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An Improved Sufficient Condition for Absence of Limit Cycles in Digital Filters

P. P. VAIDYANATHAN AND V. LIU

Abstract **—It is known that if the state transition matrix A of a digital filter structure is such that $D - A^{\dagger}DA$ is positive definite for some diagonal matrix D of positive elements, then all zero-input limit cycles can be suppressed. This paper shows that positive semidefiniteness of $D - A^{\dagger}DA$ is in fact sufficient. As a result, it is now possible to explain the absence of limit cycles in Gray–Markel lattice structures based only on the state-space viewpoint.**

I. A Property of the State Transition Matrix

Consider an IIR digital filter realization with the state-space description

$$x(n+1) = Ax(n) + Bu(n) \quad (1)$$

$$y(n) = Cx(n) + du(n) \quad (2)$$

where A is $N \times N$, B is $N \times 1$, C is $1 \times N$, and d is a scalar. In this paper, the input $u(n)$ is assumed to be zero. Fig. 1 shows a realistic model of the system, with quantizers in the feedback loop. The quantizers are such that each state variable is quantized independently of others:

$$x_k(n+1) = Q[w_k(n+1)]. \quad (3)$$

The operation $Q[x]$ represents magnitude-truncation arithmetic when $-1 \leqslant x < 1$ and 2's-complement overflow operation when x exceeds this range. Under this condition, it is well known [1] that if A satisfies[1]

$$A^{\dagger}A < I \quad (4)$$

then there are no self-sustained oscillations of either type (roundoff or overflow [2]) under zero input. Condition (4) is equivalent to saying that the singular values of A are strictly less than unity, or, in other words, that

$$V^{\dagger}A^{\dagger}AV < V^{\dagger}V \qquad \text{for every vector } V \neq \mathbf{0}. \quad (5)$$

Even though A is stable[2], (4) is in general not guaranteed unless

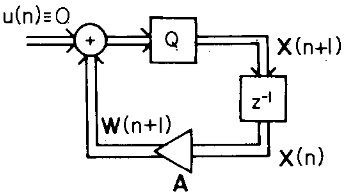

Fig. 1. The state recursion model with nonlinearity.

the state-space structure is appropriately chosen. The minimum-norm structures introduced in [1] automatically satisfy (4) because for such structures, the maximum eigenvalue of $A^{\dagger}A$ is equal to the maximum of the absolute values of the eigenvalues of A.

It is also well known in the literature that the cascaded lattice structures [4], [5] due to Gray and Markel are free from zero-input limit cycles as long as the quantizers are chosen as above. However, these structures do *not* satisfy (4). In fact, the normalized lattice structure has a state-space description [7], [8] as in (1), (2) satisfying

$$A^{\dagger}A + C^{\dagger}C = I. \quad (6)$$

Clearly, (6) implies

$$A^{\dagger}A \leqslant I. \quad (7)$$

Since C is a row vector, there exists a set of $N-1$ orthogonal vectors such that $V^{\dagger}A^{\dagger}AV = V^{\dagger}V$; hence, there are precisely $N-1$ singular values of A equal to unity, thereby violating (4). The fact that the lattice structures are free from limit cycles in spite of this can be explained based on the observation that the pair (C, A) "happens to be" completely observable [7], [8]. In this paper, we show that there is in fact a *fundamental* linear-algebraic reason why this should be so. A result is presented which shows that the complete observability in the lattice structures is by no means coincidental but is a consequence of a more basic result. This helps to obtain a formal, quantitative proof of certain useful claims made in an earlier paper [11].

The main result of this paper is the following.

Lemma 1: Let A be an $N \times N$ stable matrix satisfying (7). Then the following is true:

$$(A^{\dagger})^N A^N < I. \quad (8)$$

Condition (8) enables us to show that zero-input limit cycles will not be sustained.

Proof: First note that since (7) holds, we can always find a $p \times N$ matrix $\mathscr{C}$ such that

$$A^{\dagger}A + \mathscr{C}^{\dagger}\mathscr{C} = I. \quad (9)$$

The lemma is proved by establishing the following two properties:

Property 1: Condition (8) holds if and only if $\mathscr{C}$ satisfying (9) is such that $(\mathscr{C}, A)$ is *completely observable*.

Property 2: If A is stable, the pair $(\mathscr{C}, A)$ satisfying (9) is *necessarily* completely observable.

In order to prove Property 1, note that the pair $(\mathscr{C}, A)$ is completely observable if and only if the $pN \times N$ matrix P defined by

$$P = \begin{bmatrix} \mathscr{C} \\ \mathscr{C}A \\ \mathscr{C}A^2 \\ \vdots \\ \mathscr{C}A^{N-1} \end{bmatrix} \quad (10)$$

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[1] Superscript T denotes transposition and superscript $\dagger$ denotes transposed conjugation. The notation $P < Q$, where P and Q are Hermitian, denotes that $Q - P$ is positive definite; $P \leqslant Q$ denotes that $Q - P$ is positive semidefinite.

[2] We say that A is stable if all its eigenvalues are strictly inside the unit circle.

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has column rank equal to N. This in turn is true if and only if $P^\dagger P > 0$. Now

$$P^\dagger P = \mathscr{C}^\dagger \mathscr{C} + A^\dagger \mathscr{C}^\dagger \mathscr{C} A + \cdots + (A^\dagger)^{N-1} \mathscr{C}^\dagger \mathscr{C} A^{N-1}. \quad (11)$$

Since $(\mathscr{C}, A)$ satisfies (9), equation (11) reduces to

$$P^\dagger P = I - (A^\dagger)^N A^N \quad (12)$$

establishing Property 1.

Next, Property 2 can be proved by invoking a standard result on observability, viz., the Popov–Belevitch–Hautas criterion [6], which says that $(\mathscr{C}, A)$ is *not* completely observable if and only if there exists an eigenvector of A that is orthogonal to all rows of $\mathscr{C}$. Thus, if $(\mathscr{C}, A)$ is not observable, there exists $V \neq 0$ such that

$$AV = \lambda V \quad (13)$$

and

$$\mathscr{C} V = 0. \quad (14)$$

Since (9) holds, (14) implies

$$V^\dagger A^\dagger A V = V^\dagger V \quad (15)$$

whereas (13) implies

$$V^\dagger A^\dagger A V = |\lambda|^2 V^\dagger V. \quad (16)$$

Equations (15) and (16) together imply $|\lambda| = 1$; hence, the condition that A have all eigenvalues inside the unit circle is violated whenever $(\mathscr{C}, A)$ is not observable. This establishes Property 2. Putting together Properties 1 and 2, Lemma 1 clearly follows.

From Lemma 1, we obtain the following corollary:

Corollary: Let A be an $N \times N$ stable matrix such that

$$A^\dagger D A \leqslant D \quad (17)$$

for some Hermitian, positive definite D. Then

$$(A^\dagger)^N D A^N < D. \quad (18)$$

This result follows simply by defining a matrix

$$A_1 = TAT^{-1} \quad (19)$$

where T is such that $D = T^\dagger T$. Clearly (17) is equivalent to $A_1^\dagger A_1 \leqslant I$. Thus, by Lemma 1, $(A_1^\dagger)^N A_1^N < I$, which proves (18). Referring to the model of Fig 1, we can now obtain a formal proof of the following result (first suggested in [11]).

Theorem 1: Let A be an $N \times N$ stable matrix such that

$$A^\dagger D A \leqslant D \quad (20)$$

for some *diagonal matrix* D with positive diagonal entries. Then the structure of Fig. 1 is free from zero-input limit cycles of both kinds whenever the nonlinearity $Q[x]$ is restricted to be magnitude-truncation type, coupled with 2's-complement type of overflow feature.

This is a generalization of the result in [3], where strict inequality in (20) was shown to be sufficient. The proof of the theorem follows by combining the strategy outlined in [3] with Lemma 1. Thus, from Fig. 1 we have

$$w(n+1) = Ax(n) \quad (21)$$

$$x(n+1) = Q[w(n+1)]. \quad (22)$$

Define the norm $\|V\|$ of a vector V by

$$\|V\|^2 = V^\dagger D V. \quad (23)$$

Since $|x_k(n+1)| \leqslant |w_k(n+1)|$, $1 \leqslant k \leqslant N$ (because of the assumed nature of the quantizer), and since D is a diagonal matrix of positive diagonal elements d_k, we have

$$\|x(n+1)\| \leqslant \|w(n+1)\| \quad \text{for all } n. \quad (24)$$

Next, in view of (20),

$$x^\dagger(n) A^\dagger D A x(n) \leqslant x^\dagger(n) D x(n) \quad (25)$$

i.e.,

$$\|w(n+1)\| \leqslant \|x(n)\|. \quad (26)$$

Equations (26) and (24) imply $\|x(n+1)\| \leqslant \|x(n)\|$. Now consider $N-1$ consecutive iterations of (21) and (22) with $n_0 \leqslant n \leqslant n_0 + N - 2$. If there occurs a nonlinearity in this period, then we have

$$\|x(n+1)\|^2 \leqslant \|w(n+1)\|^2 - \epsilon \quad (27)$$

for some n in the above range, where ϵ is a fixed positive number depending only on D and on the quantization step size. Thus, $\|x(n+1)\|^2 \leqslant \|x(n)\|^2 - \epsilon$, i.e., the norm necessarily decreases. On the other hand, if there is no nonlinearity during this period, then

$$w(n_0 + N) = A^N x(n_0) \quad (28)$$

whence

$$\|w(n_0+N)\|^2 = x^\dagger(n_0)(A^\dagger)^N D A^N x(n_0) < x^\dagger(n_0) D x(n_0)$$
$$= \|x(n_0)\|^2 \quad (29)$$

because of (18). Since the quantizer permits only discrete values of the norm, we eventually have

$$\|x(n_0+N)\|^2 \leqslant \|x(n_0)\|^2 - \epsilon. \quad (30)$$

Thus, during an interval equal to N units of time, the norm $\|x(n)\|$ necessarily decreases, at least by a fixed amount $\epsilon > 0$. The internal energy therefore goes to zero in a finite amount of time. Notice that with $D = I$, (20) reduces to (7); hence, (7) is sufficient to suppress zero-input limit cycles.

The main contribution of Lemma 1 in the above argument is that it tells us that the norm of $x(n)$ decreases by at least ϵ in N units of time.

Comment: It should be emphasized here that the above results hold even though the physical multipliers in the structure are not the elements of the A matrix. As long as the implementation *satisfies* the representation of Fig. 1, with the quantizers located as indicated, and as long as A satisfies the sufficient conditions of Theorem 1, limit cycles are absent. Restricting the quantizers to be located in this manner might mean that extra internal word length is required to avoid other quantizations. This, however, is not a serious requirement if the filter is a second-order section to be used in a cascade or parallel form structure.

II. Relation to the Gray and Markel Lattice Structures

The overall appearance of this family of structures is shown in Fig. 2. The function $G(z)$ is all pass. (If arbitrary transfer functions are desired, tap coefficients are used in order to obtain these, and these do not alter the limit-cycle properties.) The exact appearance of the 2×2 building blocks varies depending on the type of the lattice structure [4], [5], [9]. For the *normalized* structure, these are as shown in Fig. 3(b), whereas Fig. 3(a) shows a denormalized, two-multiplier version. One- and three-multiplier versions can be found in [4] and [5]. The transfer matrices of the building blocks in Figs. 3(b) and (a) are, respectively, given by

$$T_b = \begin{bmatrix} k_m & \hat{k}_m \\ \hat{k}_m & -k_m \end{bmatrix} \quad T_a = \begin{bmatrix} k_m & 1-k_m^2 \\ 1 & -k_m \end{bmatrix} \quad (31)$$

where $\hat{k}_m = \sqrt{1-k_m^2}$. It is assumed throughout that $-1 < k_m < 1$,

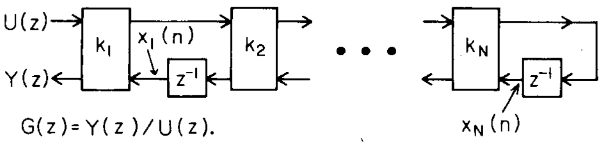

Fig. 2. The cascaded lattice structure due to Gray and Markel.

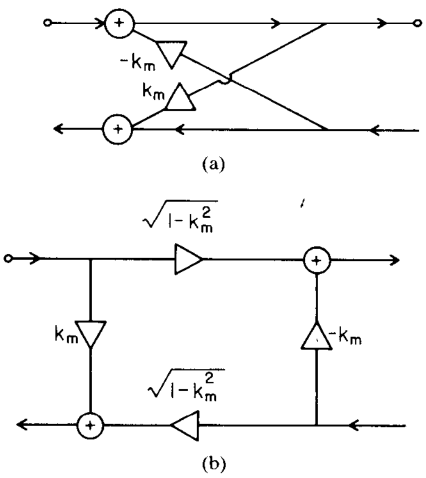

Fig. 3. Typical building blocks in Fig. 2. (a) The two-multiplier building block. (b) The four-multiplier (or normalized) building block.

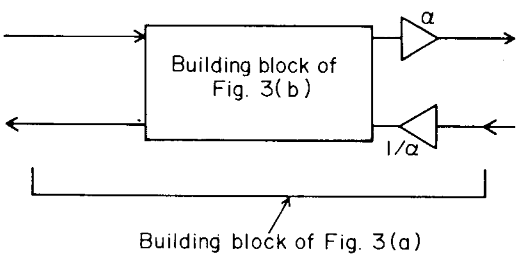

Fig. 4. Relation between the two building blocks in Fig. 3.

so that A is a stable matrix. Notice that the matrices satisfy

$$(T_a)_{ii} = (T_b)_{ii} \quad (T_a)_{12}(T_a)_{21} = (T_b)_{12}(T_b)_{21} \tag{32}$$

and hence the building block in Fig. 3(b) is related to that in Fig. 3(a) as in Fig. 4 for $\alpha = 1/\hat{k}_m$. In general, each one of the five building blocks in [4] and [5] is related to the others in this manner for appropriate α.

Now, if the state-space representation of Fig. 2 is as in (1) and (2), then for the normalized lattice structure, it is well known [7] that the $(N+1)\times(N+1)$ matrix

$$R = \begin{bmatrix} A & B \\ C & d \end{bmatrix} \tag{33}$$

is orthogonal. In particular, (6) and hence (7) hold. Thus, if the quantizers Q are introduced prior to the delays, there are no limit cycles of either type under zero-input conditions.

For the remaining four types of lattice structures, (7) does not hold. However, it has been shown by Gray based on pseudopassivity arguments that all these structures are free from limit cycles. We now give a second proof of this, based only on Theorem 1.

Let A, B, C, d be the state-space parameters for the normalized structure. In this structure, let the mth building block (which has the form of Fig. 3(b)) be replaced by the corresponding denormalized version of Fig. 3(a), and let the other building blocks be left unchanged. If the $N\times N$ A matrix of the normalized structure is partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{34}$$

where A_{11} is $(m-1)\times(m-1)$ and A_{22} is $(N-m+1)\times(N-m+1)$, the new state transition matrix is given by

$$A_{\text{new}} = \begin{bmatrix} A_{11} & \frac{1}{\alpha}A_{12} \\ \alpha A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I_{m-1} & 0 \\ 0 & \alpha I_{N-m+1} \end{bmatrix} A \begin{bmatrix} I_{m-1} & 0 \\ 0 & \frac{1}{\alpha} I_{N-m+1} \end{bmatrix} \tag{35}$$

which is a diagonal similarity transformation of A. Thus, if each building block of the normalized lattice structure is replaced by the corresponding denormalized version, the new state transition matrix A_1 is equal to $T^{-1}AT$, where T is diagonal. Since A satisfies (20) with $D = I$, A_1 evidently satisfies (20) with $D = T^2$. Thus, the transformed structure is free from limit cycles. In fact, a lattice structure in which each section is randomly chosen (out of the five possibilities in [4] and [5]) independent of the other sections is also free from limit cycles!

III. Relation to Orthogonal Digital Filters

A number of references to this class of filters can be found in [10]. The all-pass lattice structure of Fig. 2 with building blocks as in Fig. 3 belongs to this wider class. In general, an orthogonal digital filter has more than one input and one output terminal, and the transfer matrix is unitary on the unit circle of the z plane. Each component of the transfer matrix is typically a filtering function, (for instance, an elliptic IIR filter). An orthogonal implementation of such a system has a state-space representation such that the matrix R in (33) (with d now replaced with a matrix $\mathscr{D}$) is either orthogonal or can be converted into an orthogonal matrix by a diagonal similarity transformation of A. Accordingly, such structures satisfy (17), and as long as A is stable, can be rendered free from limit cycles by placing the quantizers as in Fig. 1.

IV. Concluding Remarks

A well-known sufficient condition for absence of zero-input limit cycles in digital filters has been simplified, with applications in the reinterpretation of the behavior of recursive lattice structures and orthogonal filters. It should be noticed that even though (4) is more stringent than (7), the system state in Fig. 1 (under zero input) for a given initial state typically goes to zero *faster* if (4) rather than (7) holds for a stable A matrix. This follows by noting that (4) ensures a decrease of state energy during each state recursion, whereas (8) guarantees such a decrease only once in N such recursions. Thus, in general, transients due to quantizer nonlinearity die out faster in minimum-norm implementations than in lattice structures and orthogonal structures.

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An Efficient Numerical Scheme to Compute 2-D Stability Thresholds

L. M. ROYTMAN, M. N. S. SWAMY, AND G. EICHMANN

Abstract —**Stability thresholds (margins) of two-dimensional (2-D) digital filters were recently defined in terms of the singularities of the transfer function. Stability thresholds hold a close relationship to the settling time of the 2-D impulse response and can therefore serve as a measure of stability of 2-D digital filters. In this paper, a simple, unified procedure for the computation of stability thresholds (margins) for 2-D digital filters based on the z_1 and z_2 resultants of 2-D polynomials is presented. To illustrate the process, simple examples are also provided.**

I. Introduction

One of the results of the advent of computers has been the emergence of 2-D digital signal processing. Because it requires large amounts of memory, 2-D signal processing has only recently been used for such applications as image processing and seismic signal processing. These applications involve the processing of input data sets using 2-D spatial-domain digital filters. It is well known that for a desired response characteristic, recursive filters are more efficient from a hardware point of view and are found in most applications. In the design and realization of these filters, a bounded-input–bounded-output (BIBO) stability requirement is often needed. In a recent paper [1], it was observed that for practical filter realization it is important to know not only that a filter is stable but also its degree of stability. In the same article [1], a practical stability measure, called the stability threshold, was introduced and a numerical procedure for its computation was proposed. These stability thresholds have been used for the minimization of the required word length of a designed digital filter [1], [2].

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The same concept of "how far" a multidimensional filter is from instability has been independently introduced subsequently in [3] and [4], where the term "stability margin" was used instead of "stability threshold." These papers also presented two methods for the computation of the stability margins. The computational algorithm in [3] uses the Schur–Cohn stability criterion to transform this problem into an optimization problem. We note that this optimization procedure is very similar to the computational process of [1]. The computation technique in [4] utilizes the geometrical property of points in the z plane. This geometrical argument leads directly, without having to solve an optimization problem, to a computational procedure which, though superior to the ones in [1] and [3], could sometimes lead to complicated vector space computation. We also note an interesting method, presented in [5], to obtain in state-space the lower stability margin bounds of a 2-D digital system. This method can be used for approximate stability analysis.

Recently [6], [7], using the technique of 2-D resultants, some new results on l_1 and l_2 stabilities of 2-D digital filters have been derived. The concept of z_1 and z_2 resultants is used in this article to develop a simple, unified computational algorithm for the evaluation of 2-D stability thresholds. We also illustrate by numerical examples the power of this algorithm.

II. 2-D Stability Thresholds

Let a 2-D digital filter transfer function be defined as

$$F(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)}.$$

For $F(z_1, z_2)$ to be a stable filter, we have[1]

$$B(z_1, z_2) \neq 0 \text{ in } \{(z_1, z_2) : |z_1| \leqslant 1, |z_2| \leqslant 1\}.$$

For such a stable filter, consider the three stability thresholds (margins) T_1, T_2, and T, defined by the largest bidisk for which

$$B(z_1, z_2) \neq 0 \text{ in } \{(z_1, z_2) : |z_1| < 1 + T_1, |z_2| \leqslant 1\} \quad (1)$$

$$B(z_1, z_2) \neq 0 \text{ in } \{(z_1, z_2) : |z_1| \leqslant 1, |z_2| < 1 + T_2\} \quad (2)$$

$$B(z_1, z_2) \neq 0 \text{ in } \{(z_1, z_2) : |z_1| < 1 + T, |z_2| < 1 + T\}. \quad (3)$$

We first consider the stability threshold described in (1). Let

$$k_1 = 1 + T_1$$
$$z_1 = e^{j\phi_1} \text{ and } z_2 = e^{j\phi_2}. \quad (4)$$

It follows from [3] that a minimum positive k_1 can be found such that

$$B(k_1 z_1, z_2) = 0. \quad (5)$$

The minimum value of k_1 defined by (5) defines a 2-D stability threshold (margin) of the 2-D filter. In the next section, we give a method to find this constant.

III. Computational Technique

Invoking De Moivre's theorem, (5) can be conceptually decomposed into two simultaneous real equations in k_1, ϕ_1, and ϕ_2. Thus, k_1 is implicitly defined as a function of the two real variables ϕ_1 and ϕ_2. Hence, at the point where the minimal value

[1]We intentionally exclude the special case of filter transfer functions having nonessential singularities of the second kind on the distinguished boundary since it is of no interest for the present discussion. (N.B. all the three stability thresholds are zero.)