solution may belong. We illustrate this by an example in the single unknown y. Let

$$F = y_2^2 + \prod_{j=1}^{m} [(x + j)y_1 - y]$$

where m is any integer greater than unity. Now $(x + j)y_1 - y$ has $(x + j)y_2$ as derivative and therefore has, for every j, a manifold which is a component of F. The solution $y = 0$ belongs to every such component.

DIFFERENTIALS OF FUNCTIONS WITH ARGUMENTS AND VALUES IN TOPOLOGICAL ABELIAN GROUPS[1]

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1. *Introduction.*—By a topological abelian group T (t.a.g. T) we shall mean an abstract abelian group—written additively—such that (*a*) the function $x + y$ and the inverse function $-x$ are continuous functions (neighborhood continuity) of both variables x and y and of the variable x, respectively, with respect to a postulated Hausdorff topology; (*b*) given any $y \in T$ and any Hausdorff neighborhood U of $0 \in T$, there exists a "positive integer" n such that[2] $y \in nU$.

In this note we shall give brief indications of a differential calculus for functions $f(x)$ with $x \in$ t.a.g. T_1 and values in a t.a.g. T_2. Proofs and further developments will appear elsewhere.

The real number system has entered into the various differential calculi studied so far in one or more of three ways: (1) through the independent and dependent variables; (2) in the topology via a numerically valued metric or norm; (3) as a multiplicative domain. *The differential calculus announced in the present paper does not employ the real numbers, thus giving a new flavor to an ancient subject and its modern generalizations.*

2. *First Order Differentials.*—A first order differential $f(x_0; \delta x)$ is defined as follows. In the definition of an M-differential[3] given in the paper *LTS* interpret T_1 and T_2 to be t.a.g. and not necessarily linear topological spaces. Then replace condition 2 (*b*) by the condition

$$\epsilon(x_0, x_1, nx_2) = n\, \epsilon(x_0, x_1, x_2)$$

for all "positive integers" n, for all x_1 in some Hausdorff neighborhood of $0 \in T_1$, *and for all* $x_2 \in T_1$. To complete the definition add the following condition:

2 (*d*). *There exists a Hausdorff neighborhood W of* $0 \in T_1$ *with respect to*

which the following property holds: given a Hausdorff neighborhood V of $0 \epsilon T_2$, there exists a Hausdorff neighborhood $U(V)$ of $0 \epsilon T_1$ such that

$$\epsilon(x_0, x_1, x_2) \epsilon V \text{ for } x_1 \epsilon U(V) \text{ and } x_2 \epsilon W.$$

The following theorems have been proved.

THEOREM I. *Let $f(x)$ be a function with values in a t.a.g. T_2 and defined on a Hausdorff neighborhood of $x_0 \epsilon$ t.a.g. T_1. If $f(x)$ has a first order differential $f(x_0; \delta x)$ at $x = x_0$, then $f(x_0; \delta x)$ is unique[4] for all $\delta x \epsilon T_1$.*

COROLLARY. *If $f(x)$ has a first order differential at $x = x_0$, then $f(x)$ is continuous at $x = x_0$.*

THEOREM II. *If $f_1(x)$ and $f_2(x)$ have first order differentials at $x = x_0$, then $f_3(x) = \pm n_1 f_1(x) \pm n_2 f_2(x)$ (n_1 and n_2 positive integers) has a first order differential at $x = x_0$ given by*

$$f_3(x_0;\ \delta x) = \pm n_1 f_1(x_0;\ \delta x) \pm n_2 f_2(x_0;\ \delta x).$$

THEOREM III. *Let T_1, T_2, T_3 be t.a.g., not necessarily distinct, and U_{x_0} a Hausdorff neighborhood of $x_0 \epsilon T_1$. If $f(x)$ on U_{x_0} to T_2 has a first order differential at $x = x_0$ and if $\phi(y)$ on $f(U_{x_0})$ to T_3 has a first order differential at $y_0 = f(x_0)$, then $\psi(x) = \phi(f(x))$ has a first order differential at $x = x_0$ given by*

$$\psi(x_0;\ \delta x) = \phi(f(x_0);\ f(x_0;\ \delta x)).$$

THEOREM IV. *The property of first order differentiability of a function with arguments and values in topological abelian groups is invariant under topological isomorphisms of the topological abelian groups. In particular, the invariance[5] is maintained under a passage to equivalent Hausdorff topologies of the topological abelian groups.*

THEOREM V. *If the topological abelian groups T_1 and T_2 are linear topological spaces and if $f(x)$ has a first order differential at $x = x_0 \epsilon T_1$, then the M-differential of $f(x)$ at $x = x_0$ exists and the two differentials are equal. The validity of the converse statement is an open question.*

THEOREM VI. *If the topological abelian groups T_1 and T_2 are complete normed linear spaces (Banach spaces) and if $f(x)$ has a first order differential at $x = x_0 \epsilon T_1$, then the Fréchet differential of $f(x)$ at $x = x_0$ exists and the two differentials are equal.[5] Conversely if $f(x)$ has a Fréchet differential at $x = x_0$, then a first order differential (in our sense) of $f(x)$ at $x = x_0$ exists and the two differentials are equal.*

We remark here that if we dispense with condition 2 (c) in the definition of a first order differential, then all the above theorems except Theorem V continue to hold.

3. *Second Order Differentials.*—nth successive first order differentials can be defined inductively whenever the $(n - 1)$st successive first order differential exists for x in a neighborhood of an element x_0. In this section,

however, we shall turn our attention to an inductive definition of nth order differentials at $x = x_0$ by assuming the existence of the $(n - 1)$st order differentials and the preceding differentials merely at the element $x = x_0$. For the purposes of our brief exposition, we shall do this here only for second order differentials.

Let T_1 be a t.a.g. and T_2 a t.a.g. with 0 as the only element of finite order. Let $f(x)$ possess a first order differential $f(x_0;\ \delta x)$ at $x = x_0$. A function $f(x_0;\ \delta_1 x;\ \delta_2 x)$ will be called a second order differential of $f(x)$ at $x = x_0$ with increments $\delta_1 x$ and $\delta_2 x$, if

(1) $f(x_0;\ \delta_1 x;\ \delta_2 x)$ is a 2-uniform symmetric bilinear function;

(2) there exists a function $\epsilon(x_0, x_1, x_2, x_3)$ with arguments in T_1 and values in T_2 such that

(a) $\epsilon\ (x_0, 0, x_2, x_3) = 0$ for all $x_2, x_3 \epsilon T_1$,

(b) $\epsilon\ (x_0, x_1, nx_2, mx_3) = nm\ \epsilon(x_0, x_1, x_2, x_3)$ for all "positive integers" n and m, for all x_1 in some Hausdorff neighborhood of $0 \epsilon T_1$, and for all $x_2, x_3 \epsilon T_1$,

(c) $\epsilon(x_0, x_1, x_2, x_3)$ is continuous in (x_1, x_2, x_3) at $x_1 = 0$, $x_2 = x_2$, $x_3 = x_3$ for all $x_2, x_3 \epsilon T_1$,

(d) there exist neighborhoods W_1 and W_2 of $0 \epsilon T_1$ with respect to which the following property holds: given a neighborhood V of $0 \epsilon T_2$, there exists a neighborhood $U(V)$ of $0 \epsilon T_1$ such that $\epsilon\ (x_0, x_1, x_2, x_3) \epsilon V$ for $x_1 \epsilon U(V)$, $x_2 \epsilon W_1$, $x_3 \epsilon W_2$;

(3) there exists some neighborhood N of $0 \epsilon T_1$ such that for all $\delta x \epsilon N$, $f(x_0;\ \delta_1 x;\ \delta_2 x)$ is a second order approximation to the difference $f(x_0 + \delta x) - f(x_0)$ in the sense that $2[f(x_0 + \delta x) - f(x_0) - f(x_0;\ \delta x)] - f(x_0;\ \delta x;\ \delta x) = \epsilon\ (x_0, \delta x, \delta x, \delta x)$ for all $\delta x \epsilon N$.

THEOREM VII. *If a second order differential* $f(x_0;\ \delta_1 x;\ \delta_2 x)$ *of* $f(x)$ *exists at* $x = x_0$, *then it is unique*[6] *for all* $\delta_1 x, \delta_2 x \epsilon T_1$.

THEOREM VIII. *If* $f_1(x)$ *and* $f_2(x)$ *possess second order differentials at* $x = x_0$, *then the second order differential of* $\pm n_1 f_1(x) \pm n_2 f_2(x)$ *exists at* $x = x_0$ *and is given by* $\pm n_1 f_1(x_0;\ \delta_1 x;\ \delta_2 x) \pm n_2 f_2(x_0;\ \delta_1 x;\ \delta_2 x)$.

THEOREM IX. *Let* T_1 *be a t.a.g., and* T_2 *and* T_3 *two t.a.g. without elements of finite order other than their* 0 *elements, and let* U_{x_0} *be a Hausdorff neighborhood of* $x_0 \epsilon T_1$. *If* $f(x)$ *on* U_{x_0} *to* T_2 *possesses a second order differential at* $x = x_0$ *and if* $\phi(y)$ *on* $f(U_{x_0})$ *to* T_3 *possesses a second order differential at* $y_0 = f(x_0)$, *then* $\psi(x) = \phi(f(x))$ *possesses a second order differential at* $x = x_0$ *given by the formula*

$$\psi(x_0;\ \delta_1 x;\ \delta_2 x) = \phi(f(x_0);\ f(x_0;\ \delta_1 x);\ f(x_0;\ \delta_2 x)) + \phi(f(x_0);\ f(x_0;\ \delta_1 x;\ \delta_2 x)).$$

COROLLARY. *The correspondent of Theorem IV for second order differentiability of a function.*

In conclusion, we wish to remark that with the aid of the concept of a product topological space it is possible to treat total differentials of functions of several t.a.g. variables.

[1] The results on first order differentials were presented to the American Math. Society at the Pasadena meeting, Dec. 2, 1939.

[2] By nx we understand the group sum $x + x + \ldots + x$ with n summands. Similarly $-nx$ will stand for the group difference $-x - x - \ldots - x$. If $S \subset T$, then by nS we mean the set of all elements nx with $x \in S$. Clearly, condition (b) in the definition of a t.a.g. T becomes redundant whenever T is specialized to be a linear topological space with real number multipliers. It is of interest to note here that throughout the whole paper, the Hausdorff separation axiom can be replaced by the weaker Fréchet separation axiom.

[3] Michal, A. D., "Differential Calculus in Linear Topological Spaces," these PROCEEDINGS, **24**, 340–342 (1938). The abbreviation *LTS* will be used to refer to this paper. See also Michal, A. D., "General Differential Geometries and Related Topics," *Bull. Amer. Math. Soc.*, **45**, 529–563 (1939).

[4] Condition (b) in the definition of a topological abelian group is used in this paper only to prove the uniqueness of the differentials for all values of the increment. Moreover the uniqueness Theorem I continues to hold even if T_2 does not satisfy condition (b). This makes possible a treatment of differentials of set-valued functions of a t.a.g. variable. We also plan to study differentials of functions of point set variables.

[5] If T_1 and T_2 are Banach spaces, then Fréchet differentiability is invariant only under a passage to equivalent Banach topologies of T_1 and T_2 whereas our differentiability is invariant under a passage to equivalent Hausdorff topologies of T_1 and T_2.

[6] We treat briefly polynomials and their polars with t.a.g. variables and then apply them to the proof of Theorem VII and its extensions.

TRIANGULATED MANIFOLDS WHICH ARE NOT BROUWER MANIFOLDS

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The following results, which will be elaborated elsewhere, grew from a study of Brouwer's definition[1] of a manifold, in its connection with questions of differentiability, analyticity and polyhedral imbedding. We first enumerate several definitions:

(1) *Topological m-manifold:* A connected topological space which can be covered by a denumerable set of neighborhoods, each of which is an m-cell.

(2) *Triangulable manifold:* A topological manifold which can be subdivided into the cells of a complex.

(3) *Star m-manifold:* A triangulated m-manifold on which the region covered by the star of any vertex is an m-cell.