, $k \neq M$ are assumed to be fixed. Let $S'(z)$, $A'(z)$, and $G'(z)$ be defined in the usual way. Since $D(z)$ is a convex combination (15), we see that $G'(z)$ is the following convex combination:

$$G'(z) = \lambda G'_{min}(z) + (1 - \lambda) G'_{max}(z), \qquad 0 \leq \lambda \leq 1. \tag{19}$$

If $D_{min}(z)$ and $D_{max}(z)$ are SH, then $G'_{min}(z)$ and $G'_{max}(z)$ are reactances of degree $N + L$, so $G'(z)$ is a reactance of degree $N + L$ (Property 4). As a result, $D(z)$ is SH (Theorem 4) for *any* d_M in (14). We have therefore proved the following.

Theorem 5 [3]: Let $D(z)$ be a real coefficient polynomial of degree N as in (3), and let d_M be uncertain, belonging to the interval (14), with $M \geq N/2$. Define the extreme polynomials $D_{min}(z)$ and $D_{max}(z)$ to be $D(z)$ with $d_M = d_{M,min}$ and $d_M = d_{M,max}$, respectively. Then $D(z)$ is SH for all d_M in (14) if and only if $D_{min}(z)$ and $D_{max}(z)$ are SH.

The restriction $M \geq N/2$ arises because this is necessary to ensure that L (which is taken to be $2M - N$) is nonnegative as required by Theorem 4. Notice that the "only if" part of the theorem is obvious, and has not been elaborated.

If all the coefficients d_k with $N/2 \leq k \leq N$ are uncertain, then by repeated application of the above idea it can be shown that $D(z)$ is SH for all d_k in the uncertain region, provided the 2^K extreme polynomials, defined in an obvious way, are SH, where K is the number of uncertain coefficients. See [3] for details.

III. New Discrete-Time Kharitonov-Type Results

Given an Nth degree real-coefficient polynomial $D(z)$ as in (3), consider again the decomposition into symmetric and antisymmetric components $S(z)$ and $A(z)$ as in (10). We have

$$D(z) = S(z) + A(z) \tag{20}$$

where

$$S(z) = \sum_{n=0}^{N} s_n z^{-n}, \qquad A(z) = \sum_{n=0}^{N} a_n z^{-n}. \tag{21}$$

The coefficients s_n and a_n satisfy the symmetric and antisymmetric properties, respectively:

$$s_n = s_{N-n}, \qquad a_n = -a_{N-n}. \tag{22}$$

The polynomials $S(z)$ and $A(z)$ are therefore causal linear-phase FIR filters [14]. In what follows, we shall assume that N is even. Toward the end of the section, this assumption shall be removed.

According to standard FIR filtering language, the even-order symmetric FIR filter $S(z)$ is said to be a Type 1 filter, and the even-order antisymmetric $A(z)$ is a Type 3 filter (see [14] and [15, p. 72]). It is well known that we can then express $S(e^{j\omega})$ and $A(e^{j\omega})$ as [15]

$$S(e^{j\omega}) = e^{-j\omega M}P(\omega), \qquad A(e^{j\omega}) = je^{-j\omega M}\sin(\omega)\,Q(\omega) \tag{23}$$

where $M = N/2$ and where $P(\omega)$ and $Q(\omega)$ are real-valued functions of the form

$$P(\omega) = \sum_{n=0}^{M} \alpha_n \cos(n\omega), \qquad Q(\omega) = \sum_{n=0}^{M-1} \beta_n \cos(n\omega). \tag{24}$$

The conversion from the coefficients s_n to α_n (and similarly from a_n to β_n) is a standard process described in [14] and [15].

Recall now [20] that $\cos(n\omega)$ can be expressed as $\cos(n\omega) = \sum_{k=0}^{n} c_{k,n} \cos^k(\omega)$ where $c_{k,n}$ are the coefficients of the nth-order Chebyshev polynomial (also see [15, p. 101]). Finally, $\cos(\omega)$ can be expressed as $2\cos^2(\omega/2) - 1$, so that we can rewrite $P(\omega)$ and $Q(\omega)$ as

$$P(\omega) = \sum_{n=0}^{M} p_n \cos^{2n}(\omega/2), \qquad Q(\omega) = \sum_{n=0}^{M-1} q_n \cos^{2n}(\omega/2), \tag{25}$$

where p_n and q_n are real-valued coefficients which can be computed from the coefficients d_n in (3), by the process described above. By means of the above process, we have expressed $D(e^{j\omega})$ in the form

$$D(e^{j\omega}) = e^{-j\omega M}[P(\omega) + j\sin(\omega)\,Q(\omega)]. \tag{26}$$

From the results of Section II, we immediately see that $D(z)$ is SH if and only if

$$F(e^{j\omega}) = P(\omega)/[j\sin(\omega)\,Q(\omega)] \tag{27}$$

behaves like an Nth degree reactance. Recall that $F(e^{j\omega})$ is an Nth degree reactance if and only if it satisfies the alteration property (i.e., there are N poles and zeros on the unit circle, and these are single, with poles and zeros interlacing) and the plot Im $[F(\omega)]$ has positive slope between poles (Fig. 2).

The discrete-time Kharitonov-type theorem we shall prove assumes that the coefficients p_n and q_n are uncertain. Specifically, the uncertainty intervals will be

$$p_{n,min} \le p_n \le p_{n,max} \tag{28}$$

$$q_{n,min} \le q_n \le q_{n,max} \tag{29}$$

where all the bounds in (28) and (29) are assumed to be independent. Notice that, if these uncertainties are mapped onto those of d_k in (4), the resulting uncertainty regions are complicated indeed, and the bounds on d_k's are not uncoupled from each other. Regardless of the nature, physical significance, and possible usefulness of these bounds, our only aim in this section is to show the existence of a stability result under this condition. This result is stated next.

Theorem 6: Let $D(z)$ be an Nth degree polynomial as in (3) and let N be even. Let $D(e^{j\omega})$ be expressed in the form of (26) with $P(\omega)$ and $Q(\omega)$ as in (25). With p_n and q_n defined to be uncertain and belonging to the real intervals (28), (29), define the extreme functions

$$P_1(\omega) = \sum_{n=0}^{M} p_{n,min} \cos^{2n}(\omega/2), \qquad P_2(\omega) = \sum_{n=0}^{M} p_{n,max} \cos^{2n}(\omega/2) \tag{30}$$

and

$$Q_1(\omega) = \sum_{n=0}^{M-1} q_{n,min} \cos^{2n}(\omega/2), \qquad Q_2(\omega) = \sum_{n=0}^{M-1} q_{n,max} \cos^{2n}(\omega/2) \tag{31}$$

and define the four functions

$$D_{11}(e^{j\omega}) = P_1(\omega) + j\sin(\omega)\,Q_1(\omega) \tag{32}$$

$$D_{12}(e^{j\omega}) = P_1(\omega) + j\sin(\omega)\,Q_2(\omega) \tag{33}$$

$$D_{21}(e^{j\omega}) = P_2(\omega) + j\sin(\omega)\,Q_1(\omega) \tag{34}$$

$$D_{22}(e^{j\omega}) = P_2(\omega) + j\sin(\omega)\,Q_2(\omega). \tag{35}$$

Then $D(z)$ is SH for every possible value of the coefficients p_n and q_n satisfying (28), (29), if and only if the four polynomials $D_{km}(z)$, $1 \le k, m \le 2$ are SH.

The "only if" part of this theorem (and of the lemmas to follow) is obvious and will not be elaborated. In order to prove the "if" part, we shall first assume that only p_n are uncertain [as in (28)], so that q_n are fixed at some values. Define

$$D_1(e^{j\omega}) = P_1(\omega) + j\sin(\omega)\,Q(\omega), \qquad D_2(e^{j\omega}) = P_2(\omega) + j\sin(\omega)\,Q(\omega) \tag{36}$$

and the functions

$$F_1(e^{j\omega}) = \frac{P_1(\omega)}{j\sin(\omega)\,Q(\omega)}, \qquad F_2(e^{j\omega}) = \frac{P_2(\omega)}{j\sin(\omega)\,Q(\omega)}. \tag{37}$$

Assuming that $D_1(z)$ and $D_2(z)$ are SH, we see that $F_1(z)$ and $F_2(z)$ are reactances. From the definitions of $P_1(\omega)$ and $P_2(\omega)$, we see that the following inequality holds for all ω:

$$P_1(\omega) \leq P(\omega) \leq P_2(\omega) \tag{38}$$

since the coefficients p_n of $P(\omega)$ are constrained as in (28). In other words, the function $P(\omega)$ is sandwiched between $P_1(\omega)$ and $P_2(\omega)$. Defining

$$F(e^{j\omega}) = \frac{P(\omega)}{j \sin(\omega)\, Q(\omega)}, \tag{39}$$

we see that the functions $F(e^{j\omega})$, $F_1(e^{j\omega})$, and $F_2(e^{j\omega})$ share the same set of poles. By combining this observation with (38), we conclude that the behavior of the function $F(e^{j\omega})$ is as demonstrated in Fig. 3. In other words, $F(e^{j\omega})$ satisfies the alternation property (and has positive slope) so that it is a reactance. Thus, if the coefficients q_n are fixed and p_n are uncertain as in (28), then $D(z)$ is SH for every choice of p_n in (28) if and only if the two polynomials $D_1(z)$ and $D_2(z)$ defined in (36) are SH. This is stated below as a lemma.

Lemma 2: Let $D(z)$ be an Nth degree polynomial as in (3) with N even, expressed in the form of (26), with $P(e^{j\omega})$ and $Q(e^{j\omega})$ as in (25). Let the coefficients q_n be known with certainty, and let p_n be uncertain, belonging to the intervals (28). Then $D(z)$ is SH for every p_n satisfying (28) if and only if the two extreme polynomials $D_1(z)$ and $D_2(z)$ defined as in (36) are SH, where $P_1(\omega)$ and $P_2(\omega)$ are as in (30).

By holding p_n fixed and letting q_n be uncertain, we can obtain a similar result. For this, we define the two extreme polynomials

$$D_3(e^{j\omega}) = P(\omega) + j \sin(\omega)\, Q_1(\omega),$$
$$D_4(e^{j\omega}) = P(\omega) + j \sin(\omega)\, Q_2(\omega) \tag{40}$$

and the functions

$$F_3(e^{j\omega}) = \frac{j \sin(\omega)\, Q_1(\omega)}{P(\omega)},$$
$$F_4(e^{j\omega}) = \frac{j \sin(\omega)\, Q_2(\omega)}{P(\omega)}. \tag{41}$$

Assuming that $D_3(e^{j\omega})$ and $D_4(e^{j\omega})$ are SH, we see that $F_3(e^{j\omega})$ and $F_4(e^{j\omega})$ have to be reactances. Because of the definitions of $Q_1(\omega)$ and $Q_2(\omega)$, we also see that the following property holds:

$$Q_1(\omega) \leq Q(\omega) \leq Q_2(\omega). \tag{42}$$

As a result, the ratio $j \sin(\omega)\, Q(\omega)/P(\omega)$ is a reactance if $D_3(z)$ and $D_4(z)$ are SH. This in turn means that $D(z)$ is SH for every possible set of q_n satisfying (29). This result is summarized below.

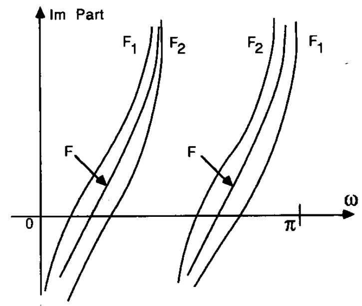

Fig. 3. Typical plots of imaginary parts of $F(e^{j\omega})$, $F_1(e^{j\omega})$, and $F_2(e^{j\omega})$ under the constraint (38).

Lemma 3: Let $D(z)$ be an Nth degree polynomial with N even, expressed in the form (26). Let the coefficients p_n be known with certainty, and let q_n be uncertain, belonging to the intervals (29). Then $D(z)$ is SH for every q_n satisfying (29) if and only if the two extreme polynomials $D_3(z)$ and $D_4(z)$ defined as in (40) are SH, where $Q_1(\omega)$ and $Q_2(\omega)$ are as in (31).

Combining the above two lemmas, we can immediately arrive at a proof of Theorem 6 as follows. Let both the sets p_n and q_n be uncertain as in (28) and (29). Define the four functions $D_{km}(z)$ as in (32)–(35). Suppose $D_{km}(z)$ are SH for $1 \leq k, m \leq 2$. Define

$$D_a(e^{j\omega}) = P(\omega) + j \sin(\omega)\, Q_1(\omega) \tag{43}$$

where $P(\omega)$ has coefficients p_n satisfying (28). By applying Lemma 2 to the pair $D_{11}(z)$, $D_{21}(z)$, we see that $D_a(z)$ is SH as long as the coefficients p_n satisfy (28). Similarly, if we define

$$D_b(e^{j\omega}) = P(\omega) + j \sin(\omega)\, Q_2(e^{j\omega}) \tag{44}$$

and apply Lemma 2 to the pair $D_{12}(z)$, $D_{22}(z)$, we see that $D_b(z)$ is SH as long as the coefficients p_n satisfy (28). Finally, by applying Lemma 3 on the pair $D_a(z)$, $D_b(z)$, we conclude that $D(z)$ is SH for any set of p_n and q_n satisfying (28) and (29). This concludes the proof of Theorem 6.

Notice that Theorem 6 works only for even N, as the decomposition (26) is valid only in this case. For odd N, $S(z)$ and $A(z)$ have odd order, so they are Type 2 and Type 4 linear-phase FIR systems, respectively [15]. From [15, p. 72] we see that in this case we can express $D(e^{j\omega})$ as

$$D(e^{j\omega}) = e^{-j\omega M}[\cos(\omega/2)\, P(\omega) + j \sin(\omega/2)\, Q(\omega)] \tag{45}$$

where

$$P(\omega) = \sum_{n=0}^{L} \alpha_n \cos(\omega n), \qquad Q(\omega) = \sum_{n=0}^{L} \beta_n \cos(\omega n), \tag{46a}$$

and $M = N/2$. Here $L = (N - 1)/2$, and α_n, β_n can be found using standard methods [15]. The functions $P(\omega)$ and $Q(\omega)$ can again be expressed in the form

$$P(\omega) = \sum_{n=0}^{L} p_n \cos^{2n}(\omega/2),$$

$$Q(\omega) = \sum_{n=0}^{L} q_n \cos^{2n}(\omega/2). \quad (46b)$$

The counterpart of Theorem 6, which holds in this case, is stated below.

Theorem 6': Let $D(z)$ be an Nth degree polynomial as in (3), and let N be odd. Let $D(e^{j\omega})$ be expressed in the form (45), with $P(\omega)$ and $Q(\omega)$ defined as in (46b). Let p_n and q_n be uncertain, belonging to the real intervals (28), (29). Define the "extreme functions"

$$P_1(\omega) = \sum_{n=0}^{L} p_{n,min} \cos^{2n}(\omega/2),$$

$$P_2(\omega) = \sum_{n=0}^{L} p_{n,max} \cos^{2n}(\omega/2) \quad (47)$$

and

$$Q_1(\omega) = \sum_{n=0}^{L} q_{n,min} \cos^{2n}(\omega/2),$$

$$Q_2(\omega) = \sum_{n=0}^{L} q_{n,max} \cos^{2n}(\omega/2) \quad (48)$$

and define the four polynomials $D_{km}(z)$ as in (32)–(35). Then $D(z)$ is SH for all p_n, q_n belonging to the range (28), (29) if and only if the four polynomials in (32)–(35) are SH.

The proof, being similar to the previous one, is omitted.

IV. Relation to Line-Spectrum-Pair (LSP) Technique Used in Speech Compression

The relation between strictly Hurwitz polynomials and reactance functions, as stated in Theorem 3, has been observed independently in a different context by the speech-processing community.[1] In 1975 Itakura [22] introduced the line-spectrum pair (LSP) as a possible parameterization of speech segments. This is an alternative to other characterizations such as, for example, the predictor coefficients d_n which appear in the traditional all-pole model $G/D(z)$ for speech segments. It was expected that there can be some potential perceptual advantages of using the LSP parameters (as confirmed later in [27]). The LSP parameterization is based on the observation that if an SH polynomial (also called a minimum-phase polynomial) $D(z)$ is used to obtain a pair of symmetric and antisymmetric polynomials $S(z)$ and $A(z)$ as in (10), then $S(z)$ and $A(z)$ have all zeros on the unit circle, and these zeros interlace with each other, and moreover, all these zeros are *simple* (i.e., multiplicity one). This is the same as saying that the ratio $S(z)/A(z)$ is a reactance.

To be specific, let the SH polynomial be $D(z) = 1 + \Sigma_{n=1}^{N} d_n z^{-n}$ with N even. Then $A(z)$ has a zero at $z = 1$ (because it is antisymmetric) and a zero at $z = -1$ (because it is of even order [14], [15]). Since the zeros of $A(z)$ and $S(z)$ are simple and interlace on the unit circle, we conclude that $S(z)$ has $N/2$ complex conjugate pairs of zeros on the unit circle. Counting the zeros of $S(z)$ and $A(z)$, there is a total of $N/2 + (N - 2)/2 = N - 1$ zeros on the unit circle with angles in the range $0 < \omega < \pi$. Let these be denoted ω_k so that

$$0 < \omega_1 < \omega_2 < \cdots < \omega_{N-1} < \pi. \quad (49)$$

We can clearly reconstruct $D(z)$ from these $N - 1$ parameters as

$$D(z) = \alpha \prod_{k=1}^{M} (1 - z_k z^{-1})(1 - z_k^* z^{-1}) + (1 - \alpha)(1 - z^{-2}) \cdot \prod_{k=1}^{M-1} \cdot (1 - z_k' z^{-1})(1 - z_k'^* z^{-1}) \quad (50)$$

where $0 < \alpha < 1$ is an appropriate constant, and $M = N/2$. Here (z_k, z_k^*) represents complex conjugate pairs of zeros of $S(z)$, whereas $(z_k', z_k'^*)$ represents those of $A(z)$. The set of N real parameters ω_k, $1 \le k \le N - 1$ and α completely characterize $D(z)$. This is called the line-spectrum-pair (LSP) parameterization in speech processing technology. A tutorial on this method appeared in [30]. This method is an alternative to a parameterization based on the well-known PARCOR coefficients or lattice coefficients k_m (see [29]). The lattice coefficients k_m have the property that even if they are quantized, $D(z)$ remains SH (provided that $k_m^2 < 1$ in spite of quantization). Similarly, the LSP parameters ω_k are such that $D(z)$ remains SH in spite of quantization of ω_k as long as (49) continues to hold. This follows from Theorem 3 or, equivalently, from Schuessler's work [11]. An independent proof in the LSP context was given by Soong and Juang [23]. The additional advantage of the LSP coefficients lies in the fact that the angle ω_k have a statistical relation to the formant frequencies and formant bandwidths of speech segments as explained by Crosmer and Barnwell [24]. Accordingly, ω_k can be quantized by taking advantage of the effect of formant properties on perception. This has actually been demonstrated recently by Kang and Fransen [27] by performing speech intelligibility tests which show that the LSP coefficients require 20 percent fewer bits than the LPC coefficients on the average. Thus, the LSP coefficients offer the same stability advantage under quantization as the LPC coefficients, while at the same time offering an improvement in compression. Optimal quantization of the LSP parameters resulting in improved efficiency is discussed in [31].

[1] I am grateful to Prof. T. Ramstad of the Norwegian Institute of Technology for bringing this to my attention.

Unlike the LPC coefficients, the LSP coefficients are more expensive to compute because the zeros of $S(z)$ and $A(z)$ should be found. But since these zeros are known to be on the unit circle, this task can be performed much more efficiently than finding the *zeros* of $D(z)$ (which would, in principle, form another parameterization of a speech segment which guarantees stability under quantization). Techniques for finding ω_k have been described in [23] and [26].

In [26] Kabal and Ramachandran have outlined a technique for finding ω_k from a given polynomial $D(z)$, by expressing $S(e^{j\omega})$ and $A(e^{j\omega})$ as Chebyshev polynomials in cos ω. This idea incidentally coincides with our development in Section III where we used a similar approach to obtain Theorem 6'. Finally it is clear that the coefficients α_n and β_n in (46a) can be used as an alternative characterization of $D(z)$ and have the property that a certain degree of quantization in these coefficients can be tolerated without violating the SH property of $D(z)$ (due to Theorem 6'). However, the perceptual advantages, if any, of such a characterization for speech signals is unclear and it might be rewarding to explore them.

V. Concluding Remarks

The continuous-time result [1] has been translated by some authors into the discrete-time domain by the use of bilinear transformation. It should be pointed out that the transformed result is different from the results of Theorems 6 and 6'. Indeed, given a polynomial $E(s)$ as in (1), if we replace s with $(1 - z^{-1})/(1 + z^{-1})$ and multiply out the denominator $(1 + z^{-1})^N$, the resulting polynomial $D(z)$ is

$$D(z) = \sum_{n=0}^{N} e_n(1 - z^{-1})^n(1 + z^{-1})^{N-n}. \quad (51)$$

On the unit circle, this becomes

$$D(e^{j\omega}) = 2^N e^{-j\omega N/2} \cdot \sum_{n=0}^{N} e_n[j^n \cos^{N-n}(\omega/2) \sin^n(\omega/2)] \quad (52)$$

which is quite different from the form (26) used in Theorem 6.

Subsequent to the publication of Kharitonov's result [1], some authors have published interesting extensions, particularly in the discrete-time domain. In this paper, we have presented a reactance-type interpretation for these, and also obtained a new discrete-time result. This result is closely analogous to Kharitonov's result [1] in the sense that all coefficients are uncertain, and that only four polynomials have to be tested in order to ensure the SH property of the complete family of polynomials. However, the bounds on the coefficients of d_n are not mutually independent (as they are in [1]), even though the bounds on p_n, q_n in (28) and (29) are. Now, the usefulness (or otherwise) of any of these results depends upon several practical considerations, none of which has been addressed in this paper. For example, if the uncertainty in d_n arises because of the uncertainties inside a plant, the bounds on d_n are interrelated, and the results in [7] are likely to be more relevant than [1]. Regardless of the applicability, or otherwise, of these results to specific situations, we do believe that this family of results is valuable from a theoretical viewpoint.

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