Measuring entanglement entropies in many-body systems

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We explore the relation between entanglement entropy of quantum many-body systems and the distribution of corresponding, properly selected, observables. Such a relation is necessary to actually *measure* the entanglement entropy. We show that, in general, the Shannon entropy of the probability distribution of certain symmetry observables gives a lower bound to the entropy. In some cases this bound is saturated and directly gives the entropy. We also show other cases in which the probability distribution contains enough information to extract the entropy: we show how this is done in several examples including BEC wave functions, the Dicke model, *XY* spin chain, and chains with strong randomness.

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I. INTRODUCTION

Entanglement entropy was first considered as a source of quantum corrections to the entropy of a black hole [1]. Since then its study expanded considerably in extent and purpose. Today its evaluation allows an estimate of the effectiveness of physical systems as quantum computers, since the entanglement entropy characterizes the resources in our disposal to perform computations. As importantly, the entanglement entropy is emerging as a tool in the field of many-body systems. Close to quantum phase transitions and at critical points it was shown to exhibit universal properties [2,3]. In particular, in one-dimensional conformaly invariant systems the entanglement entropy is a measure of the central charge of the underlying conformal field theory [4]. Consider a system partitioned into two parts A and B. The entanglement entropy $\mathcal{S}_E$ in the quantum state $\rho=|\Psi\rangle\langle\Psi|$ evaluates how many qubits in A are determined by (or entangled with) qubits in B. It is defined as the von Neumann entropy $\mathcal{S}_E=-\mathrm{Tr}[\rho_A \log_2(\rho_A)]$ of the reduced density matrix $\rho_A=\mathrm{Tr}_\mathrm{B}[\rho]$. As it stands, the notion of entanglement entropy is rather abstract.

Indeed, to the best of our knowledge, despite the large body of literature dedicated to the study of entanglement entropies in many-body systems, the actual physical possibility of *measuring* this quantity, and so investigating experimentally its properties (such as scaling behavior), has not been addressed.

In the condensed matter context one is typically interested in a state ψ which is the ground state of a many-body local Hamiltonian. Relating entropy to measurable properties of such a state is the main motivation of this paper.

In the quantum information context, entanglement entropy asymototically quantifies the number of maximally entangled qubit pairs which can be distilled under LOCC (local quantum operations and classical communication), when one has a large number of identically prepared systems at hand, which can be coherently manipulated [6]. Note that while this property may be largely irrelevant to many-body systems, still, for certain critical systems it was shown that the single-copy entanglement [7] (defined as the number of pairs that can be obtained by LOCC from a single block) scales as half the entanglement entropy [8].

It is clear that, given the entire distribution of correlation functions and observables one may always recover the density matrix, and hence $\mathcal{S}_E$. A step in this direction was taken in Ref. [9], where it was emphasized that a quantum state may be reconstructed from the set of susceptibilities with respect to a large enough class of external potentials via the Hohenberg-Kohn theorem. In Ref. [9], this was used to calculate the linear entanglement entropy and negativity of at most four-dimensional subspace (for example, two spins) with a large system. Optimal strategies for addressing the purity of systems given a large number of copies were discussed in many works [5]. It is, however, unclear how the above methods can be implemented to find the entanglement between two large subsystems where the entire density matrix cannot be realistically measured nor many copies created and coherently manipulated.

In this paper we show that, given the ground state of a quantum many-body system, one may identify a class of natural observables, whose fluctuations can be related to the entanglement entropy. After some general considerations, we provide several examples of systems where the entanglement entropy can be estimated, or directly extracted, from the probability distribution $P(x)$ of the possible outcomes x of such observables. This connection will take different forms in different problems.

The first ingredient of our analysis consists in considering directly, instead of $\mathcal{S}_E$, the so-called "measurement entropy" $S[\hat{O}]$ of properly chosen observables $\hat{O}$, i.e., the Shannon entropy $S[\hat{O}]=-\Sigma_x P(x)\log_2 P(x)$ associated with the probability distribution $P(x)$ of the outcomes x of $\hat{O}$ [10]. In classical systems this quantity, measurable by definition, is always lower than the overall entropy. On the other hand, in quantum systems it can, in general, be either larger or smaller than the entanglement entropy [11]. Below we select a class of local observables such that $S[\hat{O}]$ provides a lower bound on the entropy S_E. While in some cases the bound is saturated and $S[\hat{O}]=S_E$, in other cases, where estimates based on $S[\hat{O}]$ do not capture the scaling behavior, we are nevertheless able to extract $\mathcal{S}_E$ from the distribution $P(x)$.

II. GENERAL LOWER BOUND ON ENTROPY USING LOCAL OBSERVABLES

The class of observables whose measurement enables us to estimate the entanglement entropy is constructed as follows. Given a state ψ of interest, let us denote by $\mathcal{L}$ the set of observables $\hat{O}=\hat{O}_A\otimes I+I\otimes\hat{O}_B$, acting locally on A and B, for which ψ is an eigenstate. Note that $\mathcal{L}$ is nonempty [12]. The Schmidt decomposition of ψ can be written as $\psi=\Sigma c_i^\alpha|\alpha,i\rangle\otimes|s-\alpha,i\rangle$, where s is the eigenvalue of $\hat{O}$ acting on ψ, and such that $\hat{O}_A|\alpha,i\rangle=\alpha|\alpha,i\rangle$ and $\hat{O}_B|s-\alpha,i\rangle=(s-\alpha)|s-\alpha,i\rangle$ (here i ranges over the degeneracy of eigenstates of $\hat{O}_A$ with value α).

One may write the reduced density matrix as $\rho_A=\mathrm{tr}_B\,\rho=\Sigma P_\alpha\rho_\alpha$, where we have defined $\rho_\alpha=1/P_\alpha\Sigma|c_i^\alpha|^2|\alpha,i\rangle\langle\alpha,i|$ with $P_\alpha=\Sigma_i|c_i^\alpha|^2$. Therefore, for the entanglement entropy one has

$$\mathcal{S}_E=S[\hat{O}_A]-\sum P_\alpha\mathrm{tr}\rho_\alpha\log_2\rho_\alpha\geqslant S[\hat{O}_A],\tag{1}$$

where $S[\hat{O}_A]$ is the measurement entropy associated with the probability distribution P_α.

The equality $\mathcal{S}_E=S[\hat{O}_A]$ is realized if and only if $\mathrm{tr}\,\rho_\alpha\log_2\rho_\alpha=0$ for all α, as, for example, in the case of no degeneracy of the eigenvalue α, or in more interesting cases where α is degenerate but the ρ_α's still describe pure states. Such systems do indeed exist: we will consider, for example, a BEC-like state, a two mode squeezed state, and the Dicke model. As remarked above the measurement entropy has the advantage of being directly measurable. By repeated measurements of $\hat{O}_A$ we can recover the distribution of outcomes, and so extract $S[\hat{O}_A]$. Moreover, we note that $\mathcal{S}_E=Max_{O\in\mathcal{L}}S[\hat{O}_A]$. To see this, choose the operator $\hat{O}_A$ diagonal with nondegenerate eigenvalues λ_i in the basis $|i_A\rangle$ appearing in the Schmidt decomposition of ψ, and $\hat{O}_B$ diagonal with eigenvalues $-\lambda_i$ in the basis $|i_B\rangle$, ψ is an eigenstate of eigenvalue 0 of $\hat{O}_A\otimes I+I\otimes\hat{O}_B$, and $S[\hat{O}_A]=\mathcal{S}_E$. Note that since $[O_A,\rho_A]=0$, Eq. (1) may be understood as an information loss due to coarse graining of classical probability distributions. As such it holds also for other entropy measures. For example, the *linear measurement entropy* of O, $S_L[\hat{O}]=\Sigma P(x)[1-P(x)]$ obeys $S_L\geqslant S_L[\hat{O}]$, where $S_L=1-\mathrm{tr}\,\rho^2$ is the linear entropy of the system. In the case $S[\hat{O}]=\mathcal{S}_E$ one also immediately has $S_L[\hat{O}]=S_L$.

Intuitively, a natural subset of $\mathcal{L}$ is the set of "conserved" operators, i.e., sums of local operators which commute with the Hamiltonian of the system and thus are in $\mathcal{L}$, for instance, the total spin operator for spin chains with rotational symmetry. These are the cases we consider. We remark, however, that since not for every Hamiltonian the set $\mathcal{L}$ necessarily contains such conserved observables [13], in the general case, the appropriate choice of $\hat{O}$ requires a more elaborate analysis.

III. EXAMPLES OF LOWER BOUND SATURATION

A. BEC

Let us now illustrate the general considerations above with two examples in which the equality $S[\hat{O}]=\mathcal{S}_E$ is actually satisfied. As a first illustrative example, consider a BEC condensate of bosons who share a particular Gross-Pitaevskii wave function $f(x)$, constrained to occupy a box of volume V. One can imagine dividing the box into two parts A and B, of volumes V_A and V_B, respectively. The condensate wave function can then be written as

$$\begin{aligned}\psi&=\frac{1}{\sqrt{N!}}(ua^\dagger+\sqrt{1-u^2}b^\dagger)^N|0\rangle\\&=\sum_{k=0}^{N}\sqrt{C_k^N}u^k(1-u^2)^{N-k}|k\rangle_A\otimes|N-k\rangle_B,\end{aligned}\tag{2}$$

where N is the number of bosons, C_k^N is the binomial coefficient, and $a^\dagger,b^\dagger$ create a particle in A,B, respectively, i.e., $a^\dagger=\frac{1}{u}\int_{x\in A}f(x)\psi^\dagger(x)dx$ and $b^\dagger=\frac{1}{\sqrt{1-u^2}}\int_{x\in B}f(x)\psi^\dagger(x)dx$, with $u^2=\int_{x\in A}|f(x)|^2dx$. Note that ψ is an eigenstate of the particle number operator $\hat{N}=\int_{x\in A}dx\psi_x^\dagger\psi_x+\int_{x\in B}dx\psi_x^\dagger\psi_x$. We may consider just the subspace consisting of applications of $a^\dagger$ and $b^\dagger$ to the vacuum, and ignore other modes. In this subspace there is no degeneracy since every occupation number appears only once and so we have simply $\mathcal{S}_E=S[a^\dagger a]=-\Sigma P_k\log_2(P_k)$, with $P_k=C_k^N(u^2)^{N-k}(1-u^2)^k$. Consequently the measurement entropy of particle numbers in A in this case is exactly the entanglement entropy. Assuming for simplicity $V_B\gg V_A$, and taking $f(x)$ as uniform $[f(x)=1/\sqrt{V_A+V_B}]$ then in the limit $N\gg1$, one obtains $\mathcal{S}_E\simeq\frac{1}{2}\log_2[Nu^2(1-u^2)]$, which, in the thermodynamic limit ($N\to+\infty$, keeping $\nu=N/V$ const), simplifies to $\mathcal{S}_E\simeq1/2\log_2(V_A\nu)$.

B. Squeezed states and Dicke model

A similar example of entangled states, but this time of two distinct degrees of freedom, is a two-mode squeezed state

$$\begin{aligned}\psi&=\exp[\xi a_1^\dagger a_2^\dagger-\bar{\xi}a_1a_2]|0\rangle\\&=\frac{1}{\cosh[\eta]}\sum_{n=0}^{+\infty}e^{in\Phi}[\tanh(\eta)]^n|n\rangle_1\otimes|n\rangle_2,\end{aligned}\tag{3}$$

where $\xi=\eta e^{i\Phi}$ is the squeezing parameter, and a_1, a_2 are the annihilation operators relative to the two modes. Identifying A and B with the two modes, ψ is an eigenstate of the sum of local operators $\hat{O}=\hat{n}_1\otimes I+I\otimes(-\hat{n}_2)$. Since the eigenstates of n_1 and n_2 are nondegenerate the entanglement entropy is $\mathcal{S}_E=S[\hat{n}_1]$. In particular, $\mathcal{S}_E=\log_2[1+\bar{n}]+\log_2[(1+\bar{n})/\bar{n}]\bar{n}$, where $\bar{n}=[\sinh(\eta)]^2$ is the average number of photons per mode. In the limit $\eta\to+\infty$, one obtains $S_{12}\approx\log_2[\bar{n}]$.

Squeezed states can be generated in a number of physical situations. A particularly interesting realization is obtained when a collection of two level atoms (subsystem A) interacts with a single mode of the EM field (subsystem B): the Dicke model. In turn, the Dicke model can be realized in a number of ways, both using traditional cavity quantum electrodynamics (QED), as well as employing solid state circuits [circuit QED] [14]. Assuming for simplicity the photon mode to be in resonance with the atoms, the Hamiltonian is

$$H = \omega_0 \hat{J}_z + \omega_0 a^\dagger a + \frac{\lambda}{\sqrt{2j}}(a^\dagger + a)(\hat{J}_+ + \hat{J}_-), \tag{4}$$

where $\hat{J}_k = \Sigma_{i=1}^{2j} s_k^i$ $[k = \pm, z]$ are collective operators describing the dynamics of the collection of $2j$ two level atoms. Working in the subspace where, at $\lambda = 0$, all atoms are in the ground state, in the large j limit, one can conveniently use the Holstein-Primakoff representation $\hat{J}_z = b^\dagger b - j$, $\hat{J}_+ = b^\dagger (\sqrt{2j - b^\dagger b})$, $\hat{J}_- = (\sqrt{2j - b^\dagger b})\, b$, where b are bosonic modes [15]. In the $j \to +\infty$ limit, one obtains $H = \omega_0 b^\dagger b + \omega_0 a^\dagger a + \lambda(a^\dagger + a)(b^\dagger + b) - j\omega_0$. This Hamiltonian describes two coupled harmonic oscillators. In particular, as a function of coupling λ, the Hamiltonian has a quantum critical point at $\lambda_c = \omega_0/2$, where the ground state symmetry with respect to parity is spontaneously broken. Setting $\lambda \leqslant \lambda_c$, it is convenient to introduce the operators $x = \frac{1}{\sqrt{2\omega_0}}(a + a^\dagger)$, $y = \frac{1}{\sqrt{2\omega_0}}(b + b^\dagger)$, $p_x = i\sqrt{\frac{\omega_0}{2}}(a^\dagger - a)$, and $p_y = i\sqrt{\frac{\omega_0}{2}}(b^\dagger - b)$. Writing down the Hamiltonian in terms of these operators, and making the rotation $q_1 = (x+y)/\sqrt{2}$, $q_2 = (x-y)/\sqrt{2}$ one obtains a quadratic Hamiltonian, the spectrum being that of two independent harmonic oscillators of frequencies $K\ e^{\pm 4\eta}$, where $K = \sqrt{\omega_0^4 - 4\lambda^2\omega_0^2}$, and $e^{4\eta} = \sqrt{(\omega_0^2 - 2\lambda\omega_0)/(\omega_0^2 + 2\lambda\omega_0)}$. The ground state can be written as

$$\psi = \left[\frac{K}{\pi^2}\right]^{1/4} e^{-1/2[e^{-2\eta}(K^{1/4}q_1)^2 + e^{2\eta}(K^{1/4}q_2)^2]}. \tag{5}$$

Introducing $\Phi_n(x)$, the eigenstates of the operator $H_x(K) = (p_x^2 + Kx^2)/2$, one may write

$$\psi = \frac{1}{\cosh(\eta)} \sum_{n=0}^{+\infty} [\tanh(\eta)]^n \Phi_n(x)\Phi_n(y). \tag{6}$$

Therefore, one immediately finds that, defining the operator $\hat{O} = H_x(K) \otimes I + I \otimes [-H_y(K)]$, the ground state of the model is an eigenstate with zero eigenvalue. In addition, $\mathcal{S}_E = S[H_x(K)]$. It is interesting to note that, for $\lambda \ll \lambda_c$, one can approximate $H_x \simeq \omega_0(a^\dagger a + \frac{1}{2})$, while for $\lambda \simeq \lambda_c$ one has $H_x \simeq p_x^2/2$. In analogy with the previously considered two-mode squeezed state, for $\lambda \simeq \lambda_c$ the entanglement entropy is given by $\mathcal{S}_E \simeq 2\log_2[\sinh(\eta)] \approx \frac{1}{4}\log_2[\omega_0/|\lambda - \lambda_c|]$ [15].

IV. LIMITATIONS OF THE LOWER BOUND

The lower bound on $\mathcal{S}_E$ provided by the measurement entropy has to be considered with care, in particular in extracting the scaling of $\mathcal{S}_E$ with subsystem size L_A in one (or higher) dimensional systems. In these physical situations, it is convenient to choose the operators $\hat{O}$ as corresponding to *extensive* observables, such as the total magnetization, or particle number. In this case, one expects the distribution of measurements outcomes x of $\hat{O}$ to be Gaussian in the limit of large subsystem, i.e., $P(x) \to \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-L)^2/2\sigma^2}$ for $L_A \to \infty$, For such observables we have $S[\hat{O}] = 1 + \frac{1}{2}\log_2(2\pi\sqrt{\langle \Delta x^2 \rangle})$. More generally, given the variance $\sqrt{\langle \Delta x^2 \rangle}$, the formula above gives always the maximal measurement entropy of a continuous variable, as one can easily check by a variational argument. In other words, the variance provides an upper bound to the measurement entropy.

While for such extensive operators one immediately obtains $S[\hat{O}] \approx \log_2(\sqrt{\langle \Delta x^2 \rangle}) \leqslant \mathcal{S}_E$, in many cases this inequality is of limited use, since, while $S[\hat{O}]$ scales at most logarithmically with the variance, the entanglement entropy scales, in fact, as the variance itself and not its logarithm. This implies that, in many cases, the variance is a useful entanglement measure. Indeed the variance is decreased by local measurements respecting superselection rules [16].

V. OTHER SCHEMES

A. *XY* spin chain and free fermions

To demonstrate this, let us now consider the *XY* spin chain [4] and cases of noninteracting fermions [17]. Under the Jordan-Wigner mapping one can treat the *XY* spin chain as a special case of noninteracting fermions on a 1D tight binding model, with no external potentials. The role of the total S_A^z observable $\Sigma_{i \in A} S_i^z$ is played by the fermion number operator $\Sigma_{i \in A} c_i^\dagger c_i$. The entanglement of free fermions is particularly interesting as it exhibits scaling violating area law in higher dimensions [18,19]. Indeed, a formula for the entropy scaling was presented in [18] in d dimensions, and was recently checked numerically for certain models in $2d$ and $3d$ [20]. The method presented here is independent of dimension and is also valid for fermions in the presence of external potentials. For a noninteracting fermion system the ground state is given by filling low energy modes ϕ_i. The entropy of a region A may be expressed as [21]

$$\mathcal{S}_E(L) = -\operatorname{tr}[M \log_2 M + (1-M)\log_2(1-M)], \tag{7}$$

where M is a matrix with elements $M_{ij} = \langle \phi_i | P_A | \phi_j \rangle$, where P_A is projection on the region A.

The distribution of fermion numbers may be extracted from the moment generating function $\chi(\lambda) = \Sigma P_\alpha e^{i\lambda\alpha} = \det(1 - M + Me^{i\lambda})$ [21]. Such functions appear in the theory of quantum optics, where one studies the distribution of detected photons. More recently the analog of χ was introduced in the theory of quantum transport in mesoscopic systems and is referred to as the full counting statistics of fermions [22].

The cummulants of fermion numbers (or, equivalently S_z) in region A are given by derivatives of $\log \chi$ at $\lambda = 0$. Taking derivatives give expressions of the form $\langle\langle(\delta S_z)^n\rangle\rangle = (-i)^n \partial_\lambda^n \chi|_{\lambda=0} = Q_{n,l} \operatorname{tr} M^l$.

Now $\operatorname{tr} M^l$ can be extracted from the cummulants by inverting Q, i.e., $\operatorname{tr} M^l = Q_{l,n}^{-1} \langle\langle(\delta S_z)^n\rangle\rangle$. The matrix Q^{-1} is a lower triangular (i.e., $Q_{l,n}^{-1} = 0$ for $l > n$) and may be calculated to any desired order. For example, $\operatorname{tr} M = \langle\langle \delta S_z \rangle\rangle$, $\operatorname{tr} M^2$

$=\langle\langle \delta S_z\rangle\rangle - \langle\langle \delta S_z^2\rangle\rangle$, and tr $M^3=\langle\langle \delta S_z\rangle\rangle - \frac{3}{2}\langle\langle \delta S_z^2\rangle\rangle + \frac{1}{2}\langle\langle \delta S_z^3\rangle\rangle$.

We may write the entropy as the series, convergent whenever the eigenvalues of M are away from 0 and 1

$$\begin{aligned}\mathcal{S}_E &= -\sum_{n=2}^{\infty} \frac{1}{n(n-1)}\text{tr}[M^n - 1 + (1-M)^n] \\ &= -\sum_{n=2}^{\infty} \frac{1}{n(n-1)}\Bigg(Q^{-1}_{l,n}\langle\langle(\delta S_Z)^n\rangle\rangle \\ &\quad + \sum_{k=1}^{n} C_k^n Q^{-1}_{k,j}(-1)^j \langle\langle(\delta S_Z)^j\rangle\rangle\Bigg). \qquad (8)\end{aligned}$$

For any *finite* matrix M this series converges as each term is smaller than dim M, it is easy to check that, in fact, $\mathcal{S}_E \leqslant \log_2 2\dim M$ as should be. One must note, however, that convergence may be slow, depending on the eigenvalues of M close to unity or zero. Fortunately, all the terms in the series are positive, so the lower bound improves by adding more terms. In fact, the first nonvanishing contribution, namely $n=2$ in the series corresponds to the variance, and turns out to scale in the same way as entropy in the free fermion case [18].

Remarkably, the coefficients $(Q^{-1})_{l,n}$ appearing are universal—i.e., will fit any noninteracting fermion theory, independent of its eigenmodes. One possibility is to measure cold fermionic atoms in a trap; by measuring the histogram of particle numbers in a certain region of the trap, one could estimate the entanglement entropy using Eq. (9) regardless of the particular trap details.

B. Random spin chains

The intimate relation between bipartite entanglement and fluctuations of a conserved quantity is not restricted to noninteracting systems, but extends also to the strongly disordered spin-$\frac{1}{2}$ XXZ model:

$$\mathcal{H} = \sum_i J_i(\hat{S}_i^x S^x{}_{i+1} + \hat{S}_i^y S^y{}_{i+1} + \lambda \hat{S}_i^z S^z{}_{i+1}) \qquad (9)$$

with $-\frac{1}{2}\leqslant\lambda\leqslant 1$ and the J_i are positive and randomly distributed. This interesting class of interacting random 1D theories was recently discussed in Ref. [23], where it was shown that the bipartite entanglement entropy of a segment of length L with the rest of the chain is $\mathcal{S}_E(L)=\frac{1}{6}\ln 2\ \log_2 L$. The ground state of the Hamiltonians in Eq. (9) consists of a frozen liquid of valence bonds, which is referred to as the random singlet phase [24–26].

The Hamiltonian (9) only commutes with $\hat{S}^z_{total}=\Sigma_i\hat{S}_i^z$ (note, however, that its ground state has a full rotational symmetry). Therefore, $\hat{S}_A=\Sigma_{i\in A}\hat{S}_i^z$ is the operator of choice for estimating the entanglement between part A and the rest of the chain. In the random singlet phase there are two types of singlets: (a) N_{AB} singlets connecting between A and B, (b) $N_{AA}+N_{BB}$ singlets connecting sites in A to other sites in A, or sites in B to other sites in B. As explained in Ref. [23], each singlet contributes 1 to the entropy, and therefore: $\mathcal{S}_E=N_{AB}$. In addition, each singlet contributes $\frac{1}{4}$ to the variance of $\hat{S}_A$. In this case indeed we have the relation

$$\mathcal{S}_E = 4\langle\Delta\hat{S}_A^2\rangle = N_{AB}. \qquad (10)$$

Note that Eq. (9) still applies in this case where the full counting statistics of the z-direction spin becomes $\chi(\lambda)=\Sigma P_\alpha e^{i\lambda\alpha}=\det[(1-M)e^{-i\lambda/2}+Me^{i\lambda/2}]$, with M now being a diagonal matrix with entries $\frac{1}{2}$ for each singlet of type (a), and 0 otherwise.

Equation (10) raises the question whether there is an even more direct relationship between variance of conserved, quantized, quantities, and entanglement in the context of spin chains. We considered this for resonating valance bond (RVB) states of six spins with varying weights for each singlet configuration, and found that although in most of the Hilbert space var$S_A^z\leqslant\mathcal{S}_E$, a small region exists where this inequality is violated. This is associated with the formation of strong ferromagnetic correlations. It is possible that by putting a few restrictions on a RVB state one can prove a general relation. Nevertheless, the variance of a conserved quantity of a quantized object can be thought of as an *ad hoc* measure of entanglement.

VI. DISCUSSION

The interest in entanglement is wide spread due to the prospect of engineering and controlling entangled states in which two [or more] microscopic objects, although separated by a macroscopic distance, display quantum correlations. This gives even more urgency to understanding how to measure entanglement *quantitatively*. To determine whether they have such exotic states, experimenters carry out Bell inequality violation tests (c.f. Refs. [27,28]). Could the schemes presented here realistically quantify entanglement in many-body systems? So far we considered the entanglement entropy in the ground state, thus implicitly assuming that the temperature is $T=0$. Realistically, however, the temperature is finite, the state under consideration is not in a pure state, and Eq. (1) should, in principle, be generalized to account for thermal fluctuations. However, one can always imagine obtaining a lower bound on the ground state entanglement entropy by extrapolating the appropriate measurement entropy, as detailed above, to $T=0$. We leave the investigation of the efficacy of such realistic procedures for future work.

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