# Coulomb branch of $N=1$ supersymmetric $SU(N_c)\times SU(N_c)$ gauge theories

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We analyze the low energy behavior of $N=1$ supersymmetric gauge theories with an $SU(N_c)\times SU(N_c)$ gauge group and a Landau-Ginzburg type superpotential. These theories contain fundamentals transforming under one of the gauge groups as well as bifundamental matter which transforms as a fundamental under each. We obtain the parametrization of the gauge coupling on the Coulomb branch in terms of a hyperelliptic curve. The derivation of this curve involves making use of Seiberg's duality for supersymmetric QCD as well as the classical constraints for $N_f=N_c+1$ and the quantum modified constraints for $N_f=N_c$.
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## I. INTRODUCTION

Our understanding of the low energy behavior of supersymmetric gauge theories has increased substantially over the last few years. Seiberg and Witten found a complete solution for $N=2$, $N_c=2$ supersymmetric QCD (SQCD) on the Coulomb branch [1]. These results were later generalized to other gauge groups and to include fundamental matter [2]. The Coulomb branch is the segment of the moduli space of vacua where the microscopic gauge group of rank $r$ is broken to $U(1)^r$. On the Coulomb branch the massless particles in the low energy theory are the photons corresponding to the unbroken $U(1)$'s. These photons are described by the Lagrangian

$$\mathcal{L}=\frac{1}{4\pi}\,\mathrm{Im}\int d^2\theta\,\tau_{ij}W^iW^j, \tag{1.1}$$

where $\tau_{ij}$ is the matrix of the $U(1)$ gauge couplings. For $N=2$ theories this description has to be supplemented by a kinetic energy term for the adjoint scalars and at some points on the moduli space by terms describing particles that go massless there. The effective Lagrangian can be expressed as an integral of a holomorphic prepotential over half of the $N=2$ superspace. This provides a relation between the prepotential, the gauge couplings and the metric on the moduli space. Thus, determining the gauge couplings as a function of the moduli amounts to solving the low energy theory in that case.

In the $N=1$ case the low energy Lagrangian cannot be written in terms of a prepotential. Therefore, it is no longer sufficient to determine the gauge couplings as a function of the moduli in order to obtain a complete solution of the low energy theory. There is no simple relation between the $U(1)$ gauge couplings and the kinetic energy terms of the matter fields in the $N=1$ Lagrangian. Nevertheless, it was shown in [3] that in some cases the $U(1)$ gauge couplings can be determined using the same methods as in the $N=2$ case. A number of examples of such $N=1$ theories have been found [3–8]. Only one of these examples [3], which was generalized in [8], involves product gauge groups. We provide a second such example here.

In both $N=1$ and $N=2$ theories, the matrix of $U(1)$ gauge couplings $\tau_{ij}$ can be identified with the normalized period matrix of a Riemann surface. If the theories contain only fundamental matter, this surface is usually hyperelliptic and of genus $r$ where $r$ is the rank of the gauge group. In these cases it can generally be determined uniquely using symmetry and field theory arguments. In more complicated cases, solutions were obtained using $D$-brane configurations [9].

In this paper we analyze $N=1$ supersymmetric gauge theories with $SU(N_c)\times SU(N_c)$ gauge group and fundamental matter as well as bifundamental matter which transforms as a fundamental under both gauge groups. These theories have a Coulomb branch with an unbroken $U(1)^{N_c-1}$ gauge group if a Landau-Ginzburg-type superpotential is added. The values of the $U(1)$ gauge couplings can be parametrized in terms of a hyperelliptic curve.

The theories we analyze here have some new features. In order to derive the curve it is necessary to consider limits where one or the other $SU(N_c)$ is strongly coupled. Taking some of these limits involves passing to a dual description of the strongly coupled group. The duals involved are very similar to those found by Seiberg [10]. For certain numbers of fundamental flavors the classical constraints that arise in SQCD for $N_f=N_c+1$ play a role. In the other examples of theories with product gauge groups only the quantum modified constraints which arise for $N_f=N_c$ appeared. This is due to the fact that those theories did not contain matter transforming as a fundamental under only one of the gauge groups.

The paper is organized as follows: In Sec. II we describe the matter content and superpotential of the $SU(N_c)\times SU(N_c)$ theories. We also derive the dual description and constraint equations that arise if one switches off one or the other gauge group. These results will be needed in Sec. III to derive the curve for the $SU(N_c)\times SU(N_c)$ theories. Concluding remarks can be found in Sec. IV.

## II. PRELIMINARIES

In this section we analyze various features of the theory with gauge group $SU(N_c)_1\times SU(N_c)_2$ and the matter content given in Table I. The $N_f$ fields $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ transform as funda-

 

TABLE I. The matter content of the $\mathrm{SU}(N_c)_1\times\mathrm{SU}(N_c)_2$ gauge theory.

| | $\mathrm{SU}(N_c)_1$ | $\mathrm{SU}(N_c)_2$ | $\mathrm{SU}(N_f)_L$ | $\mathrm{SU}(N_f)_R$ | $\mathrm{U}(1)_B$ | $\mathrm{U}(1)_C$ | $\mathrm{U}(1)_R$ |
|---|---|---|---|---|---|---|---|
| $\mathcal{Q}$ | $\square$ | 1 | $\square$ | 1 | 1 | 0 | 1 |
| $\tilde{\mathcal{Q}}$ | $\overline{\square}$ | 1 | 1 | $\square$ | $-1$ | 0 | 1 |
| $\mathcal{R}$ | $\square$ | $\overline{\square}$ | 1 | 1 | 0 | 1 | 0 |
| $\tilde{\mathcal{R}}$ | $\overline{\square}$ | $\square$ | 1 | 1 | 0 | $-1$ | 0 |

mentals under $\mathrm{SU}(N_c)_1$ and are singlets under $\mathrm{SU}(N_c)_2$. The fields $\mathcal{R}$ and $\tilde{\mathcal{R}}$ transform as fundamentals in one and as antifundamentals in the other gauge group. The table also shows the nonanomalous assignment of the charges under the global symmetry

$$\mathrm{SU}(N_f)_L\times\mathrm{SU}(N_f)_R\times\mathrm{U}(1)_B\times\mathrm{U}(1)_C\times\mathrm{U}(1)_R,$$

where $\mathrm{U}(1)_R$ is an $R$ symmetry. If we add a tree level superpotential of the form

$$W=\sum_{k=0}^{l} h_{ij}^{(k)}\tilde{\mathcal{Q}}^i(\mathcal{R}\tilde{\mathcal{R}})^k\mathcal{Q}^j, \tag{2.1}$$

where $i,j=1,\ldots,N_f$, this theory has a Coulomb branch. For $\mathcal{Q}=\tilde{\mathcal{Q}}=0$ one can verify that the solution of the $D$-flatness conditions for the $\mathcal{R}$ and $\tilde{\mathcal{R}}$ fields has $N_c+1$ free parameters and that the vacuum expectation value (vevs) of $\mathcal{R}$ and $\tilde{\mathcal{R}}$ can be brought into diagonal form. Therefore, the low energy theory has an unbroken $\mathrm{U}(1)^{N_c-1}$ gauge group [8]. There is also a Higgs branch with nonzero vevs for the quarks on which the gauge group is broken completely. We will limit our discussion to the Coulomb branch in this paper.

The Coulomb branch cannot be lifted by a dynamically generated superpotential, since the nonanomalous $R$-charge assignment $R_{\mathcal{R}}=R_{\tilde{\mathcal{R}}}=0$ and $R_{\mathcal{Q}}=R_{\tilde{\mathcal{Q}}}=1$ requires any such superpotential to be quadratic in the quarks. The $F$-flatness condition arising from any superpotential will automatically be satisfied on the Coulomb branch [5] (but not necessarily on the Higgs branch).

The superpotential, Eq. (2.1), includes terms that make the theory nonrenormalizable. It should be viewed as an effective field theory which is defined below some scale $\Lambda$. We assume that all scales appearing in the effective theory are much smaller than $\Lambda$. The dimensionful coefficients $h^{(k)}$ in the superpotential scale as $1/\Lambda^{2k-1}$.

The low energy theory simplifies considerably if we take one of the two gauge groups to be much more strongly coupled than the other, i.e., $\Lambda_1\ll\Lambda_2$ or $\Lambda_2\ll\Lambda_1$. These limits are analyzed most easily if we switch off the weakly coupled group, discuss the resulting single gauge group theory without superpotential, and then promote the $\mathrm{SU}(N_c)$ that was switched off to a gauge symmetry again. This is the procedure followed in, e.g., [11] to find dual descriptions for theories with product gauge groups. Once a description of the theory in these limits is found, we can perturb it by adding the superpotential Eq. (2.1). Other perturbations of the $\mathrm{SU}(N_c)\times\mathrm{SU}(N_c)$ theory we described above were studied in [12].

TABLE II. The composites of the confining $\mathrm{SU}(N_c)_1$ with $N_c+1$ flavors.

| | $\mathrm{SU}(N_c)_1$ | $\mathrm{SU}(N_c)_2$ | $\mathrm{U}(1)_B$ | $\mathrm{U}(1)_C$ | $\mathrm{U}(1)_R$ |
|---|---|---|---|---|---|
| $M=\mathcal{Q}\tilde{\mathcal{Q}}$ | 1 | 1 | 0 | 0 | 2 |
| $P=\tilde{\mathcal{R}}\mathcal{Q}$ | 1 | $\square$ | 1 | $-1$ | 1 |
| $\tilde{P}=\tilde{\mathcal{Q}}\mathcal{R}$ | 1 | $\overline{\square}$ | $-1$ | 1 | 1 |
| $\Psi=\tilde{\mathcal{R}}\mathcal{R}$ | 1 | adj+1 | 0 | 0 | 0 |
| $B_0=\mathcal{R}^{N_c}$ | 1 | 1 | 0 | $N_c$ | 0 |
| $\tilde{B}_0=\tilde{\mathcal{R}}^{N_c}$ | 1 | 1 | 0 | $-N_c$ | 0 |
| $B_1=\mathcal{Q}\mathcal{R}^{N_c-1}$ | 1 | $\square$ | 1 | $N_c-1$ | 1 |
| $\tilde{B}_1=\tilde{\mathcal{Q}}\tilde{\mathcal{R}}^{N_c-1}$ | 1 | $\overline{\square}$ | $-1$ | $-N_c+1$ | 1 |

If $\mathrm{SU}(N_c)_1$ is switched off, the fields $\mathcal{R}$ and $\tilde{\mathcal{R}}$ look like $N_c$ flavors of fundamentals from the point of view of $\mathrm{SU}(N_c)_2$. The $\mathrm{SU}(N_c)_2$ gauge theory with no superpotential is in the confining phase, i.e., the low energy description should be in terms of the composite meson and baryon fields $\Psi$ and $B,\tilde{B}$ made from $\mathcal{R}$ and $\tilde{\mathcal{R}}$. These fields have to satisfy the quantum modified constraint [10]

$$\det\Psi-B\tilde{B}=\Lambda_2^{2N_c}. \tag{2.2}$$

Here $\Lambda_2$ is the strong coupling scale of $\mathrm{SU}(N_c)_2$. Note that the meson $\Psi=\mathcal{R}\tilde{\mathcal{R}}$ transforms as an adjoint plus a scalar under $\mathrm{SU}(N_c)_1$. In this limit, the superpotential, Eq. (2.1), takes the form

$$W=\sum_{k=0}^{l} h_{ij}^{(k)}\tilde{\mathcal{Q}}^i\Psi^k\mathcal{Q}^j. \tag{2.3}$$

The Coulomb branch of this theory was discussed in [5]. At scales much below $\Lambda_2$ no trace of the fact that $\Psi$ is a composite survives. Therefore, one can follow the arguments of [5] to determine that the superpotential is a relevant perturbation for $lN_f<2N_c$. We will restrict our discussion in this paper to superpotentials that satisfy this constraint.

Switching off $\mathrm{SU}(N_c)_2$, we have a $\mathrm{SU}(N_c)_1$ gauge theory with two types of flavors in the fundamental representation. This theory with no superpotential is very similar to SQCD [10] with a total of $N_c+N_f$ flavors. We need to distinguish two cases: $N_f=1$ corresponding to $N_f=N_c+1$ in [10] and $N_f>1$, in which case one expects the theory to have a dual description [10].

For $N_f=1$ the theory confines without chiral symmetry breaking, which can be verified by computing the anomalies for the elementary particles listed in Table I and for the composites in Table II. As expected from the analysis in [10], there are classical constraints on the composite fields which cannot be modified quantum mechanically. They follow as equations of motion from the superpotential

$$W=\frac{1}{\Lambda_1^{2N_c-1}}(\det\mathcal{M}-B_0M\tilde{B}_0-B_1\Psi\tilde{B}_1 \\ -B_0P\tilde{B}_1-B_1\tilde{P}\tilde{B}_0), \tag{2.4}$$

where

TABLE III. The matter content of the dual theory.

| | $SU(N_f)_1$ | $SU(N_c)_2$ | $SU(N_f)_L$ | $SU(N_f)_R$ | $U(1)_B$ | $U(1)_C$ | $U(1)_R$ |
|---|---|---|---|---|---|---|---|
| $q$ | $\square$ | 1 | $\overline{\square}$ | 1 | 0 | $N_c/N_f$ | 0 |
| $\widetilde{q}$ | $\overline{\square}$ | 1 | 1 | $\overline{\square}$ | 0 | $-N_c/N_f$ | 0 |
| $r$ | $\square$ | $\square$ | 1 | 1 | 1 | $-1+N_c/N_f$ | 1 |
| $\widetilde{r}$ | $\overline{\square}$ | $\overline{\square}$ | 1 | 1 | $-1$ | $1-N_c/N_f$ | 1 |
| $M=\mathcal{Q}\widetilde{\mathcal{Q}}$ | 1 | 1 | $\square$ | $\square$ | 0 | 0 | 2 |
| $P=\widetilde{\mathcal{R}}\mathcal{Q}$ | 1 | $\square$ | $\square$ | 1 | 1 | $-1$ | 1 |
| $\widetilde{P}=\mathcal{R}\widetilde{\mathcal{Q}}$ | 1 | $\overline{\square}$ | 1 | $\square$ | $-1$ | 1 | 1 |
| $\Psi=\widetilde{\mathcal{R}}\mathcal{R}$ | 1 | adj+1 | 1 | 1 | 0 | 0 | 0 |

$$\mathcal{M}=\begin{pmatrix} M & \widetilde{P} \\ P & \Psi \end{pmatrix}. \tag{2.5}$$

For $N_f>1$ the theory has the necessary number of flavors to be in the duality regime. However, the global symmetries of the theory under consideration here differ from those of the theory discussed by Seiberg [10]. The dual gauge group turns out to be $SU(N_f)$ as expected but the charge assignments of the magnetic quarks differs from that in [10]. In Table III we give the gauge group and particle content of the dual theory. One can verify that the anomalies of the global symmetries match. On the dual side we have to add a superpotential of the form

$$W=Mq\widetilde{q}+Pq\widetilde{r}+\widetilde{P}\widetilde{q}r+\Psi r\widetilde{r}, \tag{2.6}$$

to remove the bilinears of dual quarks from the chiral ring. The matching of baryons works as follows:

$$\mathcal{Q}^p\mathcal{R}^{N_c-p}\rightarrow r^p q^{N_f-p}. \tag{2.7}$$

Note that the dual theory has an $SU(N_c)$ global symmetry which can be gauged.

The constraints and the dual given here will be needed in the next section to discuss the limiting behavior of the curve which describes the Coulomb branch of the $SU(N_c)\times SU(N_c)$ theory.

### III. THE CURVE

The theory with gauge group $SU(N_c)_1\times SU(N_c)_2$, the superpotential given in Eq. (2.1), and the matter content in Table I has two limits in which it reduces to theories for which the curve describing the gauge couplings on the Coulomb branch is known. We can use these limits to constrain the curve for the theory we are considering here. If one integrates out all fundamentals, the resulting curve has to reproduce the curve for the $SU(N_c)\times SU(N_c)$ case given in [8]. In the limit $\Lambda_2\gg\Lambda_1$ the fields transforming under the $SU(N_c)_1$ gauge group are $N_f$ fundamentals and an adjoint with the superpotential Eq. (2.3). This is the theory discussed in [5], so in this limit the curve has to agree with the one given there. The analysis of the limit $\Lambda_1\gg\Lambda_2$ is somewhat more involved but it will turn out that this limit also yields a theory of the type studied in [5].

From the solution of the $D$-flatness conditions [8] we know that $N_c-1$ $U(1)$'s remain unbroken on the Coulomb branch. It is convenient to define

$$\Phi=\Psi-\frac{1}{N_c}\,\mathrm{Tr}\,\Psi=\mathcal{R}\widetilde{\mathcal{R}}-\frac{1}{N_c}\,\mathrm{Tr}\,\mathcal{R}\widetilde{\mathcal{R}}, \tag{3.1}$$

which transforms as an adjoint under $SU(N_c)_1$. The diagonal form $\Phi=\mathrm{diag}(\phi_1,\ldots,\phi_{N_c})$ can be used to define the gauge invariant symmetric polynomials

$$\prod_{j=1}^{N_c}(x-\phi_j)=\sum_{j=0}^{N_c}s_j x^{N_c-j}. \tag{3.2}$$

In terms of these variables the curve reads

$$y^2=\left[\sum_{j=0}^{N_c}s_j x^{N_c-j}+(-1)^{N_c}(\Lambda_2^{2N_c}+\Lambda_1^{2N_c-N_f}\det h^{(0)})\right]^2 -4\Lambda_2^{2N_c}\Lambda_1^{2N_c-N_f}\det\sum_{j=0}^{l}h^{(j)}x^j. \tag{3.3}$$

There could be other terms in this curve which are allowed by the symmetries of the theory but they can be excluded on the basis of the limits we discuss below.

If one takes all $h^{(i)}$, $i\neq0$ to vanish and the entries in the mass matrix $h^{(0)}$ to be large, one can integrate out all flavors of quarks. In this case, the curve has to reproduce that given in [8]. It is a simple matter to check that this is in fact the case.

The solution of the $D$-flatness conditions implies that the vev of the fields $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ can be brought into diagonal form [8]. Giving $\mathcal{R}$ a large diagonal vacuum expectation value (vev), i.e., $\mathcal{R}=\mathrm{diag}(v,\ldots,v)$, breaks the product gauge group to its diagonal subgroup $SU(N_c)_D$. Both bifundamentals decompose into an adjoint and a singlet under $SU(N_c)_D$, and the quarks $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ transform as fundamentals. One of the adjoints is eaten by the Higgs mechanism. Rewriting the superpotential in terms of the uneaten adjoint $\Psi_{\widetilde{\mathcal{R}}}=\widetilde{\mathcal{R}}-1/N_c\,\mathrm{Tr}\,\widetilde{\mathcal{R}}$, we find

$$W=\sum_{k=0}^{l}h_D^{(k)}\widetilde{\mathcal{Q}}\Psi_{\widetilde{\mathcal{R}}}^k\mathcal{Q}, \tag{3.4}$$

where $h_D^{(k)}=h^{(k)}v^k$. This theory has the same matter content and superpotential as the theory of [5]. Thus we have to recover the curve given there in this limit. The matching relation for the strong coupling scales determines

$$\Lambda_1^{2N_c-N_f}\Lambda_2^{2N_c}=v^{2N_c}\Lambda_D^{2N_c-N_f}. \tag{3.5}$$

Finally, we need to rewrite the gauge invariant polynomials in terms of the components of $\Psi_{\tilde{\mathcal{R}}}$

$$s_j=v^j s_j^{(D)}. \tag{3.6}$$

Substituting these expressions into the curve Eq. (3.3), rescaling $x\to x/v$, $y\to y/v^{N_c}$ and neglecting subleading terms gives

$$y^2=\left[\sum_{j=0}^{N_c} s_j^{(D)}x^{N_c-j}\right]^2-4\Lambda_D^{2N_c-N_f}\det\sum_{j=0}^{l} h_D^{(j)}x^j, \tag{3.7}$$

which agrees with [5].

In the limit $\Lambda_2\gg\Lambda_1$ the second gauge group confines and we need to rewrite the curve in terms of the composite degrees of freedom. The composites fields are related by the constraint Eq. (2.2). This constraint can be incorporated by shifting the gauge invariant polynomial $s_{N_c}\to s_{N_c}-(-1)^{N_c}\Lambda_2^{2N_c}$ [8]. We also need to rescale the $s_j$ to give them the canonical mass dimensions

$$s_j=\mu^j s_j^{(1)},\quad j=1,\dots,N_c-1,$$
$$s_{N_c}=\mu^{N_c}s_{N_c}^{(1)}-(-1)^{N_c}\Lambda_2^{2N_c}, \tag{3.8}$$

where $\mu$ is some as yet undetermined mass scale. The polynomials on the right hand side have the correct mass dimensions and incorporate the constraint automatically. Rescaling the field $\Psi$ in the superpotential, Eq. (2.1), to give it the right mass dimension requires a simultaneous redefinition of the coefficients $h^{(k)}\mu^k=h_L^{(k)}$. The matching relation for the strong coupling scales reads

$$\Lambda_1^{2N_c-N_f}\Lambda_2^{2N_c}=\mu^{2N_c}\Lambda_L^{2N_c-N_f}. \tag{3.9}$$

Substituting this, the rescaled coefficients, and the rescaled fields, Eq. (3.8), into the curve Eq. (3.3), one finds

$$y^2=\left[\sum_{j=0}^{N_c} s_j^{(1)}x^{N_c-j}\right]^2-4\Lambda_L^{2N_c-N_f}\det\sum_{j=0}^{l} h_L^{(j)}x^j, \tag{3.10}$$

which agrees with the curve given in [5]. In this expression we rescaled $x\to x/\mu$, $y\to y/\mu^{N_c}$, set $\mu=\Lambda_2$ and neglected an irrelevant subleading piece of the form $\Lambda_1^{2N_c-N_f}\det h^{(0)}/\Lambda_2^{2N_c}$. Note that the quantum piece of this curve vanishes whenever one of the quarks becomes massless classically. This can be seen as follows: The superpotential has the structure of a mass term for the quarks. Choosing a basis in which $\Psi=\mathrm{diag}(\phi_1,\dots,\phi_{N_c})$, the superpotential takes the form

$$W=\sum_{k=0}^{l} h_{ij}^{(k)}\sum_{\alpha}\phi_\alpha^k\mathcal{Q}^{i\alpha}\tilde{\mathcal{Q}}_\alpha^j. \tag{3.11}$$

Whenever

$$\det\sum_{k=0}^{l} h_{ij}^{(k)}\phi_\alpha^k=0 \tag{3.12}$$

is satisfied, at least one of the quarks charged under the corresponding U(1) becomes massless. This condition constrains the quantum piece of the curve [5]. After rescaling the composite field and the coefficients $h^{(k)}$ in the superpotential, the quantum piece is given by

$$\Lambda_L^{2N_c-N_f}\det\sum_{k=0}^{l} h_L^{(k)}x^k, \tag{3.13}$$

in agreement with Eq. (3.10). We will use similar considerations to check the curve Eq. (3.3) in other limits.

The limit $\Lambda_1\gg\Lambda_2$ is more complicated because, from the point of view of SU$(N_c)_1$, there are more flavors than colors. In order to determine whether the curve describes this limit correctly, we have to analyze the description of the low energy physics in terms of the composite degrees of freedom. The composites $\Psi$ and $\Phi$ have to be redefined by switching the order of $\mathcal{R}$ and $\tilde{\mathcal{R}}$ in Eq. (3.1), so that they transform as adjoints under SU$(N_c)_2$. However, this does not change the values of the gauge invariant polynomials $s_j$. Thus the classical piece of the curve is unchanged. We can use the techniques of [5] to find the curve corresponding to the description in terms of the composites and compare it to the appropriate limit of the curve Eq. (3.3).

If there is only one flavor of the quarks $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$, the first gauge group sees a total of $N_c+1$ flavors. It is in the confining phase and the composite degrees of freedom have to satisfy the constraints following from Eq. (2.4). In order to discuss the theory in this limit, we need to reexpress the tree level superpotential, Eq. (2.1), in terms of the confined composites and add the superpotential, Eq. (2.4), to incorporate the constraint on these fields. Using the operator maps in Table II, we find

$$W=h^{(0)}M+\sum_{k=1}^{l} h^{(k)}\tilde{P}\Psi^{k-1}P-\frac{1}{\Lambda_1^{2N_c-1}}(\det\mathcal{M}-B_0M\tilde{B}_0$$
$$-B_1\Psi\tilde{B}_1-B_0P\tilde{B}_1-B_1\tilde{P}\tilde{B}_0). \tag{3.14}$$

The matter content of the SU$(N_c)_2$ gauge theory consists of the singlets $M$ and $B_0,\tilde{B}_0$, two flavors of quarks $P,\tilde{P}$ and $B_1,\tilde{B}_1$ and the adjoint $\Psi$. The singlets do not take part in the gauge dynamics but $B_0$ and $\tilde{B}_0$ serve as off-diagonal mass terms for the two flavors of fundamentals. Except for the presence of the singlet $M$, this limit of the theory is similar to the theory considered in [5]. We can repeat the derivation given there to find the curve for this theory. To do so, we need to determine the classical condition for the quarks to become massless. The determinant in the superpotential, Eq. (3.14), can be expanded using the diagonal representation of $\Psi$:

$$\det \mathcal{M}=\left(M-\sum_{\alpha=1}^{N_c}\frac{P_\alpha\widetilde{P}_\alpha}{\phi_\alpha}\right)\prod_{\beta=1}^{N_c}\phi_\beta. \tag{3.15}$$

Substituting this into the superpotential, the equation of motion for $M$ requires

$$\det\Psi-B_0\widetilde{B}_0=h^{(0)}\Lambda_1^{2N_c-1}, \tag{3.16}$$

where we have reexpressed the product of the $\phi_i$ as a determinant. Note that this constraint involves only the composites made from the fields $\mathcal{R}$ and $\widetilde{\mathcal{R}}$. All the terms in the superpotential, Eq. (3.14), that involve fields which transform as fundamentals under $SU(N_c)_2$ have the structure of mass terms. We could analyze those, using the constraint Eq. (3.16), to determine where the composite quarks $P$ and $B_1$ become massless. However, it is much easier to impose the constraint by integrating out $M$ and analyze the resulting superpotential. This yields

$$W=\sum_{k=0}^{l}h^{(k)}\sum_\alpha P_\alpha\widetilde{P}^\alpha\phi_\alpha^{k-1}+\frac{1}{\Lambda_1^{2N_c-1}}\Bigg(B_1\Psi\widetilde{B}_1+B_0P\widetilde{B}_1+B_1\widetilde{P}\widetilde{B}_0+\frac{P_\alpha\widetilde{P}_\alpha}{\phi_\alpha}B_0\widetilde{B}_0\Bigg), \tag{3.17}$$

which has the form of a mass term for the two flavors of quarks. By writing these mass terms as a matrix in flavor space and requiring that its determinant vanishes, we find that at least one quark will become massless if

$$\sum_{k=0}^{l}h^{(k)}\phi_\alpha^k=0. \tag{3.18}$$

This implies that the quantum piece of the curve in this limit should be proportional to

$$\Lambda_L^{2N_c-2}\sum_{j=0}^{l}h_L^{(j)}x^j \tag{3.19}$$

after rescaling the composite fields and the coefficients $h^{(k)}$ in the superpotential.

In order to find the curve in the limit $\Lambda_1\gg\Lambda_2$, we need to rescale the gauge invariant polynomials and shift the highest one according to the constraint Eq. (3.16):

$$s_j=\mu^j s_j^{(2)},\quad j=1,\dots,N_c-1,$$
$$s_{N_c}=\mu^{N_c}s_{N_c}^{(2)}-(-1)^{N_c}h^{(0)}\Lambda_2^{2N_c-1}. \tag{3.20}$$

Rescaling the composites in the superpotential, Eq. (3.14), to give them the canonical mass dimension one requires that we define $h_L^{(k)}=h^{(k)}\mu^{k+1}$. Finally, the matching condition for the strong coupling scales reads

$$\Lambda_1^{2N_c-1}\Lambda_2^{2N_c}=\mu^{2N_c+1}\Lambda_L^{2N_c-2}. \tag{3.21}$$

Substituting this and the rescaled polynomials, Eqs. (3.20), into the curve, Eq. (3.3), gives

$$y^2=\left[\sum_{j=0}^{N_c}s_j^{(2)}x^{N_c-j}\right]^2-4\Lambda_L^{2N_c-2}\sum_{j=0}^{l}h_L^{(j)}x^j, \tag{3.22}$$

where we rescaled $x\to x/\mu$, $y\to y/\mu^{N_c}$, set $\mu=\Lambda_1$ and neglected subleading terms. The quantum piece of this curve agrees with Eq. (3.19), i.e., the curve describes this limit correctly.

Finally, we have to check that the curve, Eq. (3.3), gives the correct description in the limit $\Lambda_1\gg\Lambda_2$ for $N_f>1$. In this case, the $SU(N_c)_1$ gauge theory is in the dual phase if $SU(N_c)_2$ is switched off. In order to describe the low energy physics in this limit, we have to pass to the dual description. The operators in the tree level superpotential are mapped to operators on the magnetic side according to Table III. We also need to add the superpotential, Eq. (2.6), to eliminate gauge invariant combinations of the dual quarks from the chiral ring. This results in the superpotential

$$W=\mu h^{(0)}M+\sum_{k=1}^{l}\mu^{k+1}h^{(k)}P\Psi^{k-1}\widetilde{P}+Mq\widetilde{q}+Pq\widetilde{r}+\widetilde{P}\widetilde{q}r+\Psi r\widetilde{r}, \tag{3.23}$$

where we have inserted a scale $\mu$ in some of the terms to correct the mass dimensions. This is necessary because we take the mesons $M$ and $P$, $\widetilde{P}$ to have mass dimension one. The mass term for the meson $M$ forces the quarks $q$ and $\widetilde{q}$ to acquire a vev, which breaks the dual $SU(N_f)$ gauge group completely for generic values of $h^{(0)}$. There are $2N_f$ flavors of quarks which transform as fundamentals under $SU(N_c)_2$: $N_f$ magnetic bifundamentals and $N_f$ mesons $P$ and $\widetilde{P}$. All terms in the superpotential, Eq. (3.23), except those involving $M$ have the form of mass terms for the $2N_f$ flavors of quarks. Again we must determine where these go massless classically, because this determines the quantum piece of the curve. Using $\Psi=\mathrm{diag}(\phi_1,\dots,\phi_{N_c})$, we can rewrite the mass terms as a matrix in flavor space and require that its determinant vanishes. This yields the condition

$$\det\sum_{k=0}^{l}\mu^{k+1}h_{ij}^{(k)}\phi_\alpha^k=0 \tag{3.24}$$

on the vev of the adjoint after substituting $q\widetilde{q}=-\mu h^{(0)}$. We can now repeat the analysis of [5] to determine that the quantum piece of the curve is proportional to

$$\Lambda_L^{2N_c-2N_f}\det\sum_{k=0}^{l}h_L^{(k)}x^k, \tag{3.25}$$

where we defined $h_L^{(k)}=\mu^{k+1}h^{(k)}$. The quantum modified constraint on the mesons and baryons made from the fields $\mathcal{R}$ and $\widetilde{\mathcal{R}}$,

$$\det\Psi-B_0\widetilde{B}_0=\det h^{(0)}\Lambda_1^{2N_c-N_f}, \tag{3.26}$$

can be obtained from the matching relations for the strong coupling scales as one integrates in a flavor of $\mathcal{Q}$. We can now repeat the same analysis as in the confining case with the obvious modifications of Eqs. (3.20) and the matching conditions for the strong coupling scales Eq. (3.21). Taking $\mu=\Lambda_1$, we find

$$y^2=\left[\sum_{j=0}^{N_c} s_j^{(2)} x^{N_c-j}\right]^2-4\Lambda_L^{2N_c-2N_f}\,\det\sum_{j=0}^{l} h_L^{(j)} x^j, \tag{3.27}$$

which agrees with the curve obtained along the lines of [5] for the case we consider here. This concludes the checks on the curve given in Eq. (3.3).

## IV. CONCLUSION

We have investigated the Coulomb branch of $SU(N_c)\times SU(N_c)$ gauge theories with fundamental and bifundamental matter and a Landau-Ginzburg type superpotential. In order to discuss the behavior of these theories in the limit that one or the other gauge group is strongly coupled, it is necessary to use Seiberg's results on confinement in $SU(N_c)$ theories with $N_f=N_c$ and $N_f=N_c+1$ as well as a dual description for $N_f>N_c+1$. We found the curve that parametrizes the gauge couplings of the unbroken U(1)'s and demonstrated that it reproduces known results in four limits. The product gauge group can be broken to its diagonal subgroup, in which case we have to recover the curve given in [5]. If all flavors are integrated out, we obtain the curve of [8]. For both $N_f=1$ and $N_f>1$, the theory presented here reduces to theories considered in [5] if the limit $\Lambda_1\gg\Lambda_2$ or $\Lambda_2\gg\Lambda_1$ is taken. In all of these cases, we recover the curves given in [5]. While this method of finding the curve is certainly not rigorous, the evidence we have presented here strongly suggests that our curve is the correct description of the U(1) gauge couplings on the Coulomb branch.

The curve can be used to analyze which particles go massless at its singularities. Doing this explicitly for large values of $N_c$ is very cumbersome. For $N_c=2$ such an analysis reproduces the results in [5] except that one has to identify the variable $u$ in that paper with $s_2+\Lambda_2^4+\Lambda_1^3 h^{(0)}$. This shifts the location of the singularities by a finite amount. The curve has singularities corresponding to a monopole or a dyon going massless as well as singularities where the quarks go massless. One can find a number of inequivalent superconformal fixed points by tuning the coefficients $h^{(k)}$ such that some of the singularities corresponding to mutually nonlocal particles collide. Such fixed points exist for $N_c\geqslant 2$, $N_f\geqslant 1$ and $l\geqslant 1$. They are the $N=1$ analog (in the sense of [7]) of the $N=2$ fixed points analyzed in [13].

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