AIP | Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **56**, 062303 (2015); doi: 10.1063/1.4922815
View online: http://dx.doi.org/10.1063/1.4922815
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/56/6?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS **56**, 062303 (2015)

On the scalar curvature for the noncommutative four torus

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(Received 20 April 2015; accepted 6 June 2015; published online 25 June 2015)

The scalar curvature for noncommutative four tori $\mathbb{T}^4_\Theta$, where their flat geometries are conformally perturbed by a Weyl factor, is computed by making the use of a noncommutative residue that involves integration over the 3-sphere. This method is more convenient since it does not require the rearrangement lemma and it is advantageous as it explains the simplicity of the final functions of one and two variables, which describe the curvature with the help of a modular automorphism. In particular, it readily allows to write the function of two variables as the sum of a finite difference and a finite product of the one variable function. The curvature formula is simplified for dilatons of the form sp, where s is a real parameter and $p \in C^\infty(\mathbb{T}^4_\Theta)$ is an arbitrary projection, and it is observed that, in contrast to the two dimensional case studied by Connes and Moscovici, J. Am. Math. Soc. **27**(3), 639-684 (2014), unbounded functions of the parameter s appear in the final formula. An explicit formula for the gradient of the analog of the Einstein-Hilbert action is also calculated. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4922815]

I. INTRODUCTION

The computation of scalar curvature[9,16] for noncommutative two tori $\mathbb{T}^2_\theta$ was stimulated by the seminal work[10] of Connes and Tretkoff on the Gauss-Bonnet theorem for these C^*-algebras and its extension in Ref. 14 to general translation-invariant conformal structures. Flat geometries of $\mathbb{T}^2_\theta$,[3,4] whose conformal classes are represented by positive Hochschild cocycles,[5] are conformally perturbed by means of a positive invertible element e^{-h}, where $h = h^* \in C^\infty(\mathbb{T}^2_\theta)$ is a dilaton.[10] Local geometric invariants, such as scalar curvature, can then be computed by considering small time asymptotic expansions, which depend on the action of the algebra on a Hilbert space and the distribution at infinity of the eigenvalues of a relevant geometric operator, namely, the Laplacian of the conformally perturbed metric.

Following these works, the local differential geometry of noncommutative tori equipped with curved metrics has received considerable attention in recent years.[1,15,11,17,12] See also Refs. 29 and 13. It should be mentioned that conformal geometry in the noncommutative setting is intimately related to twisted spectral triples and we refer to Refs. 8, 10, 9, 27, 26, and 21 for detailed discussions. Also, it is closely related to the spectral action computations in the presence of a dilaton.[2,6] For noncommutative four tori $\mathbb{T}^4_\Theta$, the scalar curvature is computed in Ref. 17 and it is shown that flat metrics are the critical points of the analog of the Einstein-Hilbert action. Also, noncommutative residues for noncommutative tori were studied in Refs. 18, 25, and 17 (see also Ref. 30). We refer to Refs. 31, 22, and 24 for detailed discussions on noncommutative residues for classical manifolds.

A crucial tool for the local computations on noncommutative tori has been Connes' pseudodifferential calculus, developed for C^*-dynamical systems in Ref. 3, which can be employed to work in the heat kernel scheme of elliptic differential operators and index theory (cf. Ref. 20). An obstruction in these calculations, which is a purely noncommutative feature, is the appearance of integrals of functions over the positive real line that are C^*-algebra valued. This is overcome

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by the rearrangement lemma[10,9] (cf. Refs. 1, 16, and 23), which uses the modular automorphism of the state implementing the conformal perturbation and delicate Fourier analysis to reorder the integrands and computes the integrals explicitly. The integrals are then expressed as somewhat complicated functions of the modular automorphism acting on relevant elements of the C^*-algebra. This lemma has been generalized in Ref. 23, and the work in Ref. 12 is an instance where the generalization is used.

A striking fact about the final formulas for the curvature of noncommutative tori is their simplicity and their fruitful properties such as being entire. Considering the numerous functions from the rearrangement lemma that get involved in hundreds of terms in the computations, the final simplicity indicates an enormous amount of cancellations, which are carried out by computer assistance. One of the aims of this paper is to explain this simplicity by computing the scalar curvature for $\mathbb{T}^4_\Theta$ without using the rearrangement lemma. We then study the curvature formula for the dilatons that are associated with projections in $C^\infty(\mathbb{T}^4_\Theta)$. The gradient of the Einstein-Hilbert action is also calculated, which prepares the ground for studying its associated flow in future works.

This article is organized as follows. In Sec. II, we recall the formalism and notions used in Ref. 17 concerning the conformally perturbed Laplacian on $\mathbb{T}^4_\Theta$. In Sec. III, we use a noncommutative residue that involves integrations on the 3-sphere $\mathbb{S}^3$ to compute the scalar curvature. This method is quite convenient as it does not require any help from the rearrangement lemma and it is advantageous as it explains the simplicity of final formula (3). Also, it readily allows to write the function of two variables in (3) as the sum of a finite difference and a finite product of the one variable function. In Sec. IV, the curvature formula is simplified for dilatons of the form sp, where s is a real parameter and $p \in C^\infty(\mathbb{T}^4_\Theta)$ is an arbitrary projection. It is observed that, in contrast to the two dimensional case studied in Ref. 9, unbounded functions of the parameter s appear in the final formula. In Sec. V, we compute an explicit formula for the gradient of the analog of the Einstein-Hilbert action in terms of finite differences (cf. Refs. 9 and 23) of a one variable function that describes this action.[17]

II. PRELIMINARIES

The noncommutative four torus $C(\mathbb{T}^4_\Theta)$ is the universal C^*-algebra generated by four unitaries U_1, U_2, U_3, U_4, which satisfy the commutation relations

$$U_k U_j = e^{2\pi i \theta_{kj}} U_j U_k,$$

where $\Theta = (\theta_{kj}) \in M_4(\mathbb{R})$ is an antisymmetric matrix. For simplicity elements of the form $U_1^{\ell_1} U_2^{\ell_2} U_3^{\ell_3} U_4^{\ell_4} \in C(\mathbb{T}^4_\Theta)$ shall be denote by U^ℓ for any 4-tuple of integers $\ell = (\ell_1, \ell_2, \ell_3, \ell_4)$.

There is a natural action of $\mathbb{R}^4$ on this C^*-algebra, which is defined by

$$\alpha_s(U^\ell) = e^{is\cdot\ell} U^\ell, \qquad s \in \mathbb{R}^4, \qquad \ell \in \mathbb{Z}^4,$$

and is extended to a 4-parameter family of C^*-algebra automorphisms $\alpha : \mathbb{R}^4 \to \mathrm{Aut}\big(C(\mathbb{T}^4_\Theta)\big)$. The infinitesimal generators of this action, denoted by $\delta_1, \delta_2, \delta_3, \delta_4$, are defined on the smooth subalgebra

$$C^\infty(\mathbb{T}^4_\Theta) = \{a \in C(\mathbb{T}^4_\Theta);\ \text{the map } \mathbb{R}^4 \ni s \mapsto \alpha_s(a) \text{ is smooth}\},$$

which is a dense subalgebra of $C(\mathbb{T}^4_\Theta)$ and can alternatively be defined as the space of elements of the form $\sum_{\ell\in\mathbb{Z}^4} a_\ell U^\ell$ with rapidly decaying complex coefficients $(a_\ell) \in \mathcal{S}(\mathbb{Z}^4)$. These derivations are determined by the relations $\delta_j(U_j) = U_j$ and $\delta_j(U_k) = 0$, if $j \neq k$.

One can consider a complex structure on $C(\mathbb{T}^4_\Theta)$ (cf. Ref. 17) by introducing the analog of the Dolbeault operators

$$\partial_1 = \delta_1 - i\delta_3, \qquad \partial_2 = \delta_2 - i\delta_4, \qquad \bar\partial_1 = \delta_1 + i\delta_3, \qquad \bar\partial_2 = \delta_2 + i\delta_4$$

and by setting

$$\partial = \partial_1 \oplus \partial_2, \qquad \bar\partial = \bar\partial_1 \oplus \bar\partial_2,$$

which are maps from $C^\infty(\mathbb{T}^4_\Theta)$ to $C^\infty(\mathbb{T}^4_\Theta) \oplus C^\infty(\mathbb{T}^4_\Theta)$.

There is a canonical positive faithful trace $\varphi_0 : C(\mathbb{T}^4_\Theta) \to \mathbb{C}$, which is defined on the smooth algebra by

$$\varphi_0\Big(\sum_{\ell\in\mathbb{Z}^4} a_\ell U^\ell\Big) = a_0.$$

Following the method introduced in Ref. 10, φ_0 is viewed as the volume form, and conformal perturbation of the metric is implemented in Ref. 17 by choosing a dilaton $h = h^* \in C^\infty(\mathbb{T}^4_\Theta)$ and by considering the linear functional $\varphi : C(\mathbb{T}^4_\Theta) \to \mathbb{C}$ given by

$$\varphi(a) = \varphi_0(ae^{-2h}), \qquad a \in C(\mathbb{T}^4_\Theta).$$

This is a KMS state with the associated 1-parameter group $\{\sigma_t\}_{t\in\mathbb{R}}$ of inner automorphisms given by

$$\sigma_t(a) = e^{ith}ae^{-ith}, \qquad a \in C(\mathbb{T}^4_\Theta)$$

and will use the following operators substantially:

$$\Delta(a) = \sigma_i(a) = e^{-h}ae^h, \qquad \nabla(a) = \log\Delta(a) = [-h, a], \qquad a \in C(\mathbb{T}^4_\Theta).$$

Denoting the inner product associated with the state φ by

$$(a, b)_\varphi = \varphi(b^*a), \qquad a, b \in C(\mathbb{T}^4_\Theta),$$

the Hilbert space completion of $C(\mathbb{T}^4_\Theta)$ with respect to this inner product is denoted by $\mathcal{H}_\varphi$, and the analog of the de Rham differential is defined in Ref. 17 by

$$d = \partial \oplus \bar\partial : \mathcal{H}_\varphi \to \mathcal{H}_\varphi^{(1,0)} \oplus \mathcal{H}_\varphi^{(0,1)}.$$

The Hilbert spaces $\mathcal{H}_\varphi^{(1,0)}$ and $\mathcal{H}_\varphi^{(0,1)}$ are, respectively, the completions of the analogs of $(1,0)$-forms and $(0,1)$-forms, namely, the spaces $\{\sum_{i=1}^n a_i\partial b_i; a_i, b_i \in C^\infty(\mathbb{T}^4_\Theta), n \in \mathbb{N}\}$ and $\{\sum_{i=1}^n a_i\bar\partial b_i; a_i, b_i \in C^\infty(\mathbb{T}^4_\Theta), n \in \mathbb{N}\}$, with the appropriate inner product related to the conformal factor.

The Laplacian $d^*d : \mathcal{H}_\varphi \to \mathcal{H}_\varphi$ is then computed and shown to be anti-unitarily equivalent to the operator

$$\triangle_\varphi = e^h\bar\partial_1 e^{-h}\partial_1 e^h + e^h\partial_1 e^{-h}\bar\partial_1 e^h + e^h\bar\partial_2 e^{-h}\partial_2 e^h + e^h\partial_2 e^{-h}\bar\partial_2 e^h.$$

III. SCALAR CURVATURE FOR $\mathbb{T}^4_\Theta$ AND ITS FUNCTIONAL RELATIONS

The scalar curvature of the conformally perturbed metric on $\mathbb{T}^4_\Theta$ is the unique element $R \in C^\infty(\mathbb{T}^4_\Theta)$ such that

$$\mathrm{res}_{s=1}\mathrm{Trace}(a\triangle_\varphi^{-s}) = \varphi_0(aR), \qquad \forall a \in C^\infty(\mathbb{T}^4_\Theta).$$

Since the linear functional

$$⨍ P = \mathrm{res}_{s=0}\mathrm{Trace}(P\triangle_\varphi^{-s})$$

defines a trace on the algebra of pseudodifferential operators,[7,19] it follows from the uniqueness of traces on the algebra of pseudodifferential operators[18,17,25] that it coincides with the noncommutative residue defined in Ref. 17. Therefore, there exists a constant c such that for any P

$$⨍ P = c\int_{\mathbb{S}^3} \varphi_0(\rho_{-4}(\xi))\, d\Omega,$$

where ρ_{-4} is the homogeneous term of order -4 in the expansion of the symbol of P, and $d\Omega$ is the invariant measure on the sphere $\mathbb{S}^3$. Therefore, in order to compute the curvature, we can write

$$\mathrm{res}_{s=1}\mathrm{Trace}(a\triangle_\varphi^{-s}) = \mathrm{res}_{s=0}\mathrm{Trace}(a\triangle_\varphi^{-s-1}) = ⨍ a\triangle_\varphi^{-1} = c\,\varphi_0\Big(\int_{\mathbb{S}^3} a\, b_2(\xi)\, d\Omega\Big),$$

where b_j is the homogeneous term of order $-2-j$ in the asymptotic expansion of the symbol of the parametrix of $\triangle_\varphi$. Hence

$$R = c\int_{\mathbb{S}^3} b_2(\xi)\, d\Omega.$$

We compute b_2 by applying Connes' pseudodifferential calculus[3] to the symbol of $\triangle_\varphi$, which is the sum of the homogeneous components

$$a_2(\xi) = e^h\sum_{i=1}^4 \xi_i^2, \quad a_1(\xi) = \sum_{i=1}^4 \delta_i(e^h)\xi_i, \quad a_0(\xi) = \sum_{i=1}^4 \big(\delta_i^2(e^h) - \delta_i(e^h)e^{-h}\delta_i(e^h)\big).$$

That is, we solve the following equation explicitly up to b_2:

$$(b_0 + b_1 + b_2 + \cdots)\circ(a_0 + a_1 + a_2) \sim 1,$$

which in general yields

$$b_0 = a_2^{-1} = \big(e^h\sum_{i=1}^4 \xi_i^2\big)^{-1}, \qquad b_n = -\sum_{\substack{2+j+|\ell|-k=n,\\ 0\le j<n,\ 0\le k\le 2}} \frac{1}{\ell!}\partial^\ell(b_j)\delta^\ell(a_k)b_0 \qquad (n>1).$$

Here, for any $\ell = (\ell_1,\ell_2,\ell_3,\ell_4)\in\mathbb{Z}^4_{\ge 0}$, ∂^ℓ denotes $\partial_{\xi_1}^{\ell_1}\partial_{\xi_2}^{\ell_2}\partial_{\xi_3}^{\ell_3}\partial_{\xi_4}^{\ell_4}$ and δ^ℓ denotes $\delta_1^{\ell_1}\delta_2^{\ell_2}\delta_3^{\ell_3}\delta_4^{\ell_4}$. Note that the composition rule for pseudodifferential symbols,[3]

$$\rho\circ\rho' = \sum_{\ell\in\mathbb{Z}^4_{\ge 0}} \frac{1}{\ell!}\partial^\ell\rho(\xi)\,\delta^\ell(\rho'(\xi)),$$

is used in the derivation of the above recursive formula for b_n.

Computing b_2 and restricting it to $\mathbb{S}^3$ by the substitutions

$$\xi_1 = \cos(\psi), \qquad \xi_2 = \cos(\theta)\sin(\psi),$$
$$\xi_3 = \sin(\theta)\cos(\phi)\sin(\psi), \qquad \xi_4 = \sin(\theta)\sin(\phi)\sin(\psi),$$

with $0\le\psi<\pi, 0\le\theta<\pi, 0\le\phi<2\pi$, we perform its integral over the sphere and find that

$$\begin{aligned}
\int_{\mathbb{S}^3} b_2(\xi)\, d\Omega &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} b_2(\xi)\sin(\theta)\sin^2(\psi)\, d\psi\, d\theta\, d\phi \\
&= \sum_{i=1}^4 \Big(\big(-2\pi^2\big) b_0\delta_i\delta_i(e^h)b_0 + \big(2\pi^2\big) b_0\delta_i(e^h)\frac{1}{e^h}\delta_i(e^h)b_0 + \frac{5\pi^2}{2} b_0\delta_i(e^h)b_0\delta_i(e^h)b_0 \\
&\quad + \big(3\pi^2\big) b_0e^hb_0\delta_i\delta_i(e^h)b_0 + \big(-8\pi^2\big) b_0e^hb_0\delta_i(e^h)b_0\delta_i(e^h)b_0 \\
&\quad + \big(-2\pi^2\big) b_0e^hb_0e^hb_0\delta_i\delta_i(e^h)b_0 + \big(-\pi^2\big) b_0\delta_i(e^h)b_0e^hb_0\delta_i(e^h)b_0 \\
&\quad + \big(2\pi^2\big) b_0e^hb_0\delta_i(e^h)b_0e^hb_0\delta_i(e^h)b_0 + \big(4\pi^2\big) b_0e^hb_0e^hb_0\delta_i(e^h)b_0\delta_i(e^h)b_0\Big) \\
&= \pi^2\sum_{i=1}^4 \Big(-e^{-h}\delta_i^2(e^h)e^{-h} + \frac{3}{2}e^{-h}\delta_i(e^h)e^{-h}\delta_i(e^h)e^{-h}\Big).
\end{aligned} \tag{1}$$

The fact that, over $\mathbb{S}^3$, b_0 reduces to e^{-h} is crucial in the last equation, which leads to such a simple final formula.

We then use the following identities[10,9,16] to write expression (1) in terms of $\nabla = \log\Delta = -\mathrm{ad}_h$ and $\delta_i(h)$:

$$e^{-h}\delta_i(e^h) = g_1(\Delta)(\delta_i(h)), \qquad e^{-h}\delta_i^2(e^h) = g_1(\Delta)(\delta_i^2(h)) + 2g_2(\Delta,\Delta)(\delta_i(h)\delta_i(h)),$$

where

$$g_1(u) = \frac{u-1}{\log u}, \qquad g_2(u,v) = \frac{u(v-1)\log(u) - (u-1)\log(v)}{\log(u)\log(v)(\log(u)+\log(v))}. \tag{2}$$

This yields

$$\begin{aligned}\int_{\mathbb{S}^3} b_2(\xi)\,d\Omega &= \frac{1}{c}R\\ &= \pi^2\sum_{i=1}^{4}\Big(-e^{-h}\Delta^{-1}g_1(\Delta)(\delta_i(h)) - 2e^{-h}\Delta^{-1}\big(g_2(\Delta,\Delta)(\delta_i(h)^2)\big)\\ &\qquad +\frac{3}{2}e^{-h}\Delta^{-1}\big(g_1(\Delta)(\delta_i(h))g_1(\Delta)(\delta_i(h))\big)\Big)\\ &= \pi^2 e^{-h}k(\nabla)\Big(\sum_{i=1}^{4}\delta_i^2(h)\Big) + \pi^2 e^{-h}H(\nabla,\nabla)\Big(\sum_{i=1}^{4}\delta_i(h)^2\Big),\end{aligned} \tag{3}$$

where

$$k(s) = -e^{-s}g_1(e^s) = \frac{e^{-s}-1}{s},$$

$$\begin{aligned}H(s,t) &= -2e^{-s-t}g_2(e^s,e^t) + \frac{3}{2}e^{-s-t}g_1(e^s)g_1(e^t)\\ &= \frac{e^{-s-t}\left((e^s-1)(3e^t+1)t - (e^s+3)s(e^t-1)\right)}{2st(s+t)}.\end{aligned} \tag{4}$$

This formula matches with the one obtained in Ref. 17 (up to the multiplicative factor $1/c = 2\pi^2$).

Theorem 3.1. *Let*

$$\tilde{k}(s) = e^s k(s), \qquad \tilde{H}(s,t) = e^{s+t}H(s,t),$$

where k and H are the functions in the final formula for the scalar curvature. We have

$$\tilde{H}(s,t) = 2\frac{\tilde{k}(s+t)-\tilde{k}(s)}{t} + \frac{3}{2}\tilde{k}(s)\tilde{k}(t). \tag{5}$$

Proof. It follows from (4) and the following relation between the functions introduced in (2):

$$\begin{aligned}g_2(u,v) &= \int_0^1 su^s g_1(v^s)\,ds = \frac{1}{\log(v)}\Big(\frac{uv-1}{\log(uv)} - \frac{u-1}{\log(u)}\Big)\\ &= \frac{1}{\log(v)}\big(g_1(uv) - g_1(u)\big).\end{aligned}$$

□

IV. PROJECTIONS AND THE SCALAR CURVATURE

Similar to the illustration in Ref. 9 of the scalar curvature of $\mathbb{T}^2_\theta$ for dilatons associated with projections, we consider dilatons of the form $h = sp$, where $s \in \mathbb{R}$ and $p = p^* = p^2 \in C^\infty(\mathbb{T}^4_\Theta)$ is an arbitrary projection, and simplify expression (3) for these cases. We shall also study the behaviour of the functions of the parameter s that appear in the final formula (Figures 1–4).

Proposition 4.1. Let $p = p^* = p^2 \in C^\infty(\mathbb{T}^4_\Theta)$ be a projection. For the dilaton $h = sp$, $s \in \mathbb{R}$, the formula for the scalar curvature reduces to

$$R = e^{-sp}\big(f_1(s)\triangle(p) + f_2(s)\triangle(p)p + f_3(s)p\triangle(p) + f_4(s)p\triangle(p)p\big),$$

where $\triangle = \sum_{i=1}^4 \delta_i^2$ and

$$f_1(s) = \frac{1}{4}(-2\sinh(s) + \cosh(s) - 1), \qquad f_2(s) = \frac{1}{2}\sinh^2\left(\frac{s}{2}\right),$$

$$f_3(s) = \frac{-s+\sinh(s)}{2} - \frac{\cosh(s)-1}{4}, \qquad f_4(s) = s - \sinh(s).$$

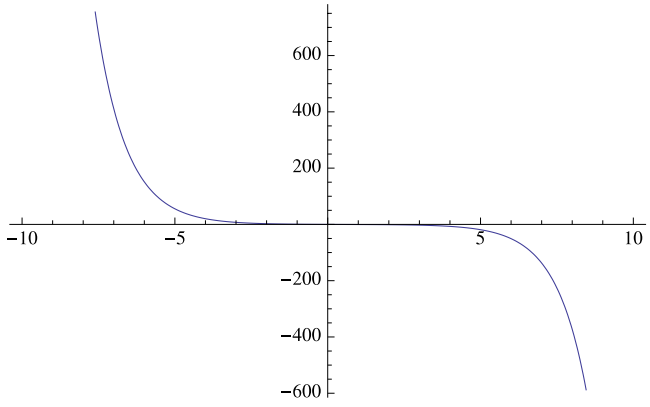

FIG. 1. Graph of the function $f_1(s) = \frac{1}{4}(-2\sinh(s) + \cosh(s) - 1)$.

Proof. Our method is quite similar to the one used in Ref. 9. That is, we first use the identity

$$\triangle(p) = p\triangle(p)p + p\triangle(p)(1-p) + (1-p)\triangle(p)p + (1-p)\triangle(p)(1-p)$$

to decompose $\triangle(p)$ to the sum of eigenvectors of $\nabla = -\mathrm{ad}_{sp}$ with eigenvalues $0, -s, s, 0$. Therefore

$$\begin{aligned} k(\nabla)(\triangle(h)) &= sk(\nabla)(\triangle(p)) \\ &= sk(0)\big(p\triangle(p)p + (1-p)\triangle(p)(1-p)\big) \\ &\quad + sk(-s)\big(p\triangle(p)(1-p)\big) + sk(s)\big((1-p)\triangle(p)p\big) \\ &= -s\big(p\triangle(p)p + (1-p)\triangle(p)(1-p)\big) \\ &\quad + (1-e^{s})\big(p\triangle(p)(1-p)\big) + (e^{-s}-1)\big((1-p)\triangle(p)p\big). \end{aligned}$$

Then, using the identity $\delta_i(p) = \delta_i(p)p + p\delta_i(p)$, one can see that

$$\begin{aligned} &H(\nabla,\nabla)(\delta_i(h)\delta_i(h)) \\ &= \frac{s^2}{2}\Big(\big(H(s,-s) + H(-s,s)\big) + \big(H(s,-s) - H(-s,s)\big)(1-2p)\Big)\big(\delta_i(p)\delta_i(p)\big) \\ &= \frac{s^2}{2}\Big(\frac{2(\cosh(s)-1)}{s^2} + \frac{4(s-\sinh(s))}{s^2}(1-2p)\Big)\big(\delta_i(p)\delta_i(p)\big) \\ &= \Big((\cosh(s)-1) + 2(s-\sinh(s))(1-2p)\Big)\big(\delta_i(p)\delta_i(p)\big) \\ &= \big(2s - 2\sinh(s) + \cosh(s) - 1 - 4(s-\sinh(s))p\big)\big(\delta_i(p)\delta_i(p)\big). \end{aligned}$$

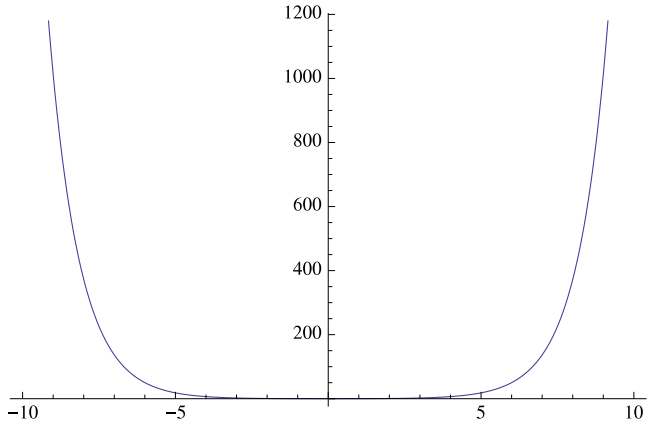

FIG. 2. Graph of the function $f_2(s) = \frac{1}{2}\sinh^2\big(\frac{s}{2}\big)$.

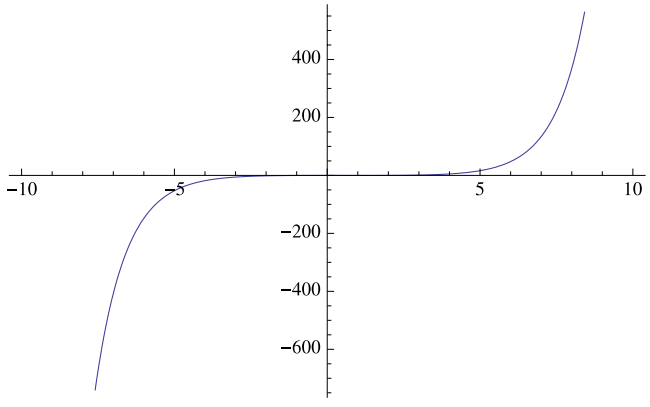

FIG. 3. Graph of the function $f_3(s) = \frac{1}{2}(\sinh(s) - s) + \frac{1}{4}(1 - \cosh(s))$.

Using the identity $2\sum \delta_i(p)^2 = (1-p)\triangle(p) - \triangle(p)p$, we sum the above expressions and find that formula (3), for the dilaton $h = sp$, reduces to

$$\frac{1}{2}(-2\sinh(s) + \cosh(s) - 1)\triangle(p) + (-s + \sinh(s) - \frac{\cosh(s)}{2} + \frac{1}{2})p\triangle(p)$$
$$+\sinh^2\left(\frac{s}{2}\right)\triangle(p)p + 2(s - \sinh(s))p\triangle(p)p,$$

up to multiplication from left by $e^{-h} = e^{-sp}$.

□

In contrast to the two dimensional case (cf. Ref. 9), the functions of the variable $s \in \mathbb{R}$ that appear in the statement of Proposition 4.1 are not bounded as they tend to $\pm\infty$ as $|s| \to \infty$. The graphs of these functions are given and some relations between these functions are investigated.

Among these functions, f_2 is the only one that is bounded below, whereas the other functions are neither bounded above nor bounded below. In fact, f_2 is a non-negative even function.

The function f_3, similar to f_1, does not satisfy any symmetry properties. The last function f_4 is obviously an odd function.

It is interesting to observe that these functions, which describe the scalar curvature for the dilaton $h = sp$, where p is an arbitrary projection, satisfy the following relations:

$$f_1(s) + f_1(-s) = f_2(s) + f_2(-s) = -\big(f_3(s) + f_3(-s)\big) = -\sinh^2\left(\frac{s}{2}\right),$$
$$f_3(s) - f_3(-s) = -\frac{1}{2}f_4(s) = \sinh(s) - s.$$

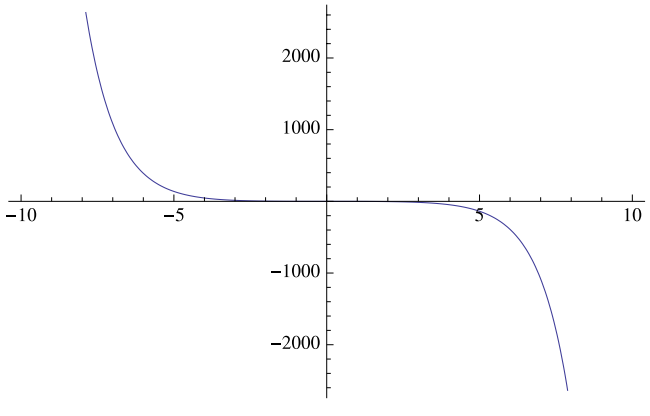

FIG. 4. Graph of the function $f_4(s) = 2(s - \sinh(s))$.

V. GRADIENT OF THE EINSTEIN-HILBERT ACTION

Denoting the Einstein-Hilbert action associated with the dilaton $h = h^* \in C^\infty(\mathbb{T}^4_\Theta)$ by $\Omega(h)$, we compute its gradient, namely, an explicit formula for an element $\mathrm{Grad}_h\Omega$ that represents the derivative at $\varepsilon = 0$ of $\Omega(h + \varepsilon a)$, where h, a are selfadjoint smooth elements. The final formula for the gradient is expressed in terms of finite differences of the function T, obtained in the proof of Theorem 5.3 of Ref. 17. We recall from the proof of this theorem that

$$\Omega(h) = \varphi_0(R) = \sum_{i=1}^{4} \varphi_0\big(e^{-h}T(\nabla)(\delta_i(h))\delta_i(h)\big),$$

where

$$T(s) = \frac{-2s + e^s - e^{-s}(2s+3) + 2}{4s^2}.$$

The fact that this function is non-negative played a crucial role in identifying the extrema of the Einstein-Hilbert action in Ref. 17.

Theorem 5.1. *For any selfadjoint $h \in C^\infty(\mathbb{T}^4_\Theta)$, we have*

$$\mathrm{Grad}_h\Omega = \sum_{i=1}^{4} \Big(e^{-h}\omega_1(\nabla)(\delta_i^2(h)) + e^{-h}\omega_2(\nabla,\nabla)(\delta_i(h)^2)\Big),$$

where

$$\omega_1(s) = \frac{-2\sinh(s) + \sinh(2s) - \cosh(2s) + 1}{4s^2},$$

$$\omega_2(s,t) =$$

$$\frac{1}{4s^2t^2(s+t)^2}(-s(-(s^2+st+3t^2)\sinh(s) + (s^2+5st+t^2)\cosh(s) - t^2)(\sinh(3(s+t)) + \cosh(3(s+t)))$$
$$+s((s^2(8t+5) + st(12t+17) + t^2(4t+5))(\sinh(s)+\cosh(s)) - 4st^2 - 4t^3 - 5t^2)(\sinh(s+t) + \cosh$$
$$(s+t)) + (-(5s^3 + s^2t(4t+15) + 2st^2(2t+5) + 5t^3)(\sinh(s)+\cosh(s)) + t^2(-(s+t)) - t^2(4s(s+t$$
$$-2) - 5t)(\sinh(2s)+\cosh(2s)) + t(s+t)(2s+t)(\sinh(3s)+\cosh(3s)))(\sinh(s+2t)+\cosh(s+2t))$$
$$+s(s\sinh(s) + s\cosh(s) + t)((s+t)(\sinh(s)+\cosh(s)) - t))(\cosh(3s+2t) - \sinh(3s+2t)).$$

Proof. We have

$$\begin{aligned}
&\Omega(h+\varepsilon a)\\
&= \sum_{i=1}^{4} \varphi_0\big(e^{-h-\varepsilon a}T(\nabla_{h+\varepsilon a})(\delta_i(h+\varepsilon a))\delta_i(h+\varepsilon a)\big)\\
&= \sum_{i=1}^{4} \Big(\varphi_0\big(e^{-h-\varepsilon a}T(\nabla_{h+\varepsilon a})(\delta_i(h))\delta_i(h)\big) + \varepsilon\varphi_0\big(e^{-h-\varepsilon a}T(\nabla_{h+\varepsilon a})(\delta_i(a))\delta_i(h)\big)\\
&\quad +\varepsilon\varphi_0\big(e^{-h-\varepsilon a}T(\nabla_{h+\varepsilon a})(\delta_i(h))\delta_i(a)\big) + \varepsilon^2\varphi_0\big(e^{-h-\varepsilon a}T(\nabla_{h+\varepsilon a})(\delta_i(a))\delta_i(a)\big)\Big).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{d}{d\varepsilon}|_{\varepsilon=0}\Omega(h+\varepsilon a)\\
&= \varphi_0\big((\frac{d}{d\varepsilon}|_{\varepsilon=0}e^{-h-\varepsilon a})T(\nabla)(\delta(h))\delta(h)\big) + \varphi_0\big(e^{-h}\frac{d}{d\varepsilon}|_{\varepsilon=0}T(\nabla_{h+\varepsilon a})(\delta(h))\delta(h)\big)\\
&\quad +\varphi_0\big(e^{-h}T(\nabla)(\delta(h))\delta(a)\big) + \varphi_0\big(e^{-h}T(\nabla)(\delta(a))\delta(h)\big).
\end{aligned}$$

Using the following lemmas, we obtain the explicit formula:

$$\mathrm{Grad}_h\Omega = \sum_{i=1}^{4} \Big(e^{-h}\omega_1(\nabla)(\delta_i^2(h)) + e^{-h}\omega_2(\nabla,\nabla)(\delta_i(h)\delta_i(h))\Big),$$

where

$$\omega_1(s) = -T(s) - T(-s)e^{-s},$$

$$\begin{aligned}\omega_2(s,t) &= E(s,t) + L(s,t) - P(s,t) - Q(s,t)\\ &= \frac{e^{-s-t}-1}{s+t}T(s) + e^{-s-t}\Big(\frac{T(-t)-T(s)}{s+t} + \frac{T(t)-T(-s)}{s+t}e^{t}\Big)\\ &\quad -\Big(T(s+t)\frac{e^{-s}-1}{s} + \frac{T(t)-T(s+t)}{s}e^{-s} + \frac{T(s+t)-T(s)}{t}\Big)\\ &\quad -\Big(T(-s-t)e^{-s-t}\frac{e^{-s}-1}{s} + \frac{T(t)-T(s+t)}{s}e^{-s} + \frac{T(s+t)-T(s)}{t}\Big).\end{aligned}$$

Note that the functions E, L, P, Q are introduced in lemmas below. Then, one can find the above explicit functions in the statement of the theorem by direct computer assisted computations (Figures 5 and 6). □

For simplicity in the notation, in the following lemmas, δ can be taken to be any of the canonical derivations δ_i introduced in Sec. II. The proofs follow closely the techniques given in Ref. 9 for the computation of the gradient of linear functionals similar to Ω (see also Ref. 23).

Lemma 5.1. For any $x \in C^\infty(\mathbb{T}^4_\Theta)$,

$$\varphi_0\big((\frac{d}{d\varepsilon}|_{\varepsilon=0}e^{-h-\varepsilon a})G(\nabla)(x)x\big) = \varphi_0\big(ae^{-h}E(\nabla,\nabla)(xx)\big),$$

where

$$E(s,t) = \frac{e^{-s-t}-1}{s+t}G(s).$$

Proof. Using

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}e^{-h-\varepsilon a} = \frac{1-e^{\nabla}}{\nabla}(a)e^{-h},$$

we have

$$\begin{aligned}\varphi_0\big((\frac{d}{d\varepsilon}|_{\varepsilon=0}e^{-h-\varepsilon a})G(\nabla)(x)x\big) &= \varphi_0\big(\frac{1-e^{\nabla}}{\nabla}(a)e^{-h}G(\nabla)(x)x\big)\\ &= \varphi_0\big(ae^{-h}\,\frac{e^{-\nabla}-1}{\nabla}(G(\nabla)(x)x)\big)\\ &= \varphi_0\big(ae^{-h}E(\nabla,\nabla)(xx)\big).\end{aligned}$$

□

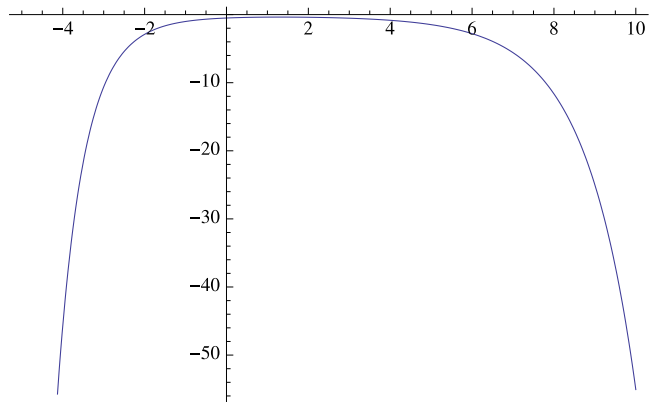

FIG. 5. Graph of the function ω_1.

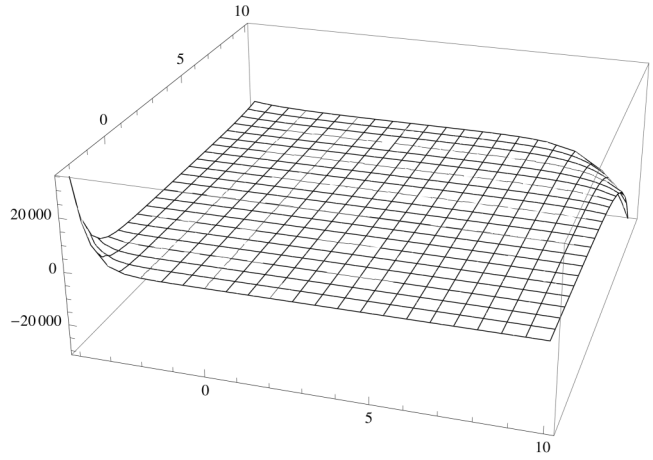

FIG. 6. Graph of the function ω_2.

Lemma 5.2. For any $x \in C^\infty(\mathbb{T}^4_\Theta)$, we have

$$\varphi_0\Big(e^{-h}\frac{d}{d\varepsilon}|_{\varepsilon=0}G(\nabla_{h+\varepsilon a})(x)x\Big) = \varphi_0(ae^{-h}\,L(\nabla,\nabla)(xx)),$$

where

$$L(s,t) = e^{-s-t}\Big(\frac{G(-t)-G(s)}{s+t} + \frac{G(t)-G(-s)}{s+t}e^t\Big).$$

Proof. Writing $G(v) = \int e^{-itv}g(t)\,dt$ and using the following identity:[9]

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}G(\nabla_{h+\varepsilon a}) = \int_0^1\int it\,\sigma_{ut}\,\mathrm{ad}_a\,\sigma_{(1-u)t}\,g(t)\,dt\,du,$$

we find that

$$\varphi_0\Big(e^{-h}\frac{d}{d\varepsilon}|_{\varepsilon=0}G(\nabla_{h+\varepsilon a})(x)x\Big) = \varphi_0(e^{-h}a\,L_0(\nabla,\nabla)(xx)),$$

where

$$L_0(s,t) = \frac{G(-t)-G(s)}{s+t} + \frac{G(t)-G(-s)}{s+t}e^t.$$

Therefore

$$\varphi_0\Big(e^{-h}\frac{d}{d\varepsilon}|_{\varepsilon=0}G(\nabla_{h+\varepsilon a})(x)x\Big) = \varphi_0(ae^{-h}\,L(\nabla,\nabla)(xx)),$$

where

$$L(s,t) = e^{-s-t}L_0(s,t).$$

□

Lemma 5.3. For any $x \in C^\infty(\mathbb{T}^4_\Theta)$, one has

$$\delta\big(G(\nabla)(x)\big) = G(\nabla)(\delta(x)) + M_1(\nabla,\nabla)(\delta(h)x) + M_2(\nabla,\nabla)(x\delta(h)),$$

where

$$M_1(s,t) = \frac{G(t)-G(s+t)}{s}, \qquad M_2(s,t) = \frac{G(s+t)-G(s)}{t}.$$

Proof. It can be seen by writing $G(v) = \int e^{-itv}g(t)\,dt$ and using the identity[9]

$$\delta_i\sigma_t = \sigma_t\delta_i + it\int_0^1 \sigma_{ut}\,\mathrm{ad}_{\delta_i(h)}\sigma_{(1-u)t}\,du.$$

□

Lemma 5.4. We have

$$\varphi_0\big(e^{-h}G(\nabla)(\delta(h))\delta(a)\big) = -\varphi_0\big(ae^{-h}G(\nabla)(\delta^2(h))\big) - \varphi_0\big(ae^{-h}P(\nabla,\nabla)(\delta(h)\delta(h))\big),$$

where

$$P(s,t) = G(s+t)\frac{e^{-s}-1}{s} + M_1(s,t)e^{-s} + M_2(s,t).$$

Proof. We start by writing

$$\begin{aligned}
&\varphi_0\big(e^{-h}G(\nabla)(\delta(h))\delta(a)\big) = -\varphi_0\big(a\,\delta(G(\nabla)(e^{-h}\delta(h)))\big)\\
&= -\varphi_0\big(a\,G(\nabla)(\delta(e^{-h}\delta(h)))\big) - \varphi_0\big(aM_1(\nabla,\nabla)(\delta(h)e^{-h}\delta(h))\big)\\
&\quad -\varphi_0\big(aM_2(\nabla,\nabla)(e^{-h}\delta(h)\delta(h))\big)\\
&= -\varphi_0\big(ae^{-h}\,G(\nabla)(e^{h}\delta(e^{-h})\delta(h))\big) - \varphi_0\big(a\,G(\nabla)(e^{-h}\delta^2(h))\big)\\
&\quad -\varphi_0\big(ae^{-h}M_1(\nabla,\nabla)(e^{-\nabla}(\delta(h))\delta(h))\big) - \varphi_0\big(ae^{-h}M_2(\nabla,\nabla)(\delta(h)\delta(h))\big).
\end{aligned}$$

Then, using the fact that $e^h\delta(e^{-h}) = \frac{e^{-\nabla}-1}{\nabla}(\delta(h))$, one can find the above expression for $\varphi_0\big(e^{-h}G(\nabla)(\delta(h))\delta(a)\big)$ in the statement of the lemma. □

Lemma 5.5. We have

$$\varphi_0\big(e^{-h}G(\nabla)(\delta(a))\delta(h)\big) = -\varphi_0\big(ae^{-h}\bar{G}(\nabla)(\delta^2(h))\big) - \varphi_0\big(ae^{-h}Q(\nabla,\nabla)(\delta(h)\delta(h))\big),$$

where

$$\begin{aligned}
\bar{G}(s) &= G(-s)e^{-s},\\
Q(s,t) &= \bar{G}(s+t)\frac{e^{-s}-1}{s} + M_1(s,t)e^{-s} + M_2(s,t).
\end{aligned}$$

Proof. It follows from the previous lemma after writing

$$\varphi_0\big(e^{-h}G(\nabla)(\delta(a))\delta(h)\big) = \varphi_0\big(e^{-h}G(-\nabla)e^{-\nabla}(\delta(h))\delta(a)\big).$$

□

The Taylor series at $s = 0$ of the one variable function ω_1 appearing in the formula for the gradient of Ω is given by

$$\omega_1(s) = -\frac{1}{2} + \frac{s}{4} - \frac{s^2}{6} + \frac{s^3}{16} - \frac{s^4}{45} + \frac{s^5}{160} + O\left(s^6\right).$$

The function of two variables ω_2 appearing in the formula for the above gradient has the following Taylor expansion at the origin:

$$\begin{aligned}
\omega_2(s,t) = &\left(\frac{1}{4} + \frac{t}{24} + \frac{13t^2}{240} - \frac{7t^3}{360} + O\left(t^4\right)\right) + s\left(-\frac{3}{8} + \frac{5t}{48} - \frac{17t^2}{120} + \frac{t^3}{14} + O\left(t^4\right)\right)\\
&+s^2\left(\frac{47}{240} - \frac{t}{6} + \frac{77t^2}{480} - \frac{1159t^3}{13440} + O\left(t^4\right)\right)\\
&+s^3\left(-\frac{83}{720} + \frac{169t}{1260} - \frac{697t^2}{5760} + \frac{151t^3}{2304} + O\left(t^4\right)\right) + O\left(s^4\right).
\end{aligned}$$

We now look at the behavior of the function ω_2 on the diagonals (Figures 7 and 8). We have

$$\omega_2(s,s) = -\frac{e^{-3s/2}\sinh\left(\frac{s}{2}\right)(8s + (8s-5)\sinh(s) - 3\sinh(2s) + \sinh(3s) - 8\cosh(s) - 3\cosh(2s) + 11)}{4s^3},$$

with the Taylor expansion

$$\omega_2(s,s) = \frac{1}{4} - \frac{s}{3} + \frac{17s^2}{48} - \frac{319s^3}{720} + \frac{623s^4}{1440} - \frac{155s^5}{448} + O\left(s^6\right).$$

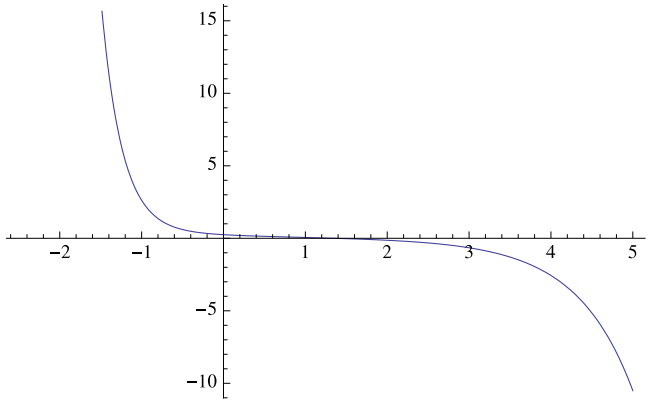

FIG. 7. Graph of the function $s \mapsto \omega_2(s, s)$.

On the other diagonal, we have

$$\omega_2(s,-s) = \frac{4s + e^{-2s} - 2e^{s} + 1}{4s^2} = \frac{1}{4} - \frac{5s}{12} + \frac{7s^2}{48} - \frac{17s^3}{240} + \frac{31s^4}{1440} - \frac{13s^5}{2016} + O\left(s^6\right).$$

VI. DISCUSSION

Considering the variety of geometric spaces that fit into the paradigm of noncommutative geometry,[4,5] it is of great importance to develop different methods for computing their local geometric invariants such as scalar curvature. Different methods of computation can also help to achieve conceptual understandings of such invariants for specific examples.

The appearance of functions of a modular automorphism in the final formulas for the scalar curvature of noncommutative tori is a purely noncommutative feature,[9,16,17,12] which accompanies the following striking facts. First, the final one and two variable functions in the curvature formulas are significantly simple, which indicates an enormous amount of cancellations in the algebraically lengthy formulas that involve hundreds of terms with numerous functions from the rearrangement lemma involved.[10,9,1,16] Second, the function of two variables for the curvature of $\mathbb{T}^2_\theta$ can be recovered from the one variable function[9] by finite differences.

Using the noncommutative residue that involves integration on the 3-sphere,[18,17] we computed the scalar curvature of $\mathbb{T}^4_\Theta$ in this paper without using the rearrangement lemma. This method avoids the complexity stemming from the functions coming from this lemma and also explains the simplicity of the final functions in the curvature formula. It should be emphasized that there

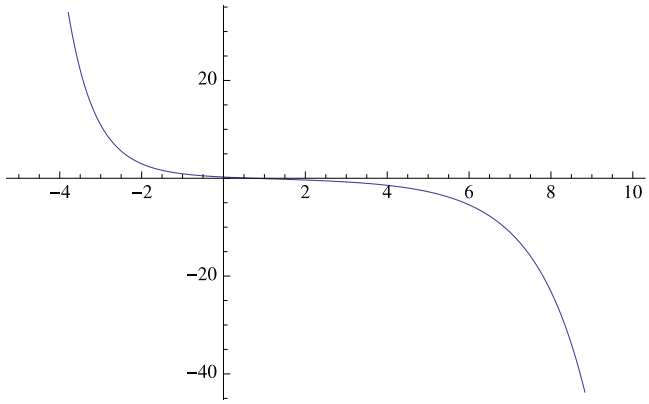

FIG. 8. Graph of the function $s \mapsto \omega_2(s, -s)$.

is another rather technical simplifying factor in this method, compared to the use of parametric pseudodifferential calculus, which is due to the fact that the first term b_0 of the parametrix of the Laplacian reduces on the sphere to a power of the Weyl factor. It can be seen in the derivation of (1) that this softens out some further complexities and leads to the final expression, whose summand consists of only a few terms. By working out a simple relation between the functions that relate the derivatives of the Weyl factor to the derivatives of the dilaton, we have then written the two variable function in the formula for the curvature of $\mathbb{T}^4_\Theta$ as the sum of a finite difference and a finite product of the one variable function.

Similar to the concrete illustration in Ref. 9 of the scalar curvature of $\mathbb{T}^2_\theta$ for dilatons associated with projections, which exist in abundance for noncommutative tori,[28] we have worked out a concrete formula in the case of $\mathbb{T}^4_\Theta$. For a dilaton of the form $h = sp$, where $s \in \mathbb{R}$ and $p \in C^\infty(\mathbb{T}^4_\Theta)$ is an arbitrary projection, the final concrete formula for the curvature involves unbounded functions of the parameter s, which is in contrast to the striking fact about the boundedness of the functions obtained in the two dimensional case.[9] The question that arises is whether there is a conceptual meaning behind this contrast. Also, the explicit computation of the gradient of the analog of the Einstein-Hilbert action for $\mathbb{T}^4_\Theta$ prepares the ground for further studies of the natural associated geometric flows in this context, cf. Refs. 1 and 9.

ACKNOWLEDGMENTS

The author thanks the Max-Planck Institute for Mathematics, the Stefan Banach International Mathematical Center at the Institute for Mathematics at the Polish Academy of Sciences, and the Hausdorff Research Institute for Mathematics for their support and hospitality in 2014.

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