

Bulk Locality and Boundary Creating Operators

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We formulate a minimum requirement for CFT operators to be localized in the dual AdS. In any spacetime dimensions, we show that a general solution to the requirement is a linear superposition of operators creating spherical boundaries in CFT, with the dilatation by the imaginary unit from their centers. This generalizes the recent proposal by Miyaji *et al.* for bulk local operators in the three dimensional AdS. We show that Ishibashi states for the global conformal symmetry in any dimensions and with the imaginary dilatation obey free field equations in AdS and that incorporating bulk interactions require their superpositions. We also comment on the recent proposals by Kabat *et al.*, and by H. Verlinde.

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Recently, Miyaji *et al.* [1] proposed a construction of bulk local states and corresponding operators in the three-dimensional AdS using Ishibashi states [2], which create spherical boundaries in the dual CFT in two dimensions. To be precise, the states they proposed preserve one half of the global conformal symmetry and not of the full Virasoro symmetry. A similar but different construction was also proposed by H. Verlinde [3]. In this paper, we will formulate a minimum requirement on bulk local operators in AdS and show that its solution generalizes the construction by Miyaji *et al.*, both in spacetime dimensions and in $1/N$ corrections. We will also comment on its relation to the earlier construction using a bulk-boundary kernel in AdS [4–8].

Consider CFT in d dimensions and its dual AdS gravity in $(d+1)$ dimensions. A bulk local operator ψ is an *operator* in the Hilbert space of CFT and is a *local* probe of the *bulk* AdS geometry. As a local probe, it must depend on the position (t, ρ, \vec{x}) in AdS with the metric,

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\vec{x}^2, \quad (1)$$

where we set the AdS radius to be 1 and use \vec{x} subject to $\vec{x}^2 = 1$ in d dimensions to parametrize the sphere S^{d-1} .

The only requirement we impose on $\hat{\psi}$ is that the actions of the conformal symmetry and the bulk isometry on $\hat{\psi}$ are compatible. Namely,

$$[J, \hat{\psi}(t, \rho, \vec{x})] = i\mathcal{L}_{\mathcal{J}}\hat{\psi}(t, \rho, \vec{x}), \quad (2)$$

where J on the left-hand side is a generator of the conformal symmetry of CFT and $\mathcal{L}_{\mathcal{J}}$ on the right-hand side is the Lie derivative in the AdS coordinates (t, ρ, \vec{x}) with respect to the Killing vector field \mathcal{J} corresponding to J . We will show that a general solution to (2) is a linear superposition of boundary creating operators in CFT with the dilatation by the imaginary unit. When $\hat{\psi}$ is a scalar field in AdS, the Lie derivative on the right-hand side of (2) acts as,

$$\mathcal{L}_{\mathcal{J}}\hat{\psi} = \mathcal{J}^\mu \partial_\mu \hat{\psi}. \quad (3)$$

In general, $\hat{\psi}$ may carry a spin, *i.e.*, belong to a finite dimensional representation of the isotropy group $SO(1, d) \subset SO(2, d)$ preserving a point in AdS.

The $SO(2, d)$ conformal generators on the cylinder $\mathbb{R} \times S^{d-1}$ can be organized as $J = (H, M_{ab}, P_a, K_a)$, where $a, b = 1, \dots, d$, H is the global Hamiltonian along \mathbb{R} , $M_{ab} = -M_{ba}$ generate rotations on S^{d-1} , and P_a and K_a are the translation and the special conformal generators when $\mathbb{R} \times S^{d-1}$ is mapped onto \mathbb{R}^d . Their commutation relations relevant in the following discussion are

$$\begin{aligned} [K_a, P_b] &= 2\delta_{ab}H - 2iM_{ab}, \\ [H, P_a] &= P_a, \quad [H, K_a] = -K_a. \end{aligned} \quad (4)$$

The Belavin-Polyakov-Zamolodchikov conjugation rule is $K_a^\dagger = P_a$. $H^\dagger = H$, $M_{ab}^\dagger = M_{ab}$.

Let us examine implications of the requirement (2) when $\hat{\psi}$ is located at the origin $\rho = 0$ on the $t = 0$ slice. The isotropy group $SO(1, d)$ of this point is generated by M_{ab} and $P_a + K_a$. Therefore, if $\hat{\psi}$ is a scalar field, (2) combined with (3) gives,

$$\begin{aligned} [M_{ab}, \hat{\psi}(0)] &= 0, \\ [P_a + K_a, \hat{\psi}(0)] &= 0. \end{aligned} \quad (5)$$

More generally, if $\hat{\psi}$ is in a finite dimensional representation of the isotropy group $SO(1, d)$,

$$\begin{aligned} [M_{ab}, \hat{\psi}(0)] &= s_{ab}\hat{\psi}(0), \\ [P_a + K_a, \hat{\psi}(0)] &= s_{0a}\hat{\psi}(0), \end{aligned} \quad (6)$$

for some matrices $s_{ab} = -s_{ba}$ and s_{0a} characterizing the spin of $\hat{\psi}$.

Acting $\hat{\psi}(0)$ on the conformally invariant vacuum $|0\rangle$, we obtain the state $|\psi(0)\rangle \equiv \hat{\psi}(0)|0\rangle$, which satisfies

$$\begin{aligned} M_{ab}|\psi(0)\rangle &= s_{ab}|\psi(0)\rangle, \\ (P_a + K_a)|\psi(0)\rangle &= s_{0a}|\psi(0)\rangle. \end{aligned} \quad (7)$$

The Hilbert space of CFT is decomposed into a sum of irreducible highest weight representations of the conformal algebra. These equations have a unique solution within each of the representations.

For example, when $\hat{\psi}$ is a scalar field, the corresponding state $|\psi(0)\rangle$ satisfies,

$$\begin{aligned} M_{ab}|\psi(0)\rangle &= 0, \\ (P_a + K_a)|\psi(0)\rangle &= 0. \end{aligned} \quad (8)$$

We can solve these equations by starting with any conformal primary state $|\phi\rangle$ satisfying,

$$H|\phi\rangle = \Delta_\phi|\phi\rangle, \quad K_a|\phi\rangle = 0, \quad M_{ab}|\phi\rangle = 0, \quad (9)$$

and by adding its descendants as,

$$|\phi\rangle = \sum_{n=0}^{\infty} (-1)^n C_n (P^2)^n |\phi\rangle, \quad (10)$$

where $P^2 = \sum_{a=1}^d P_a P_a$ and

$$C_n = \prod_{k=1}^n \frac{1}{4k\Delta_\phi + 4k^2 - 2kd}, \quad (11)$$

up to an overall normalization independent of n . It turns out that the sum over n can be expressed in term of the Bessel function $J_\nu(x)$ of the first kind as,

$$|\phi\rangle = \Gamma\left(\Delta_\phi - \frac{d}{2} + 1\right) \left(\frac{\sqrt{P^2}}{2}\right)^{\frac{d}{2} - \Delta_\phi} J_{\Delta_\phi - \frac{d}{2}}(\sqrt{P^2})|\phi\rangle, \quad (12)$$

We note that it is related to the Fourier transform of a bulk-boundary Green function in AdS. A general solution to (7) is then a linear combination of $|\phi\rangle$ over primary states ϕ ,

$$|\psi(0)\rangle = \sum_{\phi} \psi_\phi |\phi\rangle. \quad (13)$$

These states are naturally related to boundary conditions in CFT. To see this, perform time evolution on $|\phi\rangle$ to define a new state, $|\phi_{\text{Ishibashi}}\rangle = e^{i\frac{\pi}{2}H}|\phi\rangle$. It satisfies,

$$\begin{aligned} M_{ab}|\phi_{\text{Ishibashi}}\rangle &= 0, \\ (P_a - K_a)|\phi_{\text{Ishibashi}}\rangle &= 0, \end{aligned} \quad (14)$$

and is expressed as,

$$|\phi_{\text{Ishibashi}}\rangle = e^{i\frac{\pi}{2}\Delta_\phi} \sum_{n=0}^{\infty} C_n (P^2)^n |\phi\rangle, \quad (15)$$

with the coefficients C_n given by (11). We recognize that (14) are exactly the conditions on conformal boundary states located at the equator of the Euclidean S^d related to the the $t = 0$ slice of the Lorentzian $\mathbb{R} \times S^{d-1}$ by

the Wick rotation $t = -i\tau$. They preserve $SO(1, d) \subset SO(1, d+1)$ of the Euclidean conformal group.

Our boundary states $|\phi_{\text{Ishibashi}}\rangle$, defined in any dimensions, generalize Ishibashi states for two dimensional CFT [2]. For this reason, we call $|\phi\rangle = e^{-i\frac{\pi}{2}H}|\phi_{\text{Ishibashi}}\rangle$ as a ‘‘twisted Ishibashi state.’’ The global Hamiltonian H acting on a boundary state is the dilatation operator from the center of the spherical boundary generated by the state. Therefore, we may interpret $e^{-i\frac{\pi}{2}H}$ as the dilatation by the imaginary unit, $e^{i\pi/2} = i$.

When $d = 2$, the state $|\phi\rangle = e^{-i\frac{\pi}{2}H}|\phi_{\text{Ishibashi}}\rangle$ reduces to the one proposed by [1] as a bulk state localized at the origin $\rho = 0$ on the $t = 0$ slice. Note, however, that a general solution to (8) is a superposition of twisted Ishibashi states (13). As we shall see below, bulk interactions and the microscopic causality require a non-trivial superposition since each twisted Ishibashi state obeys a free field equation in AdS.

More generally, when the bulk local operator $\hat{\psi}$ carries a non-trivial spin, $|\psi(t = -\pi/2)\rangle = e^{i\frac{\pi}{2}H}|\psi(0)\rangle$ satisfies,

$$\begin{aligned} M_{ab}|\psi(t = -\pi/2)\rangle &= s_{ab}|\psi(t = -\pi/2)\rangle, \\ (P_a - K_a)|\psi(t = -\pi/2)\rangle &= i s_{0a}|\psi(t = -\pi/2)\rangle. \end{aligned} \quad (16)$$

A solution to these equations can also be found by starting with a primary state $|\phi\rangle$ in the same representation $\{s_{ab}, s_{0a}\}$ of $SO(1, d)$ and by adding conformal descendants, whose coefficients are fixed by (16) iteratively, except when there are null vectors in the representation, in which case the solution is not unique (this correspond to gauge degrees of freedom in the bulk). Existence of these states suggests that conformal boundary conditions (14) can be generalized so that boundary states are not invariant under the $SO(1, d)$ subgroup of the conformal symmetry but are in its finite dimensional representation. It would be interesting to explore their interpretation from the point of view of CFT.

The relation between the bulk local state $|\psi(0)\rangle$ at the origin of AdS and a boundary state of CFT can be explained intuitively as follows. The conformal generators M_{ab} and $P_a - K_a$, which annihilate boundary states, generate isometry on the $t = 0$ slice of AdS. Therefore, if a boundary state is dual to a gas of free massive particles in AdS, these particles must be uniformly distributed at $t = 0$. Now, geodesics in AdS are 2π periodic in the global time t . If a massive particle has zero orbital angular momentum, it comes to the origin at $\rho = 0$ at every half period π . Therefore, the uniformly distributed gas of free massive particles at $t = 0$ will converge at the origin $\rho = 0$ on the $t = \pi/2$ slice, showing how time evolution of the boundary state by a quarter of the period gives a localized state.

Once a state $|\psi(0)\rangle$ localized at the origin of AdS is constructed, it can be moved to an arbitrary point by the AdS isometry. More explicitly, we can map the origin $\rho = 0$ to any point (ρ, \vec{x}) on the $t = 0$ slice using the

generators $P_a - K_a$, and we can then use H to move to a different time slice as,

$$|\psi(t, \rho, \vec{x})\rangle = e^{-iHt} e^{\rho(P_a - K_a)x^a} |\psi(0)\rangle. \quad (17)$$

Since the AdS coordinates (t, ρ, \vec{x}) are coupled to the generators H and $P_a - K_a$ of the coset $SO(2, d)/SO(1, d)$ by the exponential map in (17), their infinitesimal variations automatically give,

$$J|\psi(t, \rho, \vec{x})\rangle = i\mathcal{L}_J|\psi(t, \rho, \vec{x})\rangle, \quad (18)$$

for each generator $J = (H, M_{ab}, P_a, K_a)$ of $SO(2, d)$. Therefore, the corresponding operator $\hat{\psi}(r, \rho, \vec{x})$ satisfies our requirement (2) for bulk local operators.

So far, we have studied solutions to (8) and found that they are linear superpositions of twisted Ishibashi states as in (13). We would like to discuss how the superposition coefficients ψ_ϕ are determined.

For each primary state $|\phi\rangle$,

$$|\phi(t, \rho, \vec{x})\rangle = e^{-iHt} e^{\rho(P_a - K_a)x^a} |\phi\rangle, \quad (19)$$

obeys a free field equation in AdS. To see this, note that $|\phi(t, \rho, \vec{x})\rangle$ is constructed in a single irreducible representation and therefore is an eigenstate of the quadratic Casimir operator of $SO(2, d)$ with the eigenvalue $m^2 = \Delta_\phi(\Delta_\phi - d)$. The compatibility condition (18) of the conformal symmetry of CFT and the isometry of AdS then turns the operator into the Laplace-Beltrami operator on $|\phi(t, \rho, \vec{x})\rangle$, and the free field equation with mass m follows.

For example, we can compute an overlap of a scalar primary state $|\phi\rangle$ and the corresponding twisted Ishibashi state $|\phi(t, \rho, \vec{x})\rangle$ as,

$$\begin{aligned} \langle\phi|\phi(t, \rho, \vec{x})\rangle &= \langle\phi|e^{-iHt} e^{\rho(P_a - K_a)x^a} |\phi\rangle \\ &= \frac{e^{-i\Delta_\phi t}}{\cosh \rho^{\Delta_\phi}}, \end{aligned} \quad (20)$$

which indeed satisfies the Klein-Gordon equation in AdS.

Thanks to the state-operator correspondence of CFT, we can construct an operator $\hat{\phi}(t, \rho, \vec{x})$ for the twisted Ishibashi state $|\phi(t, \rho, \vec{x})\rangle$. Could it be a bulk local operator? We can answer this question by computing the two-point function,

$$\langle 0|\hat{\phi}(t, \rho, \vec{x})\hat{\phi}(t', \rho', \vec{x}')|0\rangle = \langle\langle\phi(t, \rho, \vec{x})|\phi(t', \rho', \vec{x}')\rangle\rangle. \quad (21)$$

Since $|\phi(t, \rho, \vec{x})\rangle$ belongs to a highest weight representation, it is a sum of positive energy states. Together with the free field equation on $|\phi(t, \rho, \vec{x})\rangle$ and the boundary condition on the two-point function at $t = t'$, the two-point function is uniquely determined to be the Wightman function of a free field in AdS (see also appendix B of [1]). If $\hat{\phi}(t, \rho, \vec{x})$ is a truly local operator in the

bulk, the Reeh-Schlieder theorem would imply that its higher point correlation functions are trivial [9]. Namely, $\hat{\phi}(t, \rho, \vec{x})$ would be a free field in the bulk.

At this point, it is instructive to compare our requirement on bulk local operators with the proposal by Kabat, Lifshytz, and Lowe (KLL) [4–6] (see also [10–12] for earlier papers). Their construction at the leading order in $1/N$ expansion is

$$\hat{\phi}_0^{\text{KLL}}(t, \rho, \vec{x}) = \int dt' dx' K(t, \rho, \vec{x}|t', \vec{x}') \phi(t', \hat{x}'), \quad (22)$$

where $\phi(t', \hat{x}')$ is the primary field in CFT corresponding to $|\phi\rangle$. The bulk-boundary kernel $K(t, \rho, \vec{x}|t', \vec{x}')$, which is called as a smearing function in [4–6], can be extracted from the boundary behavior $\rho' \rightarrow 0$ of a bulk Green's function $G(t, \rho, \vec{x}|t', \rho', \vec{x}')$ as,

$$\begin{aligned} G(t, \rho, \vec{x}|t', \rho', \vec{x}') \\ \sim \frac{\rho'^{\Delta_\phi} L(t, \rho, \vec{x}|t', \vec{x}') + \rho'^{d-\Delta_\phi} K(t, \rho, \vec{x}|t', \vec{x}')}{2\Delta_\phi - d}. \end{aligned} \quad (23)$$

We can choose the Green's function G so that $K(t, \rho, \vec{x}|t', \vec{x}')$ is non-zero only when (t, ρ, \vec{x}) and (t', \vec{x}') are space-like separated [4, 7].

We claim that the operator $\hat{\phi}(t, \rho, \vec{x})$ corresponding to the twisted Ishibashi state $|\phi(t, \rho, \vec{x})\rangle$ is identical to $\hat{\phi}_0^{\text{KLL}}(t, \rho, \vec{x})$ given by (22). This follows from the facts that both satisfy the compatibility condition (2) and that both generate states in the same irreducible representation with the highest weight state $|\phi\rangle$. Since these conditions uniquely determine the twisted Ishibashi state $|\phi(t, \rho, \vec{x})\rangle$, the two states must be identical, and so are the corresponding operators,

$$\hat{\phi}(t, \rho, \vec{x}) = \hat{\phi}_0^{\text{KLL}}(t, \rho, \vec{x}), \quad (24)$$

by the state-operator correspondence of CFT.

Due to the periodicity of the Green's function in t , the KLL state $\hat{\phi}_0^{\text{KLL}}(t, \rho, \vec{x})|0\rangle$ at t and $t + 2\pi$ are identical modulo a phase factor $\exp(-2\pi i\Delta_\phi)$. The twisted Ishibashi state $|\phi(t, \rho, \vec{x})\rangle$ has the same periodicity since it consists of eigenstates of the global Hamiltonian H with eigenvalues equal to Δ_ϕ plus integers.

When the bulk gravity theory is interacting, the leading order KLL operator $\hat{\phi}_0^{\text{KLL}}$ does not satisfy the microscopic causality. One can incorporate effects due to interactions by modifying the bulk-boundary map (22) perturbatively so that the microscopic causality is satisfied [5–7]. Recently, it was shown in [8] that effects in the next leading order in $1/N$ can be expressed as a sum of (22) for different primary fields ϕ , whose coefficients are determined by the operator product expansion. This procedure can be repeated order by order in perturbation. Our result guarantees that these perturbative corrections take the form (22) to all order in perturbation since we have shown that a bulk local operator is a superposition

of twisted Ishibashi operators provided the compatibility condition (2) is satisfied. The superposition coefficients ψ_ϕ in (13) depend on details of the theory, such as the operator product expansion in CFT or the bulk interactions in AdS. On the other hand, we may encounter inconsistency between the microscopic causality and our compatibility requirement (2) at a non-perturbative level since we do not expect resolution better than the Planck length in the bulk.

The proposal by H. Verlinde [3] for local operators in AdS₃ uses Ishibashi states for the full Virasoro symmetry as opposed to the global conformal symmetry and without the imaginary dilatation $e^{-i\frac{\pi}{2}H}$. In this paper, we have shown that the imaginary dilatation is required by the compatibility condition (2), which we think is an essential feature for any bulk local operator in AdS. Each Ishibashi state for the full Virasoro symmetry can be decomposed into a sum of those for the global conformal symmetry. Therefore, modulo the imaginary dilatation, the proposal of [3] can be regarded as a particular choice of the coefficient ψ_ϕ in (13). With this choice, only global conformal primaries within a single Virasoro representation appear in the superposition. However, we expect that $1/N$ corrections would generate global conformal primaries in other Virasoro representations also.

There are several avenues for future investigations. The use of boundary states in constructing bulk local operators may be related to the computation of the bulk energy density by the Radon transform of the entanglement entropy in [13] and to the integral geometry discussed in [14]. We would also like to find out to what extent our proposal depends on the AdS background. We note that, for $d = 2$ and in the leading order in $1/N$, it has been pointed out in [15] that the same bulk operator $\hat{\phi}_0^{\text{KLL}}(t, \rho, \vec{x})$ can be used to probe the BTZ black hole as well as the pure AdS vacuum. It would also be interesting to explore consistency of the microscopic causality and our compatibility requirement (2) to find out whether bulk local operators can be defined non-perturbatively in the bulk.

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- [1] M. Miyaji, T. Numasawa, N. Shiba, T. Takayanagi and K. Watanabe, arXiv:1506.01353 [hep-th].
 - [2] N. Ishibashi, *Mod. Phys. Lett. A* **4**, 251 (1989).
 - [3] H. Verlinde, arXiv:1505.05069 [hep-th].
 - [4] A. Hamilton, D. N. Kabat, G. Lifschytz and D. A. Lowe, *Phys. Rev. D* **74**, 066009 (2006) [hep-th/0606141].
 - [5] D. Kabat, G. Lifschytz and D. A. Lowe, *Phys. Rev. D* **83**, 106009 (2011) [arXiv:1102.2910 [hep-th]].
 - [6] D. Kabat and G. Lifschytz, *Phys. Rev. D* **89**, no. 6, 066010 (2014) [arXiv:1311.3020 [hep-th]].
 - [7] I. Heemskerck, D. Marolf, J. Polchinski and J. Sully, *JHEP* **1210**, 165 (2012) [arXiv:1201.3664 [hep-th]].
 - [8] D. Kabat and G. Lifschytz, arXiv:1505.03755 [hep-th].
 - [9] I. A. Morrison, *JHEP* **1405**, 053 (2014) [arXiv:1403.3426 [hep-th]].
 - [10] T. Banks, M. R. Douglas, G. T. Horowitz and E. J. Martinec, hep-th/9808016.
 - [11] V. Balasubramanian, P. Kraus and A. E. Lawrence, *Phys. Rev. D* **59**, 046003 (1999) [hep-th/9805171].
 - [12] I. Bena, *Phys. Rev. D* **62**, 066007 (2000) [hep-th/9905186].
 - [13] J. Lin, M. Marcolli, H. Ooguri and B. Stoica, *Phys. Rev. Lett.* **114**, no. 22, 221601 (2015) [arXiv:1412.1879 [hep-th]].
 - [14] B. Czech, L. Lamprou, S. McCandlish and J. Sully, arXiv:1505.05515 [hep-th].
 - [15] A. Hamilton, D. N. Kabat, G. Lifschytz and D. A. Lowe, *Phys. Rev. D* **75**, 106001 (2007) [*Phys. Rev. D* **75**, 129902 (2007)] [hep-th/0612053].